Chapter 9 Uniform Circular Motion

9.1 Introduction

Special cases often dominate our study of physics, and circular motion is certainly no exception. We see circular motion in many instances in the world; a bicycle rider on a circular track, a ball spun around by a string, and the rotation of a spinning wheel are just a few examples. Various planetary models described the motion of planets in circles before any understanding of gravitation. The motion of the moon around the earth is nearly circular. The motions of the planets around the sun are nearly circular. Our sun moves in nearly a circular orbit about the center of our galaxy, 50,000 light years from a massive black hole at the center of the galaxy.

We shall describe the kinematics of circular motion, the position, velocity, and acceleration, as a special case of two-dimensional motion. We will see that unlike linear motion, where velocity and acceleration are directed along the line of motion, in circular motion the direction of velocity is always tangent to the circle. This means that as the object moves in a circle, the direction of the velocity is always changing. When we examine this motion, we shall see that the direction of change of the velocity is towards the center of the circle. This means that there is a non-zero component of the acceleration directed radially inward, which is called the *centripetal acceleration*. If our object is increasing its speed or slowing down, there is also a non-zero *tangential acceleration* in the direction of motion. But when the object is moving at a constant speed in a circle then only the centripetal acceleration is non-zero.

In all of these instances, when an object is constrained to move in a circle, there must exist a force $\vec{\mathbf{F}}$ acting on the object directed towards the center.

In 1666, twenty years before Newton published his *Principia*, he realized that the moon is always "falling" towards the center of the earth; otherwise, by the First Law, it would continue in some linear trajectory rather than follow a circular orbit. Therefore there must be a *centripetal force*, a radial force pointing inward, producing this centripetal acceleration.

Since Newton's Second Law $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ is a vector equality, it can be applied to the radial direction to yield

$$F_{\text{radial}} = m\,a_{\text{radial}}. \tag{9.1.1}$$

9.2 Cylindrical Coordinate System

We first choose an origin and an axis we call the z-axis with unit vector $\hat{\mathbf{z}}$ pointing in the increasing z-direction. The level surface of points such that $z = z_P$ define a plane. We shall choose coordinates for a point P in the plane $z = z_P$ as follows.

One coordinate, r, measures the distance from the z-axis to the point P. The coordinate r ranges in value from $0 \le r \le \infty$. In Figure 9.2.1 we draw a few surfaces that have constant values of r. These `level surfaces' are circles.

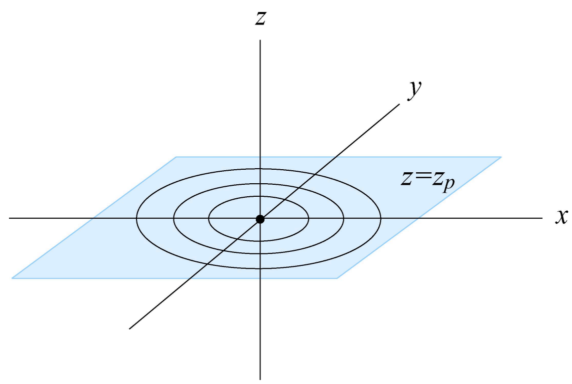


Figure 9.2.1 level surfaces for the coordinate r

Our second coordinate measures an angular distance along the circle. We need to choose some reference point to define the angle coordinate. We choose a 'reference ray', a horizontal ray starting from the origin and extending to $+\infty$ along the horizontal direction to the right. (In a typical Cartesian coordinate system, our 'reference ray' is the positive x-direction). We define the angle coordinate for the point P as follows. We draw a ray from the origin to the point P. We define the angle θ as the angle in the counterclockwise direction between our horizontal reference ray and the ray from the origin to the point P, (see Figure 9.2.2):

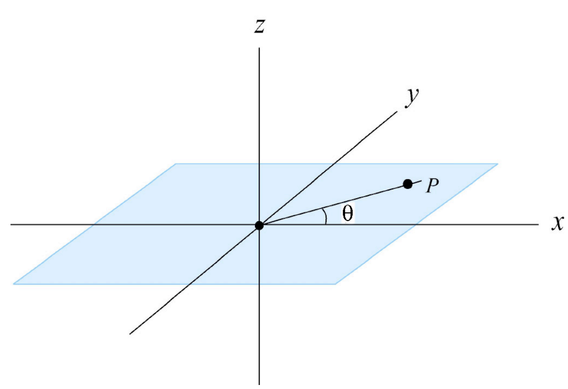


Figure 9.2.2 the angle coordinate

All the other points that lie on a ray from the origin to infinity passing through P have the same value as θ. For any arbitrary point, our angle coordinate θ can take on values from $0 \le \theta < 2\pi$. In Figure 9.2.3 we depict other `level surfaces' which are lines in the plane for the angle coordinate. The coordinates (r,θ) in the plane $z = z_P$ are called *polar coordinates*.

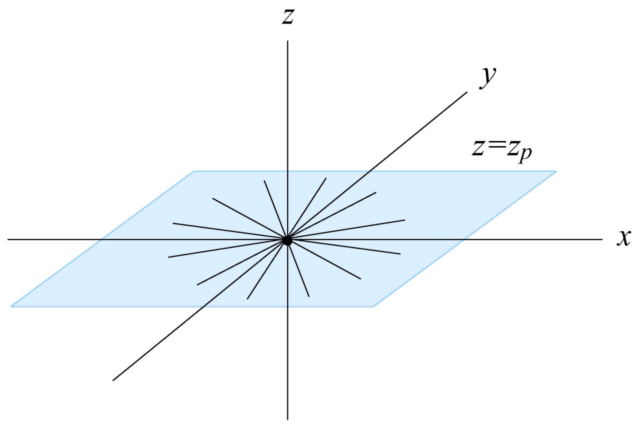


Figure 9.2.3 Level surfaces for the angle coordinate

Unit Vectors: We choose two unit vectors in the plane at the point P as follows. We choose $\hat{r}$ to point in the direction of increasing r, radially away from the z-axis. We choose $\hat{\theta}$ to point in the direction of increasing θ. This unit vector points in the counterclockwise direction, tangent to the circle. Our complete coordinate system is shown in Figure 9.2.4. This coordinate system is called a 'cylindrical coordinate system'. Essentially we have chosen two directions, radial and tangential in the plane and a perpendicular direction to the plane.

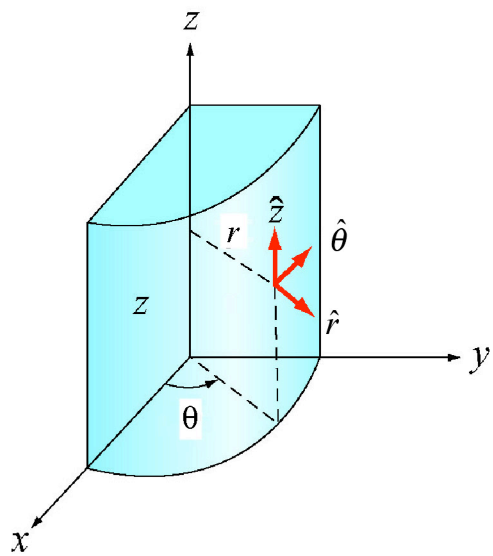


Figure 9.2.4 Cylindrical coordinates

If you are given polar coordinates (r,θ) of a point in the plane, the Cartesian coordinates (x,y) can be determined from the coordinate transformations

$$x = r\cos\theta \qquad (9.2.1)$$

$$y = r\sin\theta \qquad (9.2.2)$$

Conversely, if you are given the Cartesian coordinates (x, y), the polar coordinates (r, θ) can be determined from the coordinate transformations

$$r = +(x^2 + y^2)^{1/2} \tag{9.2.3}$$

$$\theta = \tan^{-1}(y/x) \tag{9.2.4}$$

Note that $r \geq 0$ so you always need to take the positive square root. Note also that $\tan\theta = \tan(\theta + \pi)$. Suppose that $0 \leq \theta \leq \pi/2$, then $x \geq 0$ and $y \geq 0$. Then the point $(-x, -y)$ will correspond to the angle $\theta + \pi$.

The unit vectors also are related by the coordinate transformations

$$\hat{r} = \cos\theta\, \hat{i} + \sin\theta\, \hat{j} \tag{9.2.5}$$

$$\hat{\theta} = -\sin\theta\, \hat{\mathbf{i}} + \cos\theta\, \hat{\mathbf{j}} \tag{9.2.6}$$

Similarly

$$\hat{i} = \cos\theta\, \hat{r} - \sin\theta\, \hat{\theta} \tag{9.2.7}$$

$$\hat{j} = \sin\theta\, \hat{r} + \cos\theta\, \hat{\theta} \tag{9.2.8}$$

One crucial difference between polar coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point S in the plane. The unit vectors in Cartesian coordinates $(\hat{\mathbf{i}}_S, \hat{\mathbf{j}}_S)$ at the point S have the same magnitude and point in the same direction as the unit vectors $(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P)$ at P. Any two vectors that are equal in magnitude and point in the same direction are equal; therefore

$$\hat{\mathbf{i}}_S = \hat{\mathbf{i}}_P, \quad \hat{\mathbf{j}}_S = \hat{\mathbf{j}}_P \tag{9.2.9}$$

A Cartesian coordinate system is the unique coordinate system in which the set of unit vectors at different points in space are equal. In polar coordinates, the unit vectors at two different points are not equal because they point in different directions. We show this in Figure 9.2.5.

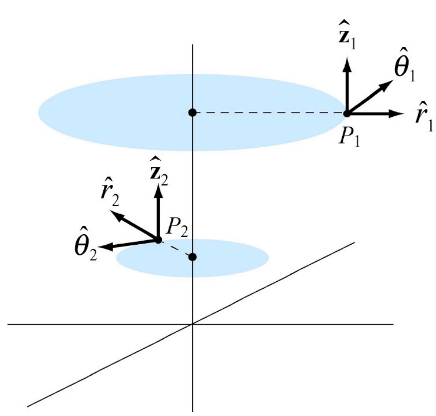


Figure 9.2.5 Unit vectors at two different points in polar coordinates.

Infinitesimal Line Elements: Consider a small infinitesimal displacement $d\vec{\mathbf{s}}$ between two points P_1 and P_2 (Figure 9.2.6). This vector can be decomposed into

$$d\vec{\mathbf{s}} = dr\,\hat{r} + rd\theta\,\hat{\theta} + dz\,\hat{\mathbf{k}} \tag{9.2.10}$$

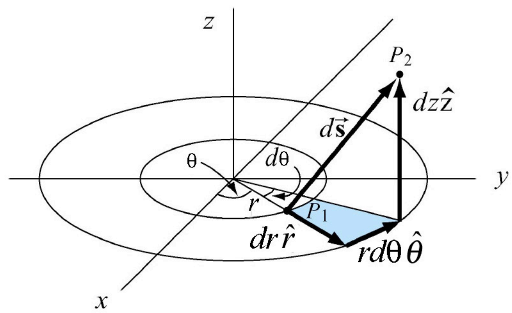


Figure 9.2.6 displacement vector $d\vec{\mathbf{s}}$ between two points

Infinitesimal Area Element:

Consider an infinitesimal area element on the surface of a cylinder of radius r (Figure 9.2.7).

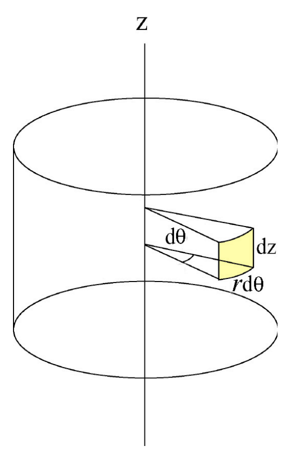


Figure 9.2.7 Area element for a cylinder

The area of this element has magnitude

$$dA = rd\theta dz \tag{9.2.11}$$

Area elements are actually vectors where the direction of the vector $d\vec{\mathbf{A}}$ points perpendicular to the plane defined by the area. Since there is a choice of direction, we shall choose the area vector to always point outwards from a closed surface. So for the surface of the cylinder, the infinitesimal area vector is

$$d\vec{\mathbf{A}} = r d\theta dz\, \hat{r} \tag{9.2.12}$$

Consider an infinitesimal area element on the surface of a disc (Figure 9.2.8) in the x-y plane.

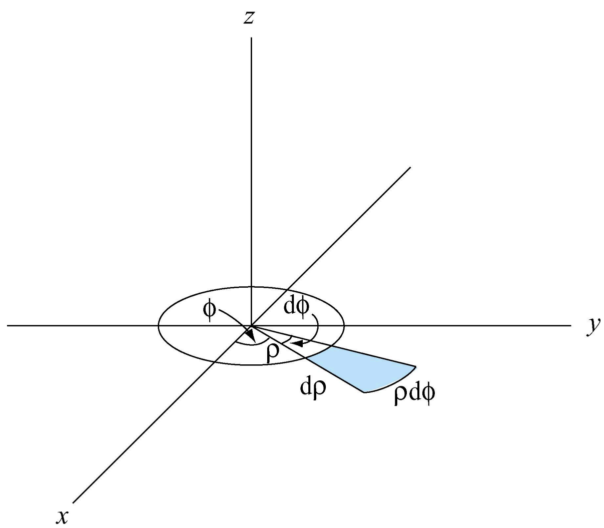


Figure 9.2.8 Area element for a disc

This area element is given by the vector

$$d\vec{\mathbf{A}} = r d\theta dr\, \hat{\mathbf{k}} \tag{9.2.13}$$

Infinitesimal volume element:

An infinitesimal volume element (Figure 9.2.9) is given by

$$dV = r d\theta\; dr\; dz \tag{9.2.14}$$

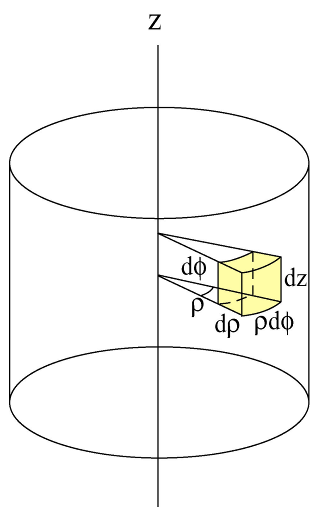


Figure 9.2.9 Volume element

The motion of objects moving in circles motivates the use of the cylindrical coordinate system. This is ideal, as the mathematical description of this motion makes use of the radial symmetry of the motion. Consider the central radial point and a vertical axis passing perpendicular to the plane of motion passing through that central point. Then any rotation about this vertical axis leaves circles invariant (unchanged), making this system ideal for use for analysis of circular motion exploiting of the radial symmetry of the motion.

9.3 Circular Motion: Velocity and Angular Velocity

We can now begin our description of circular motion. In Figure 9.2 we sketch the position vector $\vec{\mathbf{r}}(t)$ of the object moving in a circular orbit of radius R.

$$\vec{\mathbf{r}}(t) = R\hat{\mathbf{r}} \tag{9.3.1}$$

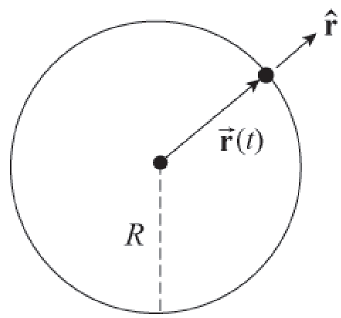


Figure 9.2 A circular orbit.

The magnitude of the displacement, $|\Delta\vec{\mathbf{r}}|$, is the represented by the length of the horizontal vector $\Delta\vec{\mathbf{r}}$ joining the heads of the displacement vectors in Figure 9.3 and is given by

$$|\Delta\vec{\mathbf{r}}| = 2R\sin(\Delta\theta / 2) \tag{9.3.2}$$

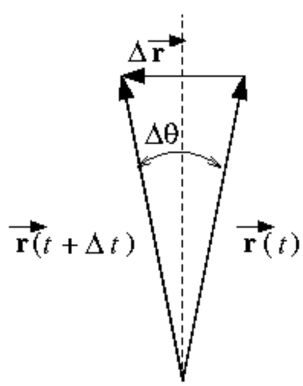


Figure 9.3 Change in position vector for circular motion.

When the angle $\Delta\theta$ is small, we can approximate

$$\sin(\Delta\theta / 2) \cong \Delta\theta / 2 . \tag{9.3.3}$$

This is called the *small angle approximation*, where the angle $\Delta\theta$ (and hence $\Delta\theta/2$) is measured in radians (see Section 1.6). This fact follows from an infinite power series expansion for the sine function given by

$$\sin\left(\frac{\Delta\theta}{2}\right) = \frac{\Delta\theta}{2} - \frac{1}{3!}\left(\frac{\Delta\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\Delta\theta}{2}\right)^5 - \cdots . \tag{9.3.4}$$

When the angle $\Delta\theta/2$ is small, only the first term in the infinite series contributes, as successive terms in the expansion become much smaller. For example, when $\Delta\theta/2 = \pi/30 \cong 0.1$, corresponding to 6°, $(\Delta\theta/2)^3/3! \cong 1.9\times10^{-4}$; this term in the power series is three orders of magnitude smaller than the first and can be safely ignored for small angles.

Using the small angle approximation, the magnitude of the displacement is

$$\left|\Delta\vec{\mathbf{r}}\right| \cong R\,\Delta\theta . \tag{9.3.5}$$

This result should not be too surprising since in the limit as $\Delta\theta$ approaches zero, the length of the chord approaches the arc length $R\Delta\theta$.

The magnitude of the velocity, $\left|\vec{\mathbf{v}}\right| \equiv v$, is then seen to be proportional to the rate of change of the magnitude of the angle with respect to time,

$$v \equiv \left|\vec{\mathbf{v}}\right| = \lim_{\Delta t\to 0}\frac{\left|\Delta\vec{\mathbf{r}}\right|}{\Delta t} = \lim_{\Delta t\to 0}\frac{R\left|\Delta\theta\right|}{\Delta t} = R\lim_{\Delta t\to 0}\frac{\left|\Delta\theta\right|}{\Delta t} = R\left|\frac{d\theta}{dt}\right| . \tag{9.3.6}$$

***Definition:* Angular Velocity**

The rate of change of angle with respect to time is called the *angular velocity* and is denoted by the Greek letter ω,

$$\omega \equiv \frac{d\theta}{dt} . \tag{9.3.7}$$

The SI units of angular velocity are $[\mathrm{rad}\cdot\mathrm{s}^{-1}]$.

Thus the magnitude of the velocity for circular motion is given by

$$v = R\left|\omega\right| . \tag{9.3.8}$$

The direction of the velocity can be determined by considering that in Figure 9.3, with $\left|\vec{\mathbf{r}}(t)\right| = \left|\Delta\vec{\mathbf{r}}(t+\Delta t)\right| = R$ a constant, the two position vectors put tail-to-tail form two sides of an isosceles triangle, and the third side, corresponding to $\Delta\vec{\mathbf{r}}$, must be perpendicular to the bisector of the triangle, regardless of the value of $\Delta\theta$; thus, in the limit $\Delta\theta \to 0$, $\Delta\vec{\mathbf{r}} \perp \vec{\mathbf{r}}$, and so the direction of the velocity $\vec{\mathbf{v}}$ at time t is perpendicular to position vector $\vec{\mathbf{r}}(t)$ and tangent to the circular orbit in the $+\hat{\boldsymbol{\theta}}$-direction.

There's a rigorous way to show this mathematically: Consider that since $R^2 = \vec{\mathbf{r}} \cdot \vec{\mathbf{r}}$ is a constant,

$$\frac{d}{dt}R^2 = \frac{d}{dt}\left(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}\right) = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = 0\,, \tag{9.3.9}$$

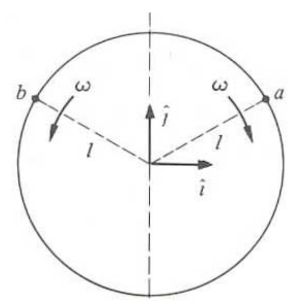
and so $\vec{\mathbf{r}} \perp \vec{\mathbf{v}}$ (the trivial case $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ will not concern us).

The velocity vector is therefore

$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}(t) = R\omega\,\hat{\boldsymbol{\theta}}(t)\,. \tag{9.3.10}$$

Example 2: Relative Motion and Polar Coordinates

By relative velocity we mean velocity with respect to a specified coordinate system. (The term velocity, alone, is understood to be relative to the observer's coordinate system.)



- **a.** A point is observed to have velocity $\vec{\mathbf{v}}_A$ relative to coordinate system A. What is its velocity relative to coordinate system B, which is displaced from system A by distance $\vec{\mathbf{R}}$? ($\vec{\mathbf{R}}$ can change in time.)

- **b.** Particles a and b move in opposite directions around a circle with the magnitude of the angular velocity ω, as shown. At $t = 0$ they are both at the point $\vec{\mathbf{r}} = l\hat{\mathbf{j}}$, where l is the radius of the circle. Find the velocity of a relative to b.

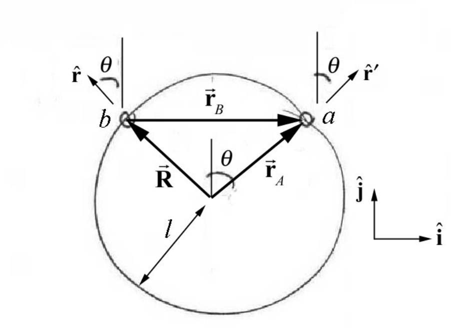
Solution: (a) The position vectors are related by

$$\vec{\mathbf{r}}_B = \vec{\mathbf{r}}_A - \vec{\mathbf{R}}\,. \tag{9.3.11}$$

Then velocities are related by the taking derivatives, (law of addition of velocities)

$$\vec{\mathbf{v}}_B = \vec{\mathbf{v}}_A - \vec{\mathbf{V}}\,. \tag{9.3.12}$$

(b) Let's choose two reference frames; frame B is centered at particle b, and frame A is centered at the center of the circle in the figure below.



Then the relative position vector between the origins of the two frames is given by

$$\vec{\mathbf{R}} = l\,\hat{\mathbf{r}}\,. \tag{9.3.13}$$

The position vector of particle a relative to frame A is given by

$$\vec{\mathbf{r}}_A = l\,\hat{\mathbf{r}}'\,. \tag{9.3.14}$$

The position vector of particle b in frame B can be found by substituting Eqs. (9.3.14) and (9.3.13) into Eq. (9.3.11),

$$\vec{\mathbf{r}}_B = \vec{\mathbf{r}}_A - \vec{\mathbf{R}} = l\,\hat{\mathbf{r}}' - l\,\hat{\mathbf{r}}\,. \tag{9.3.15}$$

We can decompose each of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ with respect to the Cartesian unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ (see figure)

$$\hat{\mathbf{r}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}} \tag{9.3.16}$$

$$\hat{\mathbf{r}}' = \sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\,. \tag{9.3.17}$$

Then Eq. (9.3.15) giving the position vector of particle b in frame B becomes

$$\vec{\mathbf{r}}_B = l\,\hat{\mathbf{r}}' - l\,\hat{\mathbf{r}} = l\,(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) - l\,(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) = 2l\sin\theta\,\hat{\mathbf{i}}\,. \tag{9.3.18}$$

In order to find the velocity vector of particle a in frame B (i.e. with respect to particle b), differentiate Eq. (9.3.18)

$$\vec{\mathbf{v}}_B = \frac{d}{dt}(2l\sin\theta)\,\hat{\mathbf{i}} = (2l\cos\theta)\frac{d\theta}{dt}\,\hat{\mathbf{i}} = 2\omega l\cos\theta\,\hat{\mathbf{i}}\,. \qquad (9.3.19)$$

9.4 Circular Motion: Tangential and Radial Acceleration

When the motion of an object is described in polar coordinates, the acceleration has two components, the tangential component, $a_{\tan} \equiv a_\theta$, and the radial component, $a_{\mathrm{rad}} \equiv a_r$. We can write the acceleration vector as

$$\vec{\mathbf{a}} = a_r\,\hat{\mathbf{r}} + a_\theta\,\hat{\boldsymbol{\theta}}\,. \qquad (9.4.1)$$

We will begin by calculating the tangential component of the acceleration for circular motion. Suppose that the tangential velocity is changing in magnitude due to the presence of some tangential force. The tangential velocity is $v_\theta = R\omega$, where ω is the angular velocity; if the angular velocity is changing, the velocity is also changing.

Since the radius is constant, the *average tangential acceleration* is just the rate of change of the magnitude of the velocity in a time interval Δt,

$$\left(a_\theta\right)_{\mathrm{ave}} = \frac{\Delta v_\theta}{\Delta t} = R\frac{\Delta\omega}{\Delta t}\,. \qquad (9.4.2)$$

The instantaneous tangential acceleration involves the same limit argument that we have previously used. Let the time interval $\Delta t \to 0$. Then the tangential acceleration is the radius times the derivative of the angular velocity with respect to time,

$$a_\theta = \lim_{\Delta t\to 0}\frac{\Delta v_\theta}{\Delta t} = R\lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = R\frac{d\omega}{dt} = R\frac{d^2\theta}{dt^2} \equiv R\alpha\,. \qquad (9.4.3)$$

Definition: **Angular Acceleration**

The *angular acceleration* is the rate of change of angular velocity with time (also the second derivative of angle with time) and is denoted by

$$\alpha \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}\,. \qquad (9.4.4)$$

The SI units of angular acceleration are $[\mathrm{rad}\cdot\mathrm{s}^{-2}]$.

The tangential component of the acceleration is then

$$a_\theta = R\alpha\,. \qquad (9.4.5)$$

Period and Frequency for Uniform Circular Motion

If the object is constrained to move in a circle and the total tangential force acting on the object is zero, $F_\theta^{\text{total}} = 0$. By Newton's Second Law, the tangential acceleration is zero,

$$a_\theta = 0 . \qquad (9.4.6)$$

This means that the magnitude of the velocity (the speed) remains constant. This motion is known as *uniform circular motion*.

Since the speed v is constant, the amount of time that the object takes to complete one circular orbit of radius R is also constant. This time interval, T, is called the *period*. In one period the object travels a distance $s = vT$ equal to the circumference, $s = 2\pi R$; thus

$$s = 2\pi R = vT . \qquad (9.4.7)$$

The period T is then given by

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{R|\omega|} = \frac{2\pi}{|\omega|}. \qquad (9.4.8)$$

The *frequency* f is defined to be the reciprocal of the period,

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \qquad (9.4.9)$$

The SI unit of frequency is the inverse second, which is defined as the hertz, $\left[\text{s}^{-1}\right] \equiv [\text{Hz}]$.

Radial Acceleration for Uniform Circular Motion

Of course, not all objects in circular orbits have constant speed. A varying speed will result in a nonzero tangential acceleration α, as described above. A racial acceleration is present due to the change in direction of the velocity vector, and may be calculated by considering the radial component of the change in the velocity vector. Considering only this change, we will assume constant speed, the case of uniform circular motion.

An object traveling in a circular orbit with a constant speed is always accelerating towards the center. Any radial inward acceleration is called *centripetal acceleration*. The magnitude of the velocity is a constant, and the direction of the velocity is always tangent to the circle. However the direction of the velocity is constantly changing because the object is moving in a circle, as can be seen in Figure 9.4. Because the velocity changes direction, the object has a nonzero acceleration.

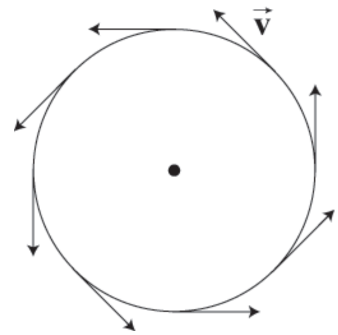


Figure 9.4 Direction of the velocity for circular motion.

The calculation of the magnitude and direction of the acceleration is very similar to the calculation for the magnitude and direction of the velocity for circular motion, but the change in velocity vector, $\Delta\vec{\mathbf{v}}$, is more complicated to visualize. The change in velocity $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)$ is depicted in Figure 9.5.

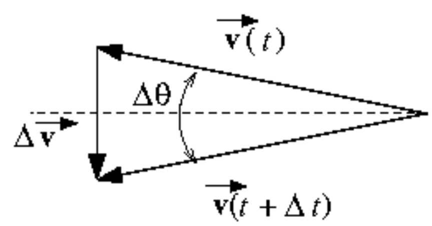


Figure 9.5 Change in velocity vector.

In Figure 9.5, the velocity vectors have been given a common point for the tails, so that the change in velocity, $\Delta\vec{\mathbf{v}}$, can be visualized. The length $\left|\Delta\vec{\mathbf{v}}\right|$ of the vertical vector can be calculated in exactly the same way as the displacement $\left|\Delta\vec{\mathbf{r}}\right|$.

The magnitude of the change in velocity is

$$\left|\Delta\vec{\mathbf{v}}\right| = 2v\sin(\Delta\theta/2). \qquad (9.4.10)$$

We can use the small angle approximation $\sin\left(\Delta\theta/2\right) \cong \Delta\theta/2$ to approximate the magnitude of the change of velocity,

$$\left|\Delta\vec{\mathbf{v}}\right| \cong v\left|\Delta\theta\right|. \qquad (9.4.11)$$

The magnitude of the radial acceleration is given by

$$a_r = \lim_{\Delta t\to 0}\frac{\left|\Delta\vec{\mathbf{v}}\right|}{\Delta t} = \lim_{\Delta t\to 0}\frac{v\left|\Delta\theta\right|}{\Delta t} = v\lim_{\Delta t\to 0}\frac{\left|\Delta\theta\right|}{\Delta t} = v\left|\frac{d\theta}{dt}\right| = v\left|\omega\right|. \qquad (9.4.12)$$

The centripetal acceleration can be expressed in several equivalent forms since both the magnitude of the velocity and the angular velocity are related by $v = R|\omega|$. Thus we have several alternative forms for the magnitude of the centripetal acceleration. The first is that in Equation (9.4.12). The second is in terms of the radius and the angular velocity,

$$|a_r| = R\omega^2 . \tag{9.4.13}$$

The third form expresses the magnitude of the centripetal acceleration in terms of the speed and radius,

$$|a_r| = \frac{v^2}{R}. \tag{9.4.14}$$

Recall that the magnitude of the angular velocity is related to the frequency by $|\omega| = 2\pi f$, so we have a fourth alternate expression for the magnitude of the centripetal acceleration in terms of the radius and frequency,

$$|a_r| = 4\pi^2 R f^2 . \tag{9.4.15}$$

A fifth form commonly encountered uses the fact that the frequency and period are related by $f = 1/T = |\omega|/(2\pi)$. Thus we have the fourth expression for the centripetal acceleration in terms of radius and period,

$$|a_r| = \frac{4\pi^2 R}{T^2} . \tag{9.4.16}$$

Other forms, such as $4\pi^2 R^2 f / T$ or $2\pi R\omega f$, while valid, are uncommon.

Often we decide which expression to use based on information that describes the orbit. A convenient measure might be the orbit's radius. We may also independently know the period, or the frequency, or the angular velocity, or the speed. If we know one, we can calculate the other three but it is important to understand the meaning of each quantity.

The direction of the acceleration is determined by the same method as the direction of the velocity; in the limit $\Delta\theta \to 0$, $\Delta\vec{\mathbf{v}} \perp \vec{\mathbf{v}}$, and so the direction of the velocity $\vec{\mathbf{a}}$ at time t is perpendicular to position vector $\vec{\mathbf{v}}(t)$ and directed inward, in the $-\hat{\mathbf{r}}$-direction.

So for an object that is undergoing circular motion the acceleration vector has radial and tangential components given by

$$\vec{\mathbf{a}}(t) = -R\left(\frac{d\theta}{dt}\right)^2 \hat{\mathbf{r}}(t) + R\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}}(t) = -R\omega^2\hat{\mathbf{r}}(t) + R\alpha\hat{\boldsymbol{\theta}}(t). \tag{9.4.17}$$

where $\omega \equiv d\theta / dt$ is the angular velocity and $\alpha \equiv d\omega / dt = d^2\theta / dt^2$ is the angular acceleration. Keep in mind that as the object moves in a circular, the unit vectors $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ change direction and hence are not constant in time.

When the motion is uniform circular motion then $d\omega / dt = 0$. Therefore the acceleration has only a radial component and the direction is towards the center of the circular orbit

$$\vec{\mathbf{a}}(t) = a_r\,\hat{\mathbf{r}}(t) = -R\omega^2\,\hat{\mathbf{r}}(t) = -\frac{v^2}{R}\hat{\mathbf{r}}(t). \tag{9.4.18}$$

For an algebraic way of deriving the above results, see Appendix A.

9.5 Summary: Circular Motion Kinematics

For an object undergoing circular motion the position vector is given by Eq. (9.3.1)

$$\vec{\mathbf{r}}(t) = R\hat{\mathbf{r}}\ . \tag{9.5.1}$$

The arc length for circular motion of radius R is

$$s = R\theta\ . \tag{9.5.2}$$

The rate of change of arc length with respect to time is the tangential speed v,

$$v = \frac{ds}{dt} = R\left|\frac{d\theta}{dt}\right| = R\left|\omega\right|, \tag{9.5.3}$$

where ω is the angular velocity. The velocity vector is given by Eq. (9.3.10)

$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = R\omega\hat{\boldsymbol{\theta}}(t). \tag{9.5.4}$$

The rate of change of the magnitude of the tangential velocity with respect to time is the tangential acceleration

$$a_\theta = \frac{dv_\theta}{dt} = R\frac{d^2\theta}{dt^2} = R\alpha\ . \tag{9.5.5}$$

where α is the angular acceleration.

The rate of change of the direction of the tangential velocity with respect to time is the centripetal acceleration; this vector is directed towards the center and has magnitude

$$\left|a_r\right| = \left|v\omega\right| = \frac{v^2}{R} = R\omega^2. \qquad (9.5.6)$$

The acceleration vector is given by Eq. (9.4.17)

$$\vec{\mathbf{a}}(t) = -R\left(\frac{d\theta}{dt}\right)^2 \hat{\mathbf{r}}(t) + R\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}}(t) = -R\omega^2\hat{\mathbf{r}}(t) + R\frac{d\omega}{dt}\hat{\boldsymbol{\theta}}(t)\,. \qquad (9.5.7)$$

For uniform circular motion, the acceleration vector is given by Eq. (9.4.18)

$$\vec{\mathbf{a}}(t) = -R\omega^2\hat{\mathbf{r}}(t)\,. \qquad (9.5.8)$$