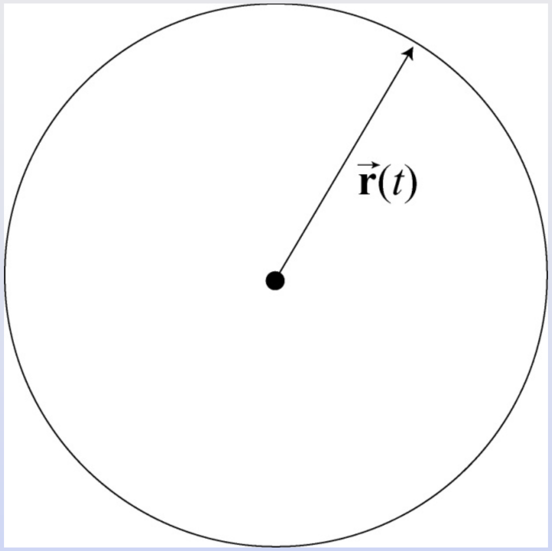
Circular Motion
8.01

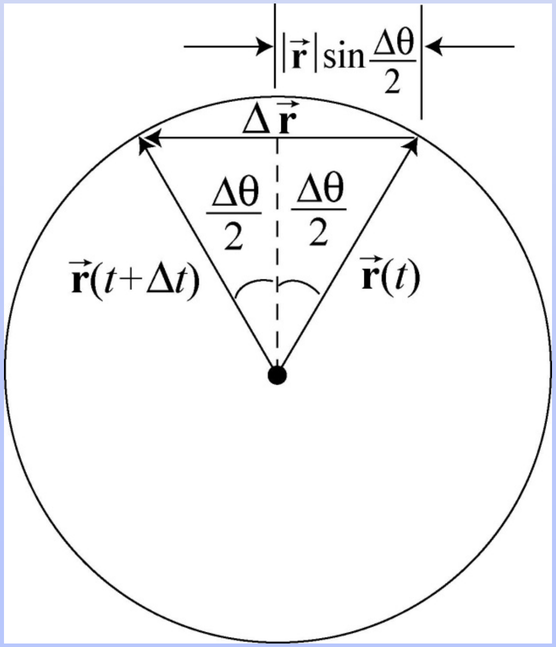
Position and Displacement

$\vec{\mathbf{r}}(t)$: position vector of an object moving in a circular orbit of radius *R*



$\Delta\vec{\mathbf{r}}(t)$: change in position between time t and time t+Δt

Position vector is changing in direction not in magnitude.

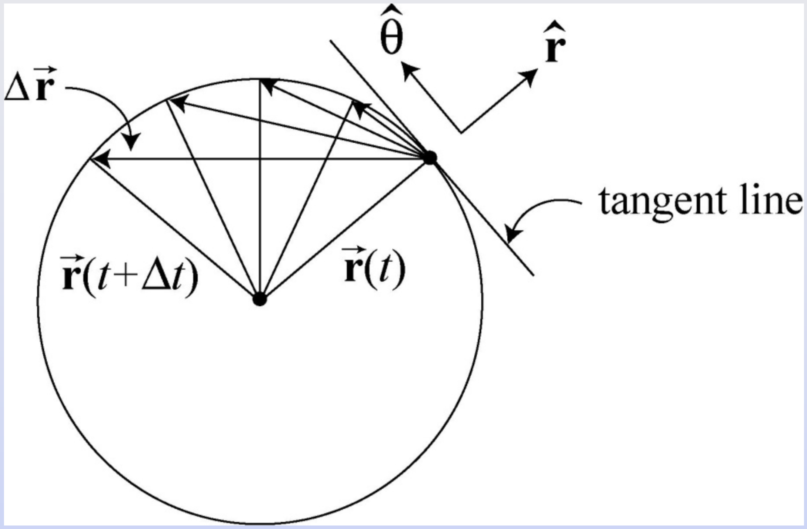


The magnitude of the displacement is the length of the chord of the circle:

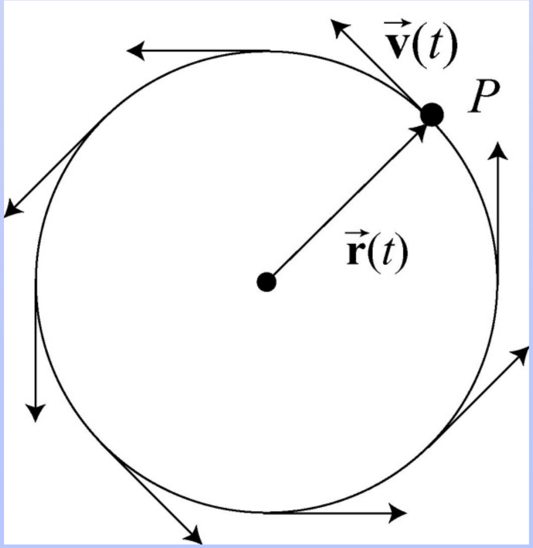
$$\boxed{\left|\Delta\vec{\mathbf{r}}\right| = 2R\sin(\Delta\theta/2)}$$

Direction of Velocity

Sequence of chord $\Delta\vec{r}$ directions approach direction of velocity as Δt approaches zero.



The direction of velocity is perpendicular to the direction of the position and tangent to the circular orbit.



Direction of velocity is constantly changing.

Small Angle Approximation

When the angle is small:

$$\sin\phi \approx \phi, \quad \cos\phi \approx 1$$

Power series expansion

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots$$

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \ldots$$

Using the small angle approximation with $\phi = \Delta\theta / 2$,
the magnitude of the displacement is

$$\left|\Delta\vec{\mathbf{r}}\right| = 2R\sin(\Delta\theta / 2) \approx R\,\Delta\theta$$

Speed and Angular Speed

The speed of the object undergoing circular motion is proportional to the rate of change of the angle with time:

$$v \equiv \left|\vec{\mathbf{v}}\right| = \lim_{\Delta t \to 0} \frac{\left|\Delta \vec{\mathbf{r}}\right|}{\Delta t} = \lim_{\Delta t \to 0} \frac{R\,\Delta\theta}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = R\frac{d\theta}{dt} = R\omega$$

Angular speed: $\omega = \dfrac{d\theta}{dt}$ $\quad (\text{units: rad} \cdot \text{s}^{-1})$

Circular Motion: Constant Speed, Period, and Frequency

In one period the object travels a distance equal to the circumference:

$$s = 2\pi R = vT$$

Period: the amount of time to complete one circular orbit of radius R

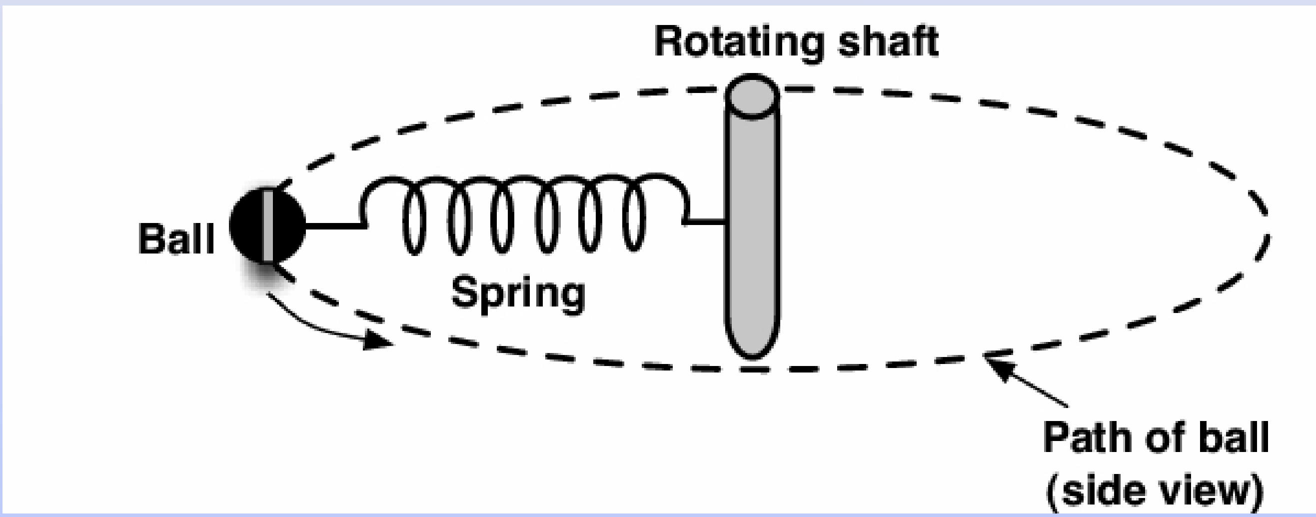
$$\boxed{T = \frac{2\pi R}{v} = \frac{2\pi R}{R\omega} = \frac{2\pi}{\omega}}$$

Frequency is the inverse of the period:

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi} \quad (\text{units: } \mathrm{s}^{-1} \text{ or Hz})}$$

Checkpoint Problem: Ball and spring

One end of a spring is attached to the central axis of a motor. The axis of the motor is in the vertical direction. A small ball of mass m_2 is then attached to the other end of the spring. The motor rotates at a constant frequency f . Neglect the gravitational force exerted on the ball. Assume that the ball and spring rotate in a horizontal plane. The spring constant is k. Let r_0 denote the unstretched length of the spring.

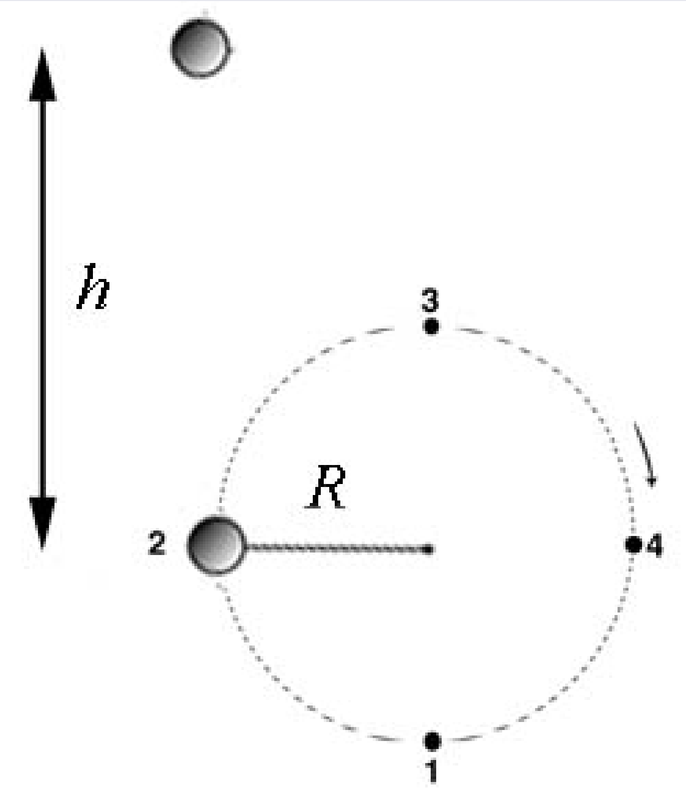


(i) How long does it take the ball to complete one rotation?

(ii) What is the angular frequency of the ball in radians per sec?

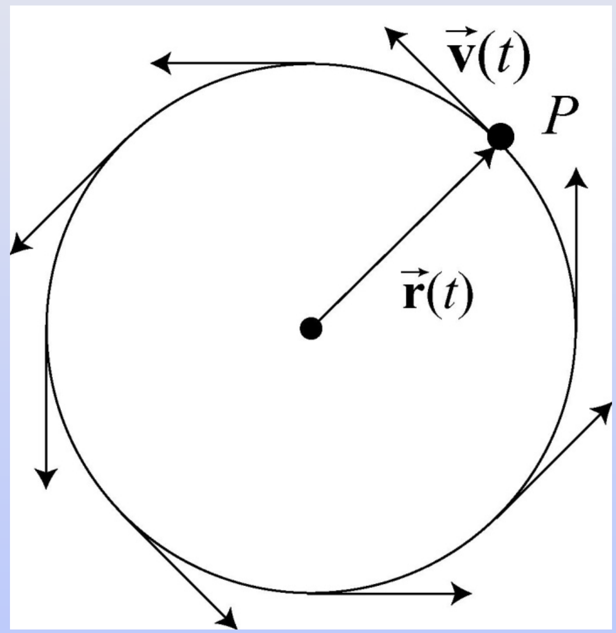
Checkpoint Problem: Whirling Stone

A stone, attached to a wheel and held in place by a string, is whirled in circular orbit of radius *R* in a vertical plane. Suppose the string is cut when the stone is at position 2 in the figure, and the stone then rises to a height h above the point at position 2. What was the angular velocity of the stone when the string was cut? Give your answer in terms of R, h and g.



Acceleration and Circular Motion

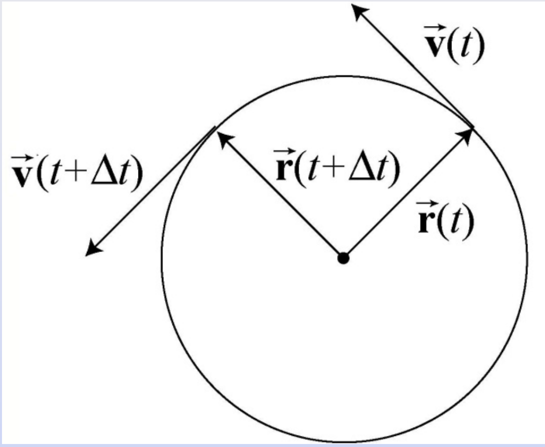
When an object moves in a circular orbit, the direction of the velocity changes and the speed may change as well.



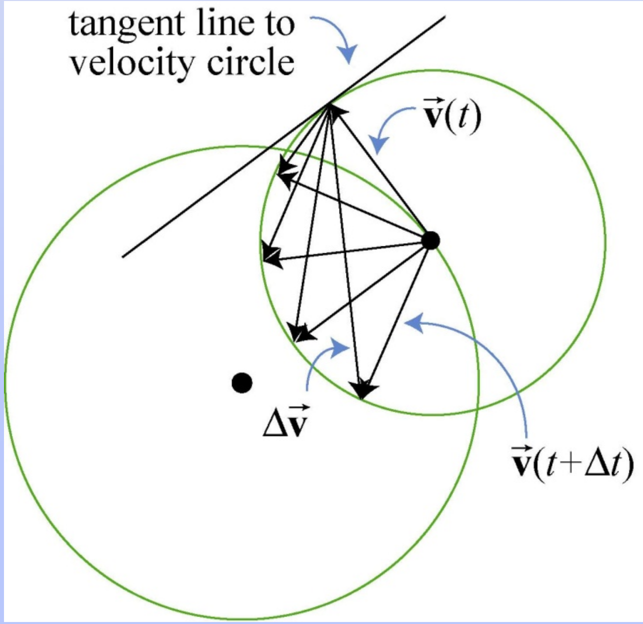
For circular motion, the acceleration will always have a radial component (a_r) due to the change in direction of velocity

The acceleration may have a tangential component if the speed changes (a_t). When $a_t = 0$, the speed of the object remains constant

Direction of Radial Acceleration: Uniform Circular Motion



Sequence of chord directions $\Delta\vec{\mathbf{v}}$ approaches direction of radial acceleration as Δt approaches zero



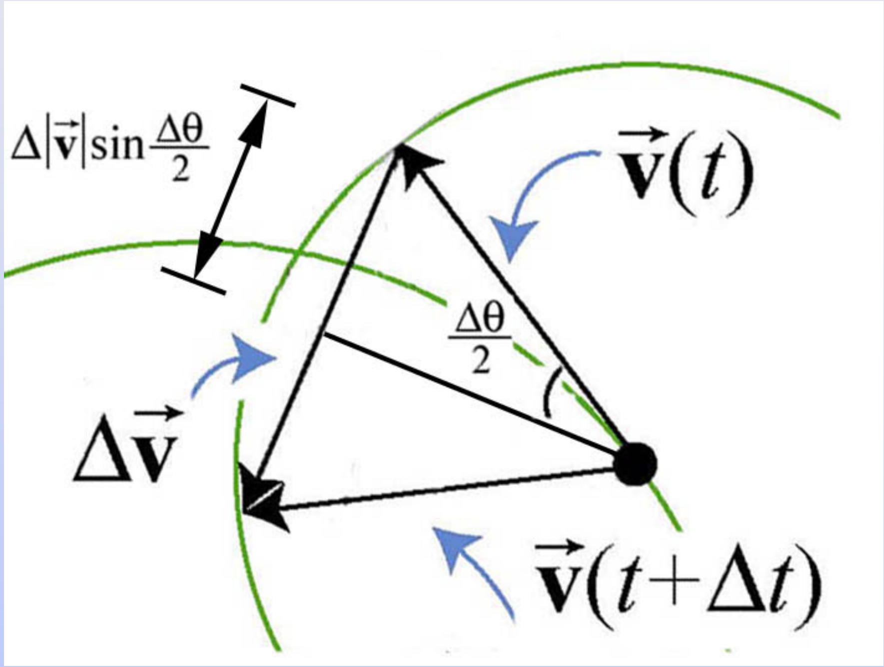
Perpendicular to the velocity vector

Points radially inward

Change in Magnitude of Velocity:Uniform Circular Motion

Change in velocity:

$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)$$



Magnitude of change in velocity:

$$\left|\Delta\vec{\mathbf{v}}\right| = 2v\sin(\Delta\theta/2)$$

Using small angle approximation

$$\boxed{\left|\Delta\vec{\mathbf{v}}\right| \cong v\Delta\theta}$$

Radial Acceleration: Constant Speed Circular Motion

Any object traveling in a circular orbit with a constant speed is always accelerating towards the center.

Direction of velocity is constantly changing.

Radial component of a_r *(minus sign indicates direction of acceleration points towards center)*

$$a_r = -\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = -\lim_{\Delta t \to 0} \frac{v \Delta \theta}{\Delta t} = -v \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = -v \frac{d\theta}{dt} = -v\omega = -\frac{v^2}{R}$$

$$\boxed{a_r = -\frac{v^2}{R} = -\omega^2 R}$$

Tangential Acceleration

The *tangential acceleration* is the rate of change of the magnitude of the velocity

$$\boxed{a_t = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = R\frac{d\omega}{dt} = R\frac{d^2\theta}{dt^2} = R\alpha}$$

Angular acceleration: rate of change of angular velocity with time

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad (\text{units: rad} \cdot \text{s}^{-2})}$$

Alternative forms of Magnitude of Radial Acceleration

Parameters: speed v, angular speed ω, angular frequency f, period T

$$\left|a_r\right| = \frac{v^2}{R} = R\omega^2 = R(2\pi f)^2 = \frac{4\pi^2 R}{T^2}$$

Checkpoint Problem: Spinning Earth

The earth is spinning about its axis with a period of 23 hours 56 min and 4 sec. The equatorial radius of the earth is $6.38 \times 10^6\ \text{m}$. The latitude of MIT in Cambridge, Mass is $42^o\ 22'$.

a) Find the velocity of a person at MIT as they undergo circular motion about the earth's axis of rotation.

b) Find the person's centripetal acceleration.

Summary: Kinematics of Circular Motion

Arc length $$s = R\theta$$

Tangential velocity $$v = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

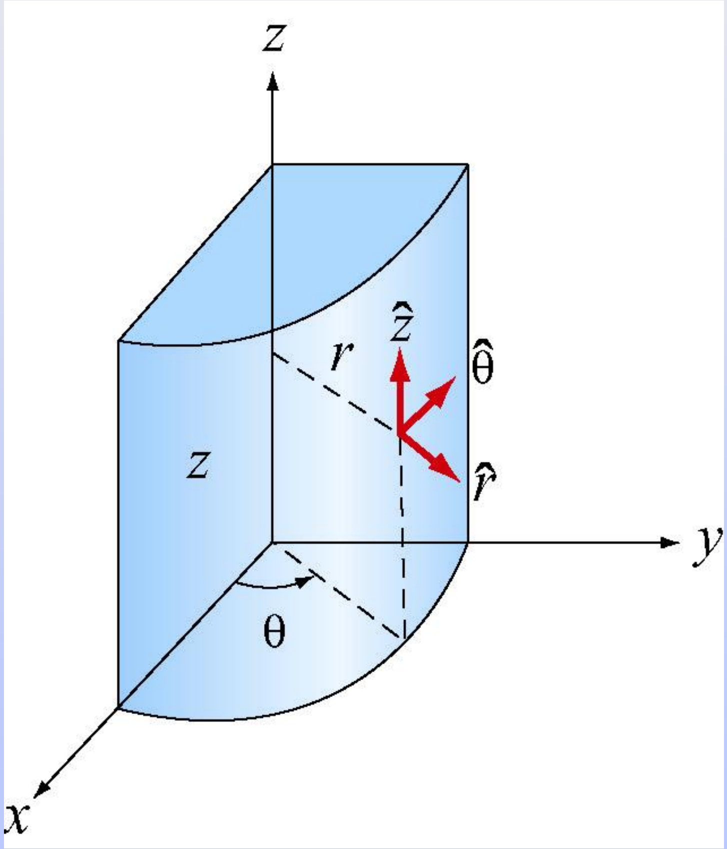
Tangential acceleration $$a_t = \frac{dv}{dt} = R\frac{d^2\theta}{dt^2} = R\alpha$$

Centripetal $$a_r = v\omega = \frac{v^2}{R} = R\omega^2$$

Cylindrical Coordinate System

Coordinates (r, θ, z)

Unit vectors $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{z}})$



Circular Motion: Vector Description

Use plane polar coordinates

Position

$$\vec{\mathbf{r}}(t) = R\,\hat{\mathbf{r}}(t)$$

Velocity

$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\,\hat{\theta}(t) = R\,\omega\,\hat{\theta}(t)$$

Acceleration

$$\vec{\mathbf{a}} = a_r\,\hat{\mathbf{r}} + a_t\,\hat{\theta}$$

$$a_t = r\alpha, \quad a_r = -r\omega^2 = -(v^2/r)$$

Checkpoint Problem: Relative Motion

Particles a and b move in opposite directions around a circle of radius R with the same angular speed ω , as shown. At t = 0 they are both at the point at the top of the circle. Find the velocity of a relative to b.