Module 14: Application of the Principle of Conservation of Energy

In the preceding chapter we consider closed systems $\Delta E_{\text{system}} = 0$ in which the only interactions on the constituents of a system were due to conservative forces. This enables us to define the concepts of potential energy and the conservation of mechanical energy. We shall now apply the Principle of Conservation of Energy to analyze the change in energy of a system and deduce how the velocity of the constituent components of a system will change between some initial state and some final state.

14.1 Principle of Conservation of Energy

Recall when a system and its surroundings undergo a transition from an initial state to a final state, the total change in energy is zero,

$$\Delta E^{\text{total}} = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0\,. \tag{14.1.1}$$

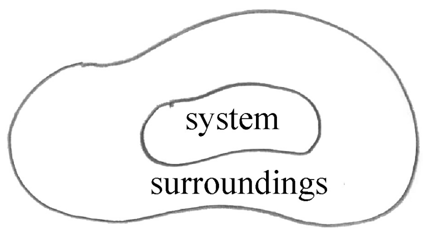


Figure 14.1 A diagram of a system and its surroundings

Energy can also flow into or out of the system across a boundary. A system in which no energy flows across the boundary is called a *closed system.* Then the total change in energy of the system is zero,

$$\Delta E^{\text{closed}}_{\text{system}} = 0\,. \tag{14.1.2}$$

For a closed system (no external forces) with only conservative internal forces, the total change in the mechanical energy is zero,

$$\Delta E_{\text{mechanical}} = \Delta K_{\text{system}} + \Delta U_{\text{system}} = 0\,. \tag{14.1.3}$$

Change of Mechanical Energy for a Closed System with Internal non-Conservative Forces

Consider a closed system that undergoes a transformation from an initial state to a final state by a prescribed set of changes.

Definition: **Non-conservative Force**

> Whenever the work done by a force in moving an object from an initial point to a final point depends on the path, the force is called a *non-conservative force*.

Suppose the internal forces are both conservative and non-conservative. The total work done by the internal forces is a sum of the internal conservative work $W_{\text{c, internal}}^{\text{total}}$, which is path-independent, and the internal non-conservative work $W_{\text{nc, internal}}^{\text{total}}$, which is path-dependent,

$$W_{\text{internal}}^{\text{total}} = W_{\text{c, internal}}^{\text{total}} + W_{\text{nc, internal}}^{\text{total}} . \tag{14.1.4}$$

The work done by the internal conservative forces is equal to the negative of the change in the total internal potential energy

$$\Delta U^{\text{total}} = -W_{\text{c, internal}}^{\text{total}} . \tag{14.1.5}$$

Substituting Equation (14.1.5) into Equation (14.1.4) yields

$$W_{\text{internal}}^{\text{total}} = -\Delta U^{\text{total}} + W_{\text{nc, internal}}^{\text{total}} . \tag{14.1.6}$$

Since the system is closed, the total work done is equal to the change in the kinetic energy,

$$\Delta K_{\text{system}} = W_{\text{internal}}^{\text{total}} = -\Delta U^{\text{total}} + W_{\text{nc, internal}}^{\text{total}} . \tag{14.1.7}$$

We can now substitute Equation (14.1.7) into our expression for the change in the mechanical energy, Equation (14.1.3), with the result

$$\Delta E_{\text{mechanical}}^{\text{total}} \equiv \Delta K_{\text{system}} + \Delta U_{\text{system}}^{\text{total}} = W_{\text{nc, internal}}^{\text{total}} . \tag{14.1.8}$$

The mechanical energy is no longer constant. The total change in energy of the system is zero,

$$\Delta E_{\text{system}}^{\text{total}} = \Delta E_{\text{mechanical}}^{\text{total}} - W_{\text{nc, internal}}^{\text{total}} = 0 . \tag{14.1.9}$$

Energy is conserved but some mechanical energy has been transferred into non-recoverable energy $W^{\text{total}}_{\text{nc, internal}}$. We shall refer to processes in which there is non-zero non-recoverable energy as *irreversible processes*.

Change of Mechanical Energy for a System in Contact with Surroundings

When the system is no longer closed but in contact with its surroundings, the total change in energy is zero, so from Equation (14.1.1) the total change in energy of the system is equal to the negative of the change in energy from the surroundings,

$$\Delta E_{\text{system}} = -\Delta E_{\text{surroundings}} \tag{14.1.10}$$

The energy from the surroundings can be the result of external work done by the surroundings on the system or by the system on the surroundings,

$$W^{\text{total}}_{\text{ext}} = \int_A^B \vec{\mathbf{F}}^{\text{total}}_{\text{ext}} \cdot d\vec{\mathbf{r}} \,. \tag{14.1.11}$$

This work will result in the system undergoing coherent motion. If the system is in thermal contact with the surroundings, then thermal energy $\Delta E_{\text{thermal}}$ can flow into or out of the system. This will result in either an increase or decrease in random thermal motion of the molecules inside the system. There may also be other forms of energy that enter the system, for example radiative energy $\Delta E_{\text{radiative}}$.

Several questions naturally arise from this set of definitions and physical concepts. Is it possible to identify all the conservative forces and calculate the associated changes in potential energies? How do we account for non-conservative forces such as friction that act at the boundary of the system?

14.2 Dissipative Forces: Friction

Suppose we consider an object moving on a rough surface. As the object slides it slows down and stops. While the sliding occurs both the object and the surface increase in temperature. The increase in temperature is due to the molecules inside the materials increasing their kinetic energy. This random kinetic energy is called *thermal energy*. Kinetic energy associated with the coherent motion of the molecules of the object has been dissipated into kinetic energy associated with the random of motion of the molecules composing the object and surface.

If we define the system to be just the object, then the friction force acts as an external force on the system and results in the dissipation of energy into both the block and the surface. Without knowing further properties of the material we cannot determine the exact changes in the energy of the system.

Friction introduces a problem in that the point of contact is not well defined because the surface of contact is constantly deforming as the object moves along the surface. If we considered the object and the surface as the system, then the friction force is an internal force, and the decrease in the kinetic energy of the moving object ends up as an increase in the internal random kinetic energy of the constituent parts of the system.

When there is dissipation at the boundary of the system, we need an additional model (thermal equation of state) for how the dissipated energy distributes itself among the constituent parts of the system. We shall return to this problem when we study the thermal properties of matter.

Source Energies

Consider a person walking. The friction force between the person and the ground does no work because the point of contact between the person's foot and the ground undergoes no displacement as the person applies a force against the ground is not displaced (there may be some slippage but that would be opposite the direction of motion of the person). However the kinetic energy of the body increases. Have we disproved the work energy theorem? The answer is no! The chemical energy stored in the body tissue is converted to kinetic energy and thermal energy. Since the person can be treated as an isolated system, we have that

$$0 = \Delta E_{\text{system}}^{\text{closed}} = \Delta E_{\text{chemical}} + \Delta E_{\text{thermal}} + \Delta E_{\text{mechanical}} . \qquad (14.2.1)$$

We can assume that there is no change in the potential energy of the system, thus $\Delta E_{\text{mechanical}} = \Delta K$. Therefore some of the internal chemical energy has been transformed into thermal energy and the rest has changed the kinetic energy of the system,

$$-\Delta E_{\text{chemical}} = \Delta E_{\text{thermal}} + \Delta K . \qquad (14.2.2)$$

14.2 Worked Examples

Example 14.2.1 ***Escape Velocity Toro***

The asteroid Toro, discovered in 1964, has a radius of about $R = 5.0\,\text{km}$ and a mass of about $m_{\text{Toro}} = 2.0 \times 10^{15}\,\text{kg}$. Let's assume that Toro is a perfectly uniform sphere. What is the escape velocity for an object of mass m on the surface of Toro? Could a person reach this speed (on earth) by running?

Solution:

The only potential energy in this problem is the gravitational potential energy. We choose the zero point for the potential energy to be when the object and Toro are an infinite distance apart, $U_{\text{gravity}}(\mathrm{r}_0 = \infty) \equiv 0$. With this choice, the potential energy when the object and Toro are a finite distance r apart is given by

$$U_{\text{gravity}}(r) = -\frac{G m_{\text{Toro}}\, m}{r} \tag{14.2.3}$$

with $U_{\text{gravity}}(\mathrm{r}_0 = \infty) \equiv 0$. The expression *escape velocity* refers to the minimum speed necessary for an object to escape the gravitational interaction of the asteroid and move off to an infinite distance away. If the object has a speed less than the escape velocity, it will be unable to escape the gravitational force and must return to Toro. If the object has a speed greater than the escape velocity, it will have a non-zero kinetic energy at infinity. The condition for the escape velocity is that the object will have exactly zero kinetic energy at infinity.

We choose our initial state, at time t_0, when the object is at the surface of the asteroid with speed equal to the escape velocity. We choose our final state, at time t_f, to occur when the separation distance between the asteroid and the object is infinite.

Initial Energy: The initial kinetic energy is $K_0 = \frac{1}{2} m v_{\text{esc}}{}^2$. The initial potential energy is $U_0 = -G\frac{m_{\text{Toro}}\, m}{R}$**,** and so the initial mechanical energy is

$$E_0 = K_0 + U_0 = \frac{1}{2} m v_{\text{esc}}^2 - G\frac{m_{\text{Toro}}\, m}{R}. \tag{14.2.4}$$

Final Energy: The final kinetic energy is $K_f = 0$, since this is the condition that defines the escape velocity. The final potential energy is zero, $U_f = 0$ since we chose the zero point for potential energy at infinity**. T**he final mechanical energy is then

$$E_f = K_f + U_f = 0. \tag{14.2.5}$$

There is no non-conservative work, so the change in mechanical energy

$$0 = W_{\text{nc}} = \Delta E_{\text{mech}}, \tag{14.2.6}$$

is then

$$0 = \frac{1}{2} m v_{\text{esc}}^2 - G\frac{m_{\text{Toro}}\, m}{R}. \tag{14.2.7}$$

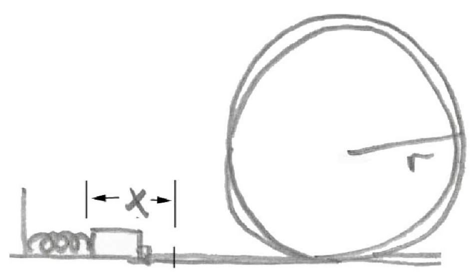
This can be solved for the escape velocity,

$$\begin{aligned} v_{\text{esc}} &= \sqrt{\frac{2Gm_{\text{Toro}}}{R}} \\ &= \sqrt{\frac{2(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2\cdot\text{kg}^{-2})(2.0\times10^{15}\,\text{kg})}{(5.0\times10^{3}\,\text{m})}} = 7.3\,\text{m}\cdot\text{s}^{-1} \end{aligned}. \tag{14.2.8}$$

Considering that Olympic sprinters typically reach velocities of $12\,\text{m}\cdot\text{s}^{-1}$, this is an easy speed to attain by running on earth. It may be harder on Toro to generate the acceleration necessary to reach this speed by pushing off the ground, since any slight upward force will raise the runner's center of mass and it will take substantially more time than on earth to come back down for another push off the ground.

Example 14.2.2 ***Spring-Loop-the-Loop***

A small block of mass m is pushed against a spring with spring constant k and held in place with a catch. The spring compresses an unknown distance x. When the catch is removed, the block leaves the spring and slides along a frictionless circular loop of radius r. When the block reaches the top of the loop, the force of the loop on the block (the normal force) is equal to twice the gravitational force on the mass.



a) Using conservation of energy, find the kinetic energy of the block at the top of the loop.

b) Using Newton's Second Law, derive the equation of motion for the block when it is at the top of the loop. Specifically, find the speed v_{top} in terms of the gravitational constant g and the loop radius r.

c) What distance was the spring compressed?

Solution:

a) Initial Energy: Choose for the initial state the instant before the catch is released. The initial kinetic energy is $K_0 = 0$. The initial potential energy is non-zero, $U_0 = (1/2)k\,x^2$. The initial mechanical energy is then

$$E_0 = K_0 + U_0 = (1/2)k\,x^2 . \qquad (14.2.9)$$

Final Energy: Choose for the final state the instant the block is at the top of the loop. The final kinetic energy is $K_f = \frac{1}{2}m\,v_{\text{top}}^2$; the mass is in motion with speed v_{top}. The final potential energy is non-zero, $U_f = (mg)(2R)$. The final mechanical energy is then

$$E_f = K_f + U_f = 2mgR + \frac{1}{2}m\,v_{\text{top}}^2 . \qquad (14.2.10)$$

Non-conservative Work: Since we are assuming the track is frictionless, there is no non-conservative work.

Change in Mechanical Energy: The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_{\text{mechanical}} = E_f - E_0 . \qquad (14.2.11)$$

Mechanical energy is conserved, $E_f = E_0$, or

$$2mgR + \frac{1}{2}m\,v_{\text{top}}^2 = \frac{1}{2}k\,x^2 . \qquad (14.2.12)$$

You may note that we pulled a fast one, sort of, in that both Equations (14.2.9) and (14.2.10) assumed that the gravitational potential energy is zero at the bottom of loop. If we had set the vertical height h above the bottom of the track to correspond to zero gravitational potential energy to be someplace else, we would have needed to subtract mgh from both equations, but this term would cancel in Equation (14.2.11) and Equation (14.2.12) remains the same.

From Equation (14.2.12), the kinetic energy at the top of the loop is

$$\frac{1}{2}m\,v_{\text{top}}^2 = \frac{1}{2}k\,x^2 - 2mgR . \qquad (14.2.13)$$

b) At the top of the loop, the forces on the block are the gravitational force of magnitude mg and the normal force of magnitude N, both directed down. Newton's Second Law in the radial direction, which is the downward direction, is

$$-mg - N = -\frac{m v_{\text{top}}^2}{R}. \tag{14.2.14}$$

In this problem, we are given that when the block reaches the top of the loop, the force of the loop on the block (the normal force, *downward* in this case) is equal to twice the weight of the block, $N = 2mg$. The Second Law, Equation (14.2.14), then becomes

$$3mg = \frac{m v_{\text{top}}^2}{R}. \tag{14.2.15}$$

We can rewrite Equation (14.2.15) in terms of the kinetic energy as

$$\frac{3}{2} mg R = \frac{1}{2} m v_{\text{top}}^2. \tag{14.2.16}$$

c) Combing Equations (14.2.13) and (14.2.16) yields

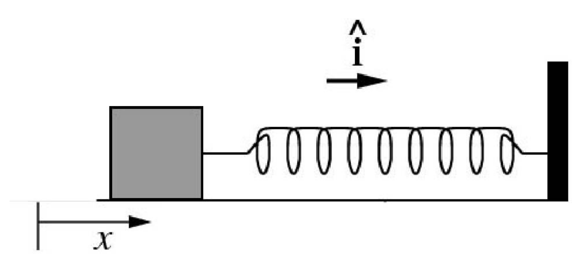
$$\frac{7}{2} mg R = \frac{1}{2} k x^2. \tag{14.2.17}$$

Thus the initial displacement of the spring from equilibrium was

$$x = \sqrt{\frac{7mg R}{k}}. \tag{14.2.18}$$

Example 14.2.3 ***Mass-Spring on a Rough Surface***

A block of mass m slides along a horizontal table with speed v_0. At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$, where b is a positive constant. Find the loss in mechanical energy when the block first momentarily comes to rest.



Solution:

From the model given for the frictional force, we could find the nonconservative work done, which is the same as the loss of mechanical energy, if we knew the position x_f where the block first comes to rest. The most direct (and easiest) way to find x_f is to use the work-energy theorem.

The initial mechanical energy is $E_0 = mv_0^2 / 2$ and the final mechanical energy is $E_f = k\,x_f^2 / 2$ (note that there is no potential energy term in E_0 and no kinetic energy term in E_f). The difference between these two mechanical energies is the nonconservative work done by the friction force,

$$\begin{aligned} W_{\mathrm{nc}} &= \int_{x=0}^{x=x_f} F_{\mathrm{nc}}\,dx = \int_{x=0}^{x=x_f} -F_{\mathrm{friction}}\,dx = \int_{x=0}^{x=x_f} -\mu\,N\,dx \\ &= -\int_0^{x_f} b\,x\,mg\,dx = -\frac{1}{2}bmg\,x_f^2. \end{aligned} \tag{14.2.19}$$

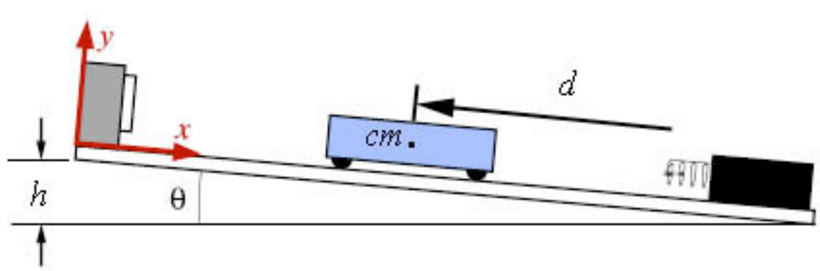
We then have that

$$\begin{aligned} \Delta E_{\mathrm{mech}} &= W_{\mathrm{nc}} \\ E_f - E_0 &= W_{\mathrm{nc}} \\ \frac{1}{2}k\,x_f^2 - \frac{1}{2}mv_0^2 &= -\frac{1}{2}bmg\,x_f^2. \end{aligned} \tag{14.2.20}$$

Solving the last of these equations for x_f^2 gives

Example 14.2.4 Cart-Spring on an Inclined Plane Solutions

An object of mass m slides down a plane that is inclined at an angle θ from the horizontal. The object starts out at rest. The center of mass of the cart is a distance d from an unstretched spring that lies at the bottom of the plane. The spring can be taken as being massless, and has a spring constant k.



a) Assume the inclined plane to be frictionless. How far will the spring compress when the mass first comes to rest?

b) Now assume that the inclined plane has a coefficient of kinetic friction μ_k. How far will the spring compress when the mass first comes to rest?

c) In case b), how much energy has been lost to friction?

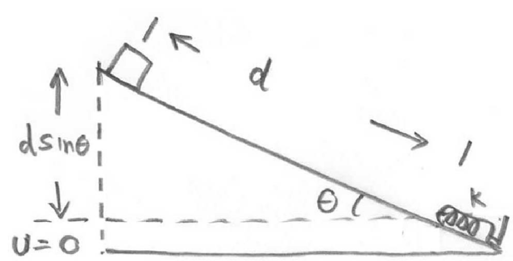
As we've seen before, the friction is primarily between the wheels and the bearings, not between the cart and the plane, but the friction force may be modeled by a coefficient of friction μ_k.

Solution:

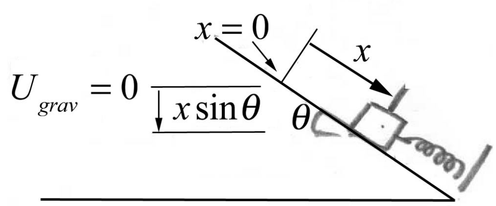
Let x denote the displacement of the spring from the equilibrium position. Choose the zero point for the gravitational potential energy $U_{\text{grav}}(x=0)=0$ not at the very bottom of the inclined plane, but at the location of the end of the unstretched spring. Choose the zero point for the spring potential energy when the spring is at its equilibrium position, $U_{\text{spring}}(x=0)=0$.

a) **Initial Energy:** Choose for the initial state the instant the object is released. The initial kinetic energy is $K_0 = 0$. The initial potential energy is non-zero, $U_0 = mg\,d\sin\theta$. The initial mechanical energy is then

$$E_0 = K_0 + U_0 = mg\,d\sin\theta \qquad (14.2.21)$$



Final Energy: Choose for the final state the instant when the object first comes to rest and the spring is compressed a distance x at the bottom of the inclined plane. The final kinetic energy is $K_f = 0$ since the mass is not in motion. The final potential energy is non-zero, $U_f = \dfrac{1}{2}kx^2 - x\,mg\sin\theta$. Notice that the gravitational potential energy is negative because the object has dropped below the height of the zero point of gravitational potential energy.



The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}k\,x^2 - x\,mg\sin\theta\,. \tag{14.2.22}$$

Non-conservative Work: Since we are assuming the track is frictionless, there is no non-conservative work.

Change in Mechanical Energy: The change in mechanical energy is therefore zero,

$$0 = W_{\mathrm{nc}} = \Delta E_{\mathrm{mechanical}} = E_f - E_0\,. \tag{14.2.23}$$

Thus mechanical energy is conserved, $E_f = E_0$, or

$$d\,mg\sin\theta = \frac{1}{2}k\,x^2 - x\,mg\sin\theta\,. \tag{14.2.24}$$

This is a quadratic equation in x,

$$x^2 - \frac{2mg\sin\theta}{k}x - \frac{2d\,mg\sin\theta}{k} = 0\,. \tag{14.2.25}$$

In the quadratic formula, we want the positive choice of square root for the solution to insure a positive displacement of the spring from equilibrium,

$$\begin{aligned} x &= \frac{mg\sin\theta}{k} + \left(\frac{m^2g^2\sin^2\theta}{k^2} + \frac{2d\,mg\sin\theta}{k}\right)^{1/2} \\ &= \frac{mg}{k}\left(\sin\theta + \sqrt{1 + 2\left(k\,d\,/\,mg\right)\sin\theta}\,\right) \end{aligned}\,. \tag{14.2.26}$$

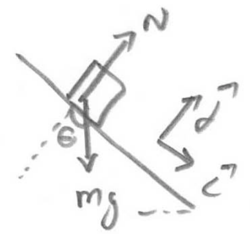
(What would the solution with the negative root represent?)

b) The effect of kinetic friction is that there is now a non-zero non-conservative work done on the object, which has moved a distance, $d + x$, given by

$$W_{\mathrm{nc}} = -f_{\mathrm{k}}\left(d + x\right) = -\mu_{\mathrm{k}}N\left(d + x\right) = -\mu_{\mathrm{k}}mg\cos\theta\left(d + x\right). \tag{14.2.27}$$

Note the normal force is found by using Newton's Second Law in the direction perpendicular to the inclined plane,

$$N - mg\cos\theta = 0\,. \tag{14.2.28}$$



The change in mechanical energy is therefore

$$W_{\text{nc}} = \Delta E_{\text{mechanical}} = E_f - E_0\,, \tag{14.2.29}$$

which becomes

$$-\mu_{\text{k}} mg\cos\theta\left(d + x\right) = \left(\frac{1}{2}k\,x^2 - x\,mg\sin\theta\right) - d\,mg\sin\theta\,. \tag{14.2.30}$$

Equation (14.2.30) simplifies to

$$0 = \left(\frac{1}{2}k\,x^2 - x\,mg\left(\sin\theta - \mu_{\text{k}}\cos\theta\right)\right) - d\,mg\left(\sin\theta - \mu_{\text{k}}\cos\theta\right). \tag{14.2.31}$$

This is the same as Equation (14.2.24) above, but with

$$\sin\theta \to \sin\theta - \mu_k\cos\theta\,.$$

The maximum displacement of the spring is when there is friction is then

$$x = \frac{mg}{k}\left(\left(\sin\theta - \mu_{\text{k}}\cos\theta\right) + \sqrt{1 + 2\left(k\,d\,/\,mg\right)\left(\sin\theta - \mu_{\text{k}}\cos\theta\right)}\right). \tag{14.2.32}$$

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c) The energy lost to friction is given by $W_{\text{nc}} = -\mu_{\text{k}} mg\cos\theta\left(d + x\right)$, where x is given in part b).

$$x_f^2 = \frac{m v_0^2}{k + bmg} \tag{14.2.33}$$

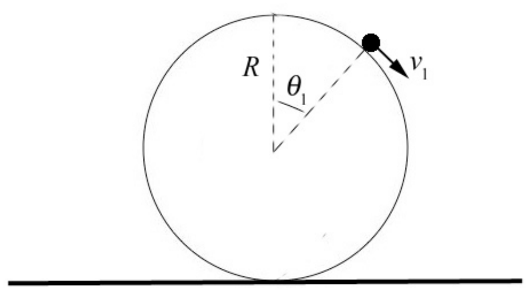
and substitution into Equation (14.2.19) gives the result

$$W_{\text{nc}} = -\frac{bmg}{2}\frac{mv_0^2}{k+bmg} = -\frac{mv_0^2}{2}\left(1+\frac{k}{bmg}\right)^{-1}. \tag{14.2.34}$$

It is worth checking that the above result is dimensionally correct. From the model, the parameter b must have dimensions of inverse length (the coefficient of friction μ must be dimensionless), and so the product bmg has dimensions of force per length, as does the spring constant k; the result is dimensionally consistent.

Example 14.2.5 Particle sliding on a sphere

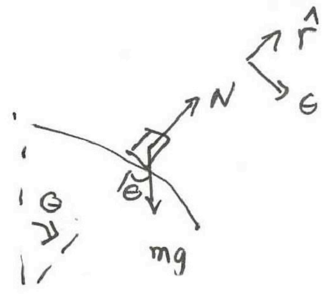
A small point like object of mass m rests on top of a sphere of radius R. The object is released from the top of the sphere with a negligible speed and it slowly starts to slide. Let g denote the gravitational constant.



a) Determine the angle θ_1 with respect to the vertical at which the particle will lose contact with the surface of the sphere.

b) What is the speed v_1 of the particle at the instant it loses contact with the surface of the sphere.

Solution: We begin by identifying the forces acting on the particle. There are two forces acting on the particle, the gravitation and radial normal force that the sphere exerts on the particle that we denote by N. We draw a free body force diagram for the particle while it is sliding on the sphere. We choose polar coordinates as shown in the figure below.



The key constraint is that when the particle just leaves the surface the normal force is zero,

$$N(\theta_1) = 0\,. \qquad (14.2.35)$$

where θ_1 denotes the angle with respect to the vertical at which the particle will just lose contact with the surface of the sphere.

Because the normal force is perpendicular to the displacement of the particle, it does no work on the particle and hence conservation of energy does not take into account the constraint on the motion imposed by the normal force. In order to analyze the effect of the normal force we must use the radial component of Newton's Second Law,

$$N - mg\cos\theta = -m\frac{v^2}{R}\,. \qquad (14.2.36)$$

Then when the particle just loses contact with the surface, Eqs. (14.2.35) and (14.2.36) require that

$$mg\cos\theta_1 = m\frac{v_1^{\,2}}{R}\,. \qquad (14.2.37)$$

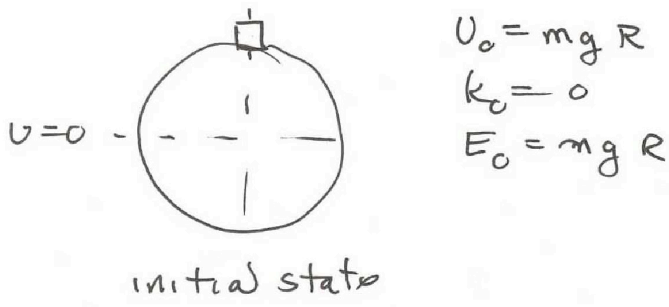
where v_1 denotes the speed of the particle at the instant it loses contact with the surface of the sphere. Note that the constrain condition Eq. (14.2.37) can be rewritten as

$$mgR\cos\theta_1 = mv_1^{\,2}\,. \qquad (14.2.38)$$

We can now apply conservation of energy. Choose the zero reference point for potential energy to be the midpoint of the sphere.

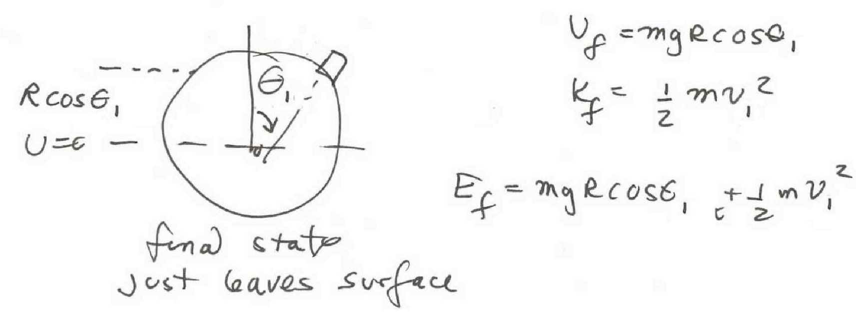
Initial State. Identify the initial state as the instant the particle is released. We can neglect the very small initial kinetic energy needed to move the particle away from the top of the sphere and so $K_0 = 0$. The initial potential energy is non-zero, $U_0 = mgR$. The initial mechanical energy is then

$$E_0 = K_0 + U_0 = mgR . \tag{14.2.39}$$



Final State. Choose for the final state the instant the particle leaves the sphere. The final kinetic energy is $K_f = \frac{1}{2} m v_1^2$; the particle is in motion with speed v_1. The final potential energy is non-zero, $U_f = mgR\cos\theta_1$. The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2} m v_1^2 + mgR\cos\theta_1 . \tag{14.2.40}$$



Because we are assuming the contact surface is frictionless, there is no nonconservative work.

Change in Mechanical Energy: The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_{\text{mechanical}} = E_f - E_0 . \tag{14.2.41}$$

Mechanical energy is conserved, $E_f = E_0$, or

$$\frac{1}{2} m v_1^2 + mgR\cos\theta_1 = mgR . \tag{14.2.42}$$

We now solve the constraint condition Eq. (14.2.38) into Eq. (14.2.42) yielding

$$\frac{1}{2}mgR\cos\theta_1 + mgR\cos\theta_1 = mgR. \qquad (14.2.43)$$

We can now solve for the angle at which the particle just leaves the surface

$$\theta_1 = \cos^{-1}(2/3). \qquad (14.2.44)$$

We now substitute this result into Eq. (14.2.38) and solve for the speed

$$v_1 = \sqrt{2gR/3}. \qquad (14.2.45)$$