PROFESSOR: Now we go to problem 6.12. That is a rather complicated problem.

I have an object A here with a certain mass mA. It's on a horizontal frictionless table. I have an object B here, and it has a mass mA. And I'm squeezing this between my two hands so that the spring, which is now relaxed becomes smaller, it becomes shorter. Makes this A and this is B. So the spring now has a length l instead of l0, which was the relaxed length. And I let it go and the system starts to oscillate.

It is key in this whole problem, absolutely essential that you recognize that the center of mass of this system always remains at the same location. No matter what happened with A and B, c remains put. Even when I squeeze it on this frictionless table, remember, action equals minus reaction. That means any force that I put on here to push A down must be exactly the same in magnitude as the force that I apply to B to push it up. Because action equals minus reaction. So there's no external force ever on this system, not even when I squeeze it down. And that means that the center of mass will stay in place. Or, which your almost saying the same thing that P total of the whole system equals 0.

Let us call the position vector of A at any moment in time rA and let's call the position vector of B at any moment in time rB.

Then, mA times rA plus mB times rB equals 0. This is the total mass of A and B times r of the center of mass relative to the center of mass. But since I used the center of mass as my origin, this r is 0. And so you see this is-- in fact, you can even say that this is the definition of the center of mass.

If I take the derivative of this equation, let's call this equation 1. Then I find mA times the velocity A. I'll put an s here to remind you that it is a velocity due to a spring in the y direction. And the reason why I do that is later we're going to have velocities in the x direction, and it becomes confusing. Plus mB times the velocity B due to the spring. That equals 0. That's equation 2. And what you see here is that this equation is exactly saying P total equals 0.

So it is important that you always recognize this throughout the entire motion, this always has to be obeyed, and this has to be obeyed. Always, at any moment in time. The center of gravity has to stay in place.

Now in our problem, we know that mB happened to be twice mA. Let's call mA little m, so that is 2m. So it's followed immediately that the magnitude of rA, the magnitude of the displacement vector of A is twice the magnitude of the displacement vector of B. That follows immediately from this equation. And it also follows that m times vA vectorially, plus 2m times vB due to the spring vectorially must be 0. And so, vB-- again, the s reminds you that we're dealing with the spring-- equals minus 1/2 vA due to the spring velocity. This is a vector equation.

And what is the minus telling me? The minus is telling me that the two are out of phase. When this one goes down, the other one goes up. Well of course it has to be that way because otherwise the center of mass could never stay in place. So they go exactly out of unison. And the displacement of A relative to the center of mass at all moments in time will always be twice as large as the displacement of B relative to the center of mass. And keep that in mind throughout this whole problem. Now

I'm going to make a plot of the positions of A, B, and C as a function of time. Not as a function of x as you have seen in your book. But I'm going to do it as a function of time, which is almost the same when you think of it. So this is C. The center of mass will always stay at the same position as a function of time.

Well let's assume that A was here originally in a relaxed position and that B, which must be 1/2 this-- B is here. This is when the situation is completely relaxed. In other words, this has length l0 and this is 2/3 l0 and this is 1/3 l0. A is going to oscillate about this position and B is going to oscillate about this position.

I pushed them in. I bring A to this position and as I do that I push B up by only 1/2 that amount. Let's push A a little further down. Let's push it there. I want to make the difference quite large. So forget this. So here is A, and I therefore push B up automatically by 1/2 this distance. That is automatic because the center of mass stays in place. And so B is going to oscillate back and forth. This is its maximum excursion in one direction and this will be its maximum excursion in the other direction. A will be here as its maximum excursion and it will be here at its maximum excursion. And this is the equilibrium line for A and this is the equilibrium line for B.

Well, I let go. I have it squeezed in and I let them go. Well, A wants to go up and B wants to go down. Well, I'll let them. B is going down. Simple harmonic motion. Nice cosine curve. Right here B is going through equilibrium. At that very same moment, A must also go through equilibrium. Otherwise the center of mass would not stay in place. So I can now draw the curve for A. And so on.

I can put in some more numbers here. This here, when A and B are both at equilibrium is of course, l0. This distance here, if we make the total length of the spring after we squeeze it l, then this is 2/3 l. And this is 1/3 l. Remember that's the only way that the center of mass can stay in place. The amplitude of the oscillation of B would then be 1/3 times l0 minus l. And the amplitude of the oscillation of A would be 2/3 times l0 minus l. And I think this is about all I can say. And everything that we've just discussed is implicit in this drawing.

As I've just shown you, v vectorially of the spring equals minus 2 point B. In other words, the speed of A-- I put bars here, so I'm not interested in the direction-- is always twice the speed at any moment in time. The speed of A is twice the speed of B. Even if the speed of A is 0, namely when they both come to a halt. Then it's still correct. Because 0 is 2 times 0. So that's still correct. So the kinetic energy in A due to the spring now alone-- I'll put the s there, obnoxiously. I will repeat the s all the time. Equals 1/2 m vA squared due to the spring plus 1/2 m-- oh, 2m because the mass of B is 2m times vB due to the spring squared. Since vA is twice vB, this is [? v ?] four times vB squared. So I can put a force here. And so I can put a 4 here. So I can include now that the kinetic energy of A at any moment in time is always twice the kinetic energy of B.

I realize that the mass of B is twice as high. But therefore, the velocity of A is always twice as high and the kinetic energy goes with v squared. So that's why the kinetic energy in A wins it from the kinetic energy in B. This holds at any moment in time. Also, at the moments that the kinetic energy of both are 0. When they come to a halt and when they return and come to a halt this still holds because 0 is still 0.

Now so far, we haven't introduced any motion in the x direction. So after I have squeezed, let's suppose that I have put in-- I've done a certain amount of work, and I have put in a certain amount of potential energy, which would be 1/2 k times the amount by which I have reduced the lengths of the spring. So that would be l0 minus l squared. That's the energy that I have put in. So at any moment in time after this because I squeeze it in and then I let it go, at any moment in time, this is the total energy in the system, which now is partially kinetic and partially potential. Here is the potential part at any moment in time and here is the kinetic energy of A as a function of time. s reminds you of the spring. Plus 1/2 2m times the speed squared of B at any moment in time times s. This must be equal always to U0, and I'll put an s here. This is the conservation of energy. We're only talking about the spring. With kinetic energy in Am kinetic energy in B and the potential energy in the spring, which is left over. And there are moments that these two are 0. Then everything is in potential energy of the spring. And that's the case when the two objects come to a halt.

There are moments that this term is 0. When the two go through equilibrium, they have a maximum speed. When they go through their equilibrium here and here, they have the maximum speed, and then all energy is converted to kinetic energy. And you can usually calculate by making this 0 that v of A maximum due to the spring equals the square root of 4/3 U0s divided by m. And you can calculate that vB max due to the spring is exactly 1/2 that. That follows immediately from this equation. So far, so good.

Now I'm introducing simultaneously a motion of the whole system in the x direction. So now we have a velocity of A in the x direction, which is the same as velocity of B in the x direction, which is the same as the velocity of the center of mass in the x direction. And so the whole system moves with a uniform velocity in the x direction while it is oscillating. And this will never change. There is no force in the x direction. There's no friction. So this motion will continue.

Now let's look at the moment in time whereby the velocity of A due to the spring in the y direction has this value. And the velocity of A in the x direction is never changing. I even drop the A's. It's just v of x. So this is A. Then, the vectorial sum is the real net velocity of A. This is the vectorial sum v of A total.

And so, if we want to know what the kinetic energy total is of A, kinetic energy of A total. That is 1/2 m times v of A total squared. I don't need the arrow because this is a scalar vA squared. But this of course, according to Pythagoras is also 1/2 m times vAs squared plus 1/2 m times vx squared. And this is the kinetic energy, which is exclusively in the spring. And this is the kinetic energy which is exclusively in the motion in the x direction. So we have here now some kind of a luxury so to speak, that the total kinetic energy of the system-- in this case, because the two directions are perpendicular to each other can be thought of as a pure component of the kinetic energy due to the motion in the x direction plus another pure component of the kinetic energy in the y direction, which is due to the spring. That's why I give it a sub index s. And that's very handy. And that's what you see here.

So if now we'll want to know what the total energy of the system is, E total of the whole system, that includes both the spring and the motion in the x direction. Well, all the energy in the spring is U0s. That's the potential energy I put in when I squeezed it down. Now in a later moment in time there may be less potential energy in the spring, but then there is more kinetic energy in the motion of A and B due to the spring. But this is the maximum that there is in the spring. But now I have to add the kinetic energy of motion for A in the x direction. And I have to add the kinetic energy of motion for B in the x direction. And that gives you these two terms.

Now, of course, I could also write for this: at any moment in time, the potential energy that is in the spring plus the kinetic energy A total. That means spring plus x. Plus kinetic energy of B total. That means spring plus x. These two are equivalent.

And so, if someone gave you this total number and this total number and you knew what E total was, well, then you can clearly calculate what the potential energy is that remains for the spring. And I think that is part of your question C.

Now I want to ask you something extra, which is to calculate the frequency of oscillation of these objects A and B. It may not be so easy, but when I dry run this session I had two minutes. Two and 1/2 minutes over. So I thought I can probably just do that in that amount of time.

So the question now is, what is the frequency omega of A. And of course, B should have the same frequency of motion. Well, it's clear that A is going to oscillate in a simple harmonic fashion. And that the amplitude of A is going to be 2/3 times l0 minus l. And it has a frequency omega, and B will be a simple harmonic oscillation. It will have an amplitude 1/3 l0 minus l. But it better have the same omega and they are 180 degrees out of phase. When one goes up, the other one goes down.

How now would I calculate the frequency omega? Well, I think of this as one spring where A is attached to it and here is C. I cut the spring here. I can do that. C is clamped. I can put C in a vice and this length is only 2/3 in its relaxed state as the total length. And for B, we have one, which is only 1/3 l. That is the separation, the distance to C. So this is B.

And so the $64 question now is, what is the equivalent spring constant if the total spring has spring constant k, what is the equivalent spring constant of a spring, which has lengths 2/3 of l0? And I will call that k off A. So this is not there. It is one spring with lengths 2/3 l0. I claim that the equivalent spring constant is 3/2 times k. And that kB equals 3 times k. And I want you to work on that and convince yourself of that. I'll help you a little bit. And that is I'll give you a problem inside a problem. I have two springs here. One with spring constant k1 and one with spring constant k2. And I displace this object over a distance x. And when I do that object has mass m. There is a spring force Fs, which is minus k.

Let's call it k prime times x. And k prime obviously follows from k1 and k2. I want you to prove that 1 over k prime equals 1 over k1 plus 1 over k2. Notice that k prime is less than k1 and that k prime is also less than k2. In other words, a shorter string is always sloppier. Sorry. The two springs together is sloppier than just one spring. And that may actually be consistent with your intuition. If you have a very long string, it is much sloppier than when you cut it in 1/2 are you try to push 1/2 the spring in. It'll take you a larger force to push it in over the same amount. That may be consistent with your experienced and with your intuition.

Once you accept this, it's now very easy to calculate the frequency. Because now I simply have here kA, which is 2/3, which is 3/2 k. And I have a mass here m and this is fixed here. And so I know that omega A equals the square root of 3/2 k over m. It's kA over m with k through to k over m. And now I have here, which is spring, which is also fixed here. Which has a kB, which is 3k. The mass is 2m, so what is omega B? That is the square root of 3k over 2m. And lo and behold, the two are the same. And they better be the same. They better be the same.

So, you see you can calculate the frequency of A and B. An easy way would be to think of the spring as being cut at point C and to think as A as making an independent oscillation. Whereby the spring is fixed at C and making B-- having B also make an independent oscillation. Of course, since the two are together-- sorry. I have a slight cold, Since the two are together, they are in opposite direction. They are out of phase 180 degrees. We've discussed that at length. And the amplitude of A is always twice the amplitude of B. In fact, the position of A from the center of mass at any moment in time is always twice the distance from B to the center of mass. And so these are the frequencies of the oscillations.