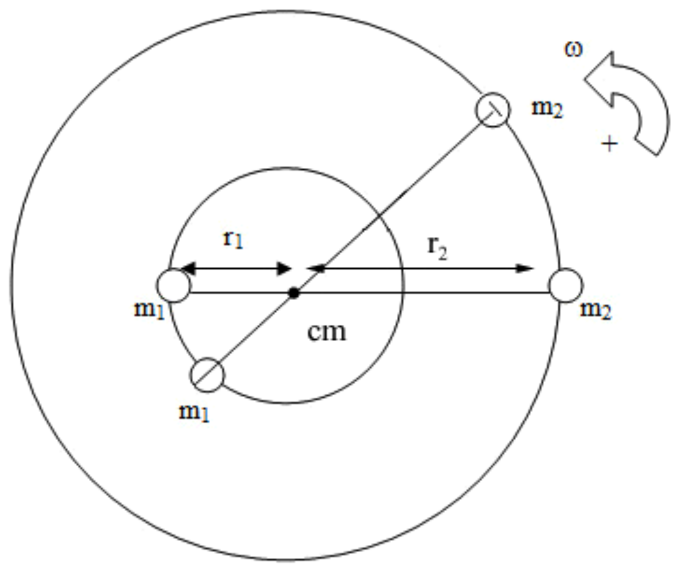
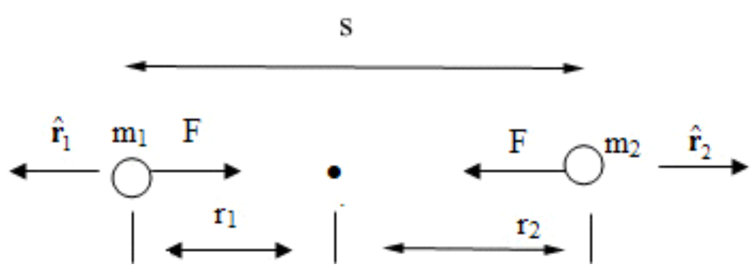
Problem Solving Circular Motion Dynamics

Problem 1: Double Star System

Consider a double star system under the influence of gravitational force between the stars. Star 1 has mass m_1 and star 2 has mass m_2. Assume that each star undergoes uniform circular motion about the center of mass of the system. If the stars are always a fixed distance s apart, what is the period of the orbit?



Solution: Choose radial coordinates for each star with origin at center of mass. Let $\hat{\mathbf{r}}_1$ be a unit vector at Star 1 pointing radially away from the center of mass. Let $\hat{\mathbf{r}}_2$ be a unit vector at Star 2 pointing radially away from the center of mass. The force diagrams on the two stars are shown in the figure below.



From Newton's Second Law, $\vec{\mathbf{F}}_1 = m_1\vec{\mathbf{a}}_1$, for Star 1 in the radial direction is

$$\hat{\mathbf{r}}_1 : -G\frac{m_1 m_2}{s^2} = -m_1 r_1 \omega^2 .$$

We can solve this for r_1,

$$r_1 = G\frac{m_2}{\omega^2 s^2}.$$

Newton's Second Law, $\vec{\mathbf{F}}_2 = m_2\,\vec{\mathbf{a}}_2$, for Star 2 in the radial direction is

$$\hat{\mathbf{r}}_2 : -G\frac{m_1 m_2}{s^2} = -m_2 r_2 \omega^2 .$$

We can solve this for r_2,

$$r_2 = G\frac{m_1}{\omega^2 s^2}.$$

Since s, the distance between the stars, is constant

$$s = r_1 + r_2 = G\frac{m_2}{\omega^2 s^2} + G\frac{m_1}{\omega^2 s^2} = G\frac{(m_2 + m_1)}{\omega^2 s^2}.$$

Thus the angular velocity is

$$\omega = \left(G\frac{(m_2 + m_1)}{s^3} \right)^{1/2}$$

and the period is then

$$T = \frac{2\pi}{\omega} = \left(\frac{4\pi^2 s^3}{G(m_2 + m_1)} \right)^{1/2}.$$

As will be seen later, this result is a variation of *Kepler's Third Law*.

Problem 2: Circular motion: banked turn

A car of mass m is going around a circular turn of radius R, that is banked at an angle θ with respect to the ground. The coefficient of static friction between the tires and the road is μ. Let g be the magnitude of the gravitational acceleration. You may neglect kinetic friction (that is, the car's tires do not slip).

a) At what speed v_0 should the car enter the banked turn if the road is very slippery (i.e. $\mu \to 0$) in order not to slide up or down the banked turn?

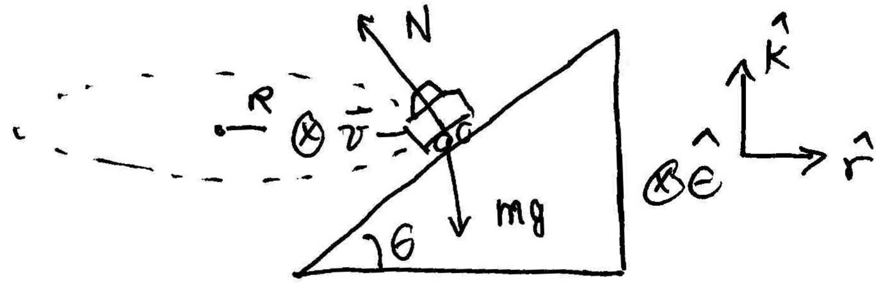
b) Suppose $\mu \tan\theta < 1$. What is the maximum speed $v_{\max}$ with which the car can enter the banked turn so that it does not slide up the banked turn?

c) Suppose $\mu \tan\theta < 1$. What is the minimum speed $v_{\min}$ with which the car can enter the banked turn so that it does not slide down the banked turn?

d) Suppose the car enters the turn with a speed v such that $v_{\max} > v > v_0$. Find an expression for the magnitude of the friction force.

Solution:

a) We will first consider the case where the friction $f_{\text{static}} \simeq 0$ is approximately zero. Denote the speed of the car by v_0. Choose cylindrical coordinates as shown in the figure below. Choose unit vectors $\hat{\mathbf{r}}$ pointing in the radial outward direction and $\hat{\mathbf{k}}$ pointing upwards. The force diagram on the car is shown in the figure below.



Newton's Second law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, becomes

$$\hat{\mathbf{r}}: \qquad -N\sin\theta = -\frac{mv_0^2}{r} \tag{2.1}$$

$$\hat{\mathbf{k}}: \quad N\cos\theta - mg = ma_z \tag{2.2}$$

Because the car is traveling in a circle the acceleration in the z -direction is zero

$$\mathrm{a_z} = 0 \tag{2.3}$$

and so the force equations (2.8) and (2.9) become

$$-N\sin\theta = -m\frac{v_0^2}{r} \tag{2.4}$$

$$N\cos\theta = mg \tag{2.5}$$

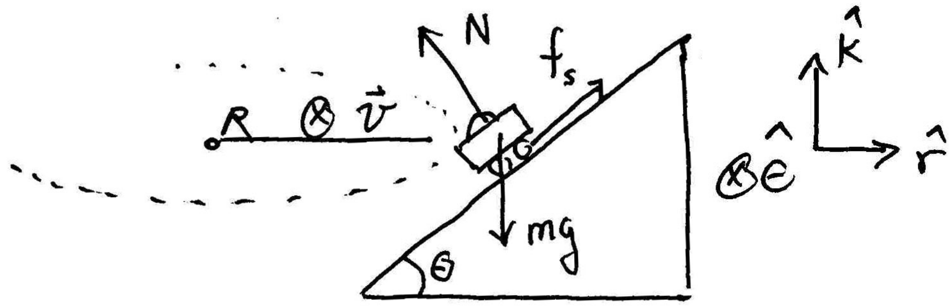
Dividing these equations yields

$$\tan\theta = \frac{v_0^2}{rg} \tag{2.6}$$

We now solve for the minimum speed, v_0, necessary to maintain a circular path

$$v_{\min} = \left(r\,g\tan\theta\right)^{\frac{1}{2}} \tag{2.7}$$

b) We now consider the case when the car is traveling slow enough, i.e. with minimum speed $v = v_{\min}$, such that it just starts to slip down the bank. The force diagram on the car is shown in the figure below.



Newton's Second law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, becomes

$$\hat{\mathbf{r}}: \qquad -N\sin\theta + f_{\text{static}}\cos\theta = -\frac{m v^2}{r} \tag{2.8}$$

$$\hat{\mathbf{k}}: \quad N\cos\theta + f_{\text{static}}\sin\theta - m\,g = m\,a_z \tag{2.9}$$

When $v = v_{\min}$, the just-slipping condition is that the acceleration in the z-direction is zero and the static friction has its maximum value:

$$\mathrm{a}_{\mathrm{z}} = 0 \tag{2.10}$$

$$f_{\text{static}} = \mu N \tag{2.11}$$

and so the force equations (2.8) and (2.9) become

$$-N\sin\theta + \mu N\cos\theta = -m\frac{v_{\min}^{\ 2}}{r} \tag{2.12}$$

$$N\cos\theta + \mu N\sin\theta = m\,g \tag{2.13}$$

Dividing these equations yields

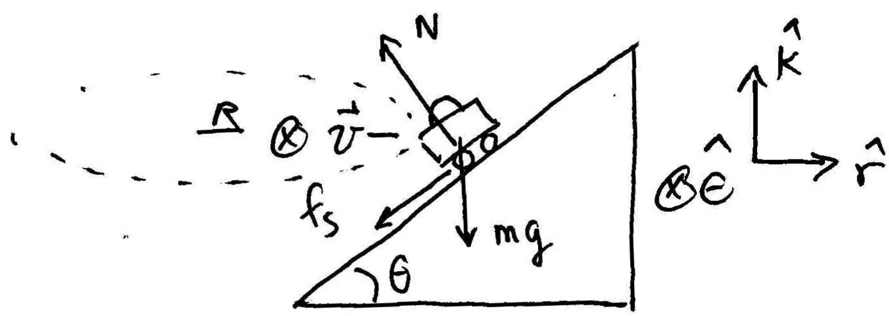
$$\frac{-\sin\theta + \mu\cos\theta}{\cos\theta + \mu\sin\theta} = -\frac{v_{\min}^2}{r\,g} \tag{2.14}$$

which can then be solved for the minimum speed, $v_{\min}$, necessary to avoid sliding down the embanked turn.

$$v_{\min} = \left(r\,g \left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} \right) \right)^{\frac{1}{2}} \tag{2.15}$$

This result should be checked for the limiting values of μ. In the limit $\mu \to 0$, $v_{\min} \to \sqrt{rg}$, the result we found in part a). In the limit $\mu \to \tan\theta$, $v_{\min} \to 0$, which is the static case of a block on an incline.

c) Now let's consider the case that the car is at the maximum speed such that it just starts to slip up the inclined plane. Then the direction of static friction points down the incline plane and the free body force diagram is shown in the figure below.



The analysis is identical to the previous case except for changing the signs of the components of static friction. Thus Newton's Second Law becomes

$$\hat{\mathbf{r}}: \quad -N\sin\theta - f_{\text{static}}\cos\theta = -\frac{m\,v^2}{r} \tag{2.16}$$

$$\hat{\mathbf{k}}: \quad N\cos\theta - f_{\text{static}}\sin\theta - m\,g = 0 \tag{2.17}$$

When $v = v_{\max}$, the static friction has its maximum value given by Eq. (2.11), and so the force equations now become

$$-N\sin\theta - \mu N\cos\theta = -m\frac{v_{\max}^{\;2}}{r} \tag{2.18}$$

$$N\cos\theta - \mu N\sin\theta = m\,g \tag{2.19}$$

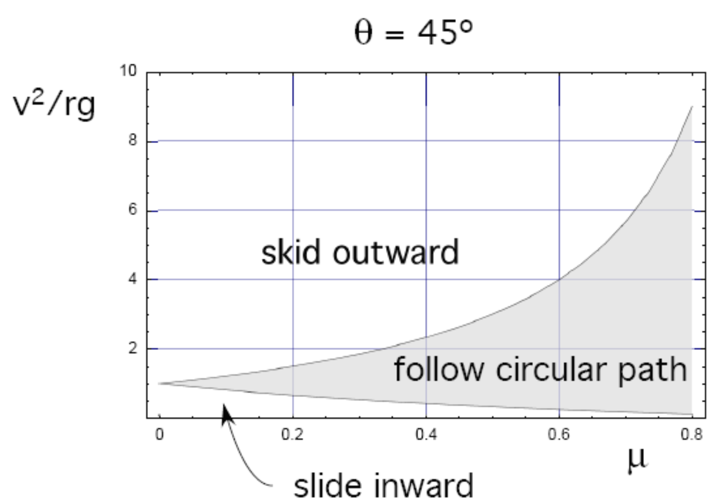
Dividing these equations yields

$$\frac{-\sin\theta-\mu\cos\theta}{\cos\theta-\mu\sin\theta}=-\frac{v_{\max}^2}{r\,g} \tag{2.20}$$

which can then be solved for the maximum speed, $v_{\max}$, necessary to avoid sliding down the embanked turn.

$$v_{\max}=\left(r\,g\left(\frac{\sin\theta+\mu\cos\theta}{\cos\theta-\mu\sin\theta}\right)\right)^{\frac{1}{2}} \tag{2.21}$$

The figure below shows a plot of $v^2/r\,g$ vs. μ when $\theta=45^\circ$. The shaded area represents the set of points $(\mu, v^2/r\,g)$ where the car remains in a circular path. Above that is the set of points in which the car will slid outward and below the set of points in which the car will slide inward.



d) The analysis is the same as in part c) but the magnitude of the static friction is less than its maximum value. Hence we can multiply Eq. (2.16) by $\cos\theta$ and Eq. (2.17) . by $\sin\theta$.

$$-N\sin\theta\cos\theta-f_{\text{static}}\cos^2\theta=-\frac{m\,v^2}{r}\cos\theta \tag{2.22}$$

$$N\cos\theta\sin\theta-f_{\text{static}}\sin^2\theta-m\,g\sin\theta=0 \tag{2.23}$$

Now add the two equations to yield

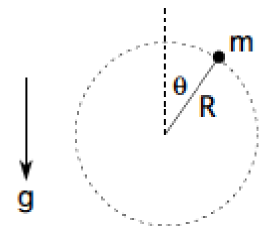
$$-f_{\text{static}}(\cos^2\theta+\sin^2\theta)-mg\sin\theta=-\frac{mv^2}{r}\cos\theta \qquad (2.24)$$

Use the identity $(\cos^2\theta+\sin^2\theta)=1$ and solve Eq. (2.24) for the magnitude of the static friction force

$$f_{\text{static}}=m\left(\frac{v^2}{r}\cos\theta-g\sin\theta\right) \qquad (2.25)$$

Problem 3

Sally swings a ball of mass m in a circle of radius R in a vertical plane by means of a massless string. The speed of the ball is constant and it makes one revolution every t_0 seconds.

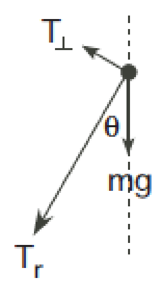


a) Find an expression for the radial component of the tension in the string $T(\theta)$ as a function of the angle θ the ball makes with the vertical[1]. Express your answer in terms of some combination of the parameters m, R, t_0 and the gravitational constant g.

b) Is there a range of values of t_0 for which this type of circular motion can not be maintained? If so, what is that range?

Solution:

a) The free body diagram is shown in the figure below.



Newton's second Law $\vec{F} = m\vec{a}$ in the radially inward direction becomes

$$-T_r - mg\cos\theta = -mR\omega^2 .$$

Thus

[1]Note added after the fact: The ball moves in a circle, but Sally's hand cannot remain at the center of the circle if a constant speed is to be maintained.

$$T_r = mR\omega^2 - mg\cos\theta$$

Because the angular speed $\omega = 2\pi / t_0$, the magnitude of the radial component of the tension in the string is

$$T_r = m(\frac{4\pi^2 R}{t_0^2} - g\cos\theta)$$

b) The magnitude of the radial component of the tension in the string T_r must always be greater than zero. When $\pi / 2 \leq \theta \leq 3\pi / 2$, $\cos(\theta) \leq 0$ hence $T_r > 0$. For the range of angle $0 \leq \theta < \pi / 2$ and $3\pi / 2 \leq \theta < 2\pi$, $\cos(\theta) > 0$ and so the condition

$$T_r = m(\frac{4\pi^2 R}{t_0^2} - g\cos\theta) > 0$$

implies that

$$2\pi\sqrt{\frac{R}{g\cos\theta}} > t_0 .$$

At $\theta = 0$, where mg makes the maximum contribution to the required radially inward directed force, $\cos(0) = 1$, for all other values of θ in the range of angle $0 < \theta < \pi / 2$ and $3\pi / 2 < \theta < 2\pi$, $\cos(\theta) < 1$ Thus to maintain circular motion the period t_0 must be less than a critical value $(t_0)_c$

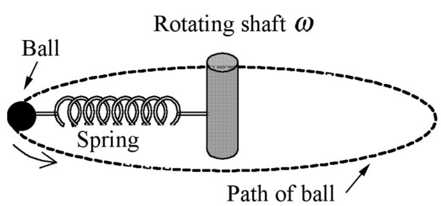
$$t_0 < (t_0)_c = 2\pi\sqrt{\frac{R}{g}} .$$

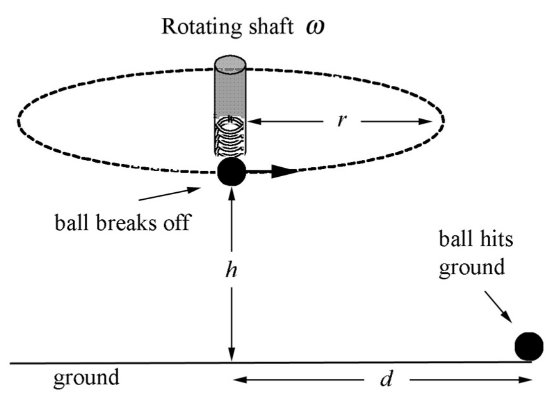
Problem 4

A ball of mass m is connected by a spring to shaft that is rotating with constant angular velocity ω. A student looking down on the apparatus sees the ball moving counterclockwise in a circular path of radius r. When the spring is unstretched, the distance from the mass to the axis of the shaft is r_0. The orbital plane of the ball is a height h above the ground. Suddenly the ball breaks loose from the spring, flies through the air, and hits the ground an unknown horizontal distance d from the point the ball breaks free from the spring. Let g be the magnitude of the acceleration due to gravity. You may ignore air resistance and the size of the ball.

a) What is the magnitude of the spring constant k? Show all your work. Answers without work will not receive credit.

b) Find an expression for the horizontal distance d the ball traveled from the point the ball breaks free from the spring until it hits the ground.





Solution:

a) Given that gravity may be neglected, the only force on the ball is the spring force. The ball is still moving with uniform circular motion, with acceleration directed inward, and so the spring force is directed inward, horizontal and perpendicular to the ball's motion. The magnitude of the spring force is given by Hooke's Law, $F_{\text{spring}} = k\,\Delta l = k(r - r_0)$. From Newton's Second Law, this force is related to the inward acceleration by

$$F_{\text{spring}} = k(r - r_0) = m\omega^2 r\,. \tag{4.1}$$

Solving for the spring constant k gives

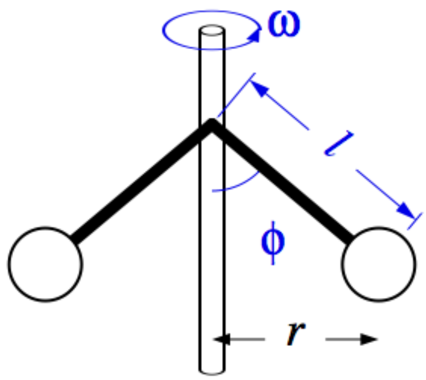
$$k = \frac{m\omega^2 r}{(r - r_0)}\,. \tag{4.2}$$

b) Just after the ball breaks free, it is moving horizontally, and its velocity has not changed from the uniform circular motion velocity. The speed is then that of uniform circular motion , $v = \omega r$. The ball is in free fall, moving with constant x-component of velocity and moving vertically with position above the ground given by $y = h - gt^2 / 2$. The ball will hit the ground when $y = 0$, at time $t_1 = \sqrt{2h / g}$. The ball will move in the horizontal direction with the constant speed found $v = \omega r$ for a time $t_1 = \sqrt{2h / g}$. The distance d is then the product of these two quantities,

$$d = r\omega\sqrt{2h / g}\,. \tag{4.3}$$

Problem 5: Circular Motion

A governor to control the rotational speed of a steam engine was invented by James Watt. Two spheres were attached to a rotating shaft by rigid arms that were free to rotate up and down about a pivot where they attached to the shaft, as in the diagram above. As the arms pivoted up and down they actuated a mechanism to control the throttle of the steam engine. Assume the rigid arms have length l and no mass. All of the mass is then concentrated in the two spheres at the end of the arms, each having mass m.



a) Describe the acceleration of the spheres. Explaining and quantifying your knowledge of the acceleration will help you model the problem. Show all relevant free body diagrams.

b) Show that there is a minimum angular velocity ω_{min} below which the governor will not function as intended.

c) Derive an expression for the radius r of the circular path followed by the spheres. Express your answer only in terms of as few of the quantities m, ω, l, and g (the acceleration due to gravity) as you can. (Do not use the angle ϕ in your answer).

Solutions:

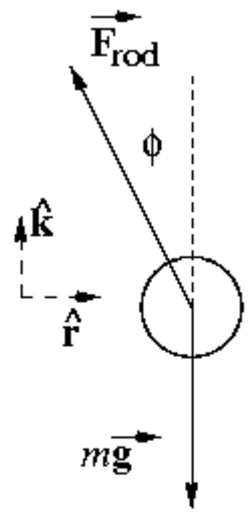
a) Each sphere moves in a horizontal circle of radius $r = l\sin\phi$ with period $T = 2\pi/\omega$, and hence an inward radial acceleration of magnitude

$$|a_r| = \omega^2 r = \omega^2 l\sin\phi\ . \tag{5.1}$$

Thus, there must be a net inward force of magnitude $F_{\text{net}} = m|a_r| = m\omega^2 l\sin\phi$. This force is the vector sum of the gravitational force $m\vec{\mathbf{g}}$ and the force $\vec{\mathbf{F}}_{\text{rod}}$ that the rod applies to the ball (since we call the period "T," we don't want to use "$\vec{\mathbf{T}}$" for a "tension force").

The gravitational force has no horizontal component, so the vertical component of $\vec{\mathbf{F}}_{\text{rod}}$ must balance the weight $m\vec{\mathbf{g}}$ and the horizontal component of $\vec{\mathbf{F}}_{\text{rod}}$ must have magnitude $F_{\text{net}} = m\left|a_r\right| = m\omega^2 l \sin\phi$.

A free body diagram for the right-hand sphere is shown below (the diagram for the left-hand sphere is of course just a reflection and won't be shown here).



b) Expressing the answer to part a) mathematically, the two components of Newton's Second Law are:

$$\begin{aligned} \hat{\mathbf{k}}: &\ F_{\text{rod}}\cos\phi - mg = 0 \\ \hat{\mathbf{r}}: &\ F_{\text{rod}}\sin\phi = m\omega^2 l \sin\phi. \end{aligned} \tag{5.2}$$

Eliminating $\sin\phi$ (why can we exclude the case $\sin\phi = 0$?) from the second expression in (5.2) yields $F_{\text{rod}} = m\omega^2 l$ and substitution into the first and rearranging yields

$$\cos\phi = \frac{g}{l\omega^2}. \tag{5.3}$$

We must have $\cos\phi < 1$, or $\omega^2 > g/l$, so $\omega_{\text{min}} = \sqrt{g/l}$.

c) Using $r = l\sin\phi = l\sqrt{1-\cos^2\phi}$ from part a) and $\cos\phi = g/l\omega^2$ from part b) above, we can eliminate the angle ϕ with the result

$$r = l\sqrt{1 - g^2/(l\omega)^2} = \sqrt{l^2 - g^2/\omega^4}\ . \tag{5.4}$$

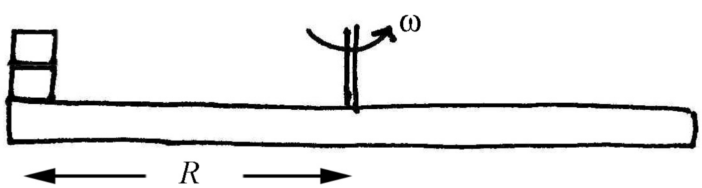
Note that this is consistent with part b); for a positive argument of the square root, we must have $\omega > \omega_{\min}$.

Problem 6:

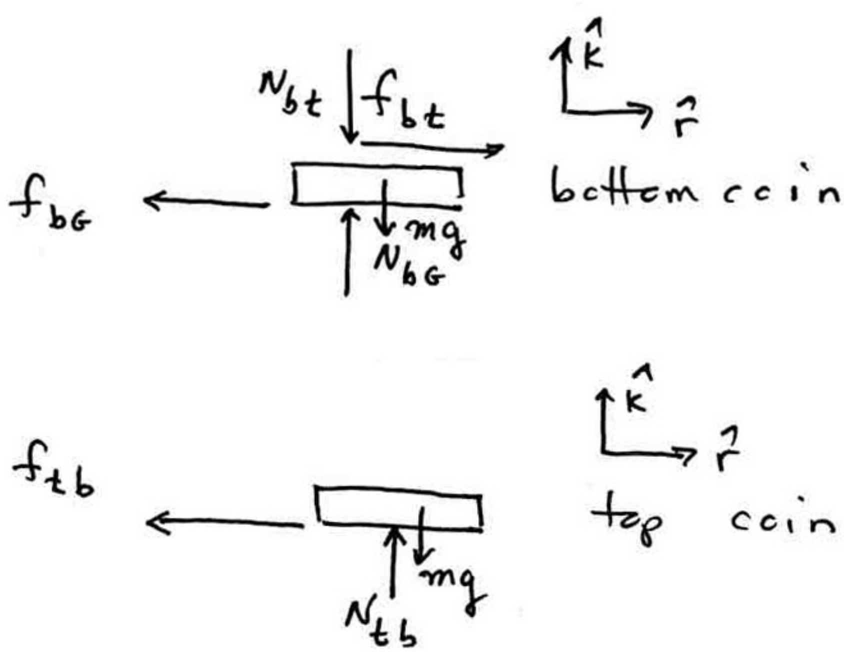
Two identical coins each of mass m are stacked on top of each other exactly at the rim of a turntable, a distance R from the center. The turntable turns at constant angular speed ω and the coins rides without slipping. Suppose the coefficient of static friction between the turntable and the coin is given by μ_1 and the coefficient of static friction between the the coins is given by μ_2 with $\mu_2 < \mu_1$. Let g be the gravitational constant.

a) What is the magnitude of the radial force (friction force) exerted by the turntable on the bottom coin?

b) As the angular speed increases which coin slips first or do they both slip at the same instant? What is the maximum angular speed ω_{max} such that no slipping occurs?



Solution: We choose a polar coordinate system and the free body force diagrams on each coin are shown in the figure below.



We will now apply Newton's Second Law to each coin and determine the magnitude of the radial force exerted by the turntable on the bottom coin, f_{bG}. The key point is that the static friction between the coins form an action-reaction pair, (neither coin is slipping). So the static friction that makes the top coin accelerate inward also acts on the bottom coin in the opposite direction (radially outward).

Newton's Second Law on the bottom coin in the radial direction, noting that the centripetal acceleration has magnitude $a_r = -R\omega^2$, is given by

$$-f_{bG} + f_{bt} = -m\,R\omega^2 . \tag{6.1}$$

Newton's Second Law on the top coin in the radial direction, noting that the centripetal is given by

$$-f_{tb} = -m\,R\omega^2 . \tag{6.2}$$

Newton's Third Law, requires that $f_{bt} = f_{tb}$. So substituting Eq. (6.2) becomes

$$f_{bt} = mR\omega^2 . \tag{6.3}$$

Substituting Eq. (6.3) into Eq. (6.1) then yields

$$-f_{bG} + m\,R\omega^2 = -m\,R\omega^2 . \tag{6.4}$$

Hence the magnitude of the radial inward force exerted on the bottom coin due to the turntable is then

$$f_{bG} = 2m\,R\omega^2 \tag{6.5}$$

So the static friction on the bottom coin is twice the magnitude of the static fiction on the upper coin.

In general slippage between surfaces will occur because static friction has a maximum possible value $f_{\max} = \mu N$. So we must find the magnitude of the normal force between the relevant surfaces.

Applying Newton's Second Law to the top coin in the z-direction, noting that the coin is static hence $a_z = 0$, yields

$$N_{tb} - mg = 0 . \tag{6.6}$$

Thus the normal force between the coins is

$$N_{tb} = mg\,. \tag{6.7}$$

The top coin will slip when the static friction between the two coins reaches its maximum value

$$(f_{tb})_{\max} = \mu_2 N_{tb} = \mu_2 mg\,. \tag{6.8}$$

We substitute Eq. (6.8) in to Eq. (6.2) and find

$$\mu_2 mg = m\,R\omega_{\max,t}^2\,. \tag{6.9}$$

So we solve for the maximum angular speed at which the top coin will slip

$$\omega_{\max,t} = \sqrt{\frac{\mu_2 g}{R}}\,. \tag{6.10}$$

Applying Newton's Second Law to the bottom coin in the z-direction,

$$N_{bg} - N_{bt} - mg = 0\,. \tag{6.11}$$

Noting that $N_{bt} = N_{tb} = mg$. Thus the normal force between the turntable and the bottom coin is

$$N_{bg} = 2mg\,. \tag{6.12}$$

The bottom coin will slip when the static friction between the turntable and the bottom coin is

$$(f_{bG})_{\max} = \mu N_{bG} = 2\mu_1 mg\,. \tag{6.13}$$

Using Eq. (6.5), we can find the maximum angular speed such that the bottom coin will slip

$$\omega_{\max,b} = \sqrt{\frac{\mu_1 g}{R}}\,. \tag{6.14}$$

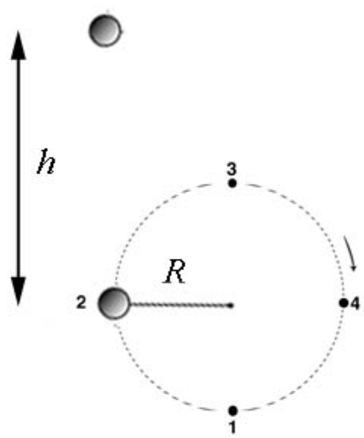
Comparing Eqs. (6.10) and (6.14), and noting that $\mu_2 < \mu_1$ we see that

$$\omega_{\max,t} < \omega_{\max,b}\,. \tag{6.15}$$

Thus as we increase the angular velocity the top coin will slip first.

Problem 7 Whirling Stone

A stone (or a ball in the demo), attached to a wheel and held in place by a string, is whirled in circular orbit of radius R in a vertical plane. Suppose the string is cut when the stone is at position 2 in the figure, and the stone then rises to a height h above the point at position 2. What was the angular velocity of the stone when the string was cut? Give your answer in terms of R, h and g.



Solution:

There are two distinct stages of motion. The first is circular motion in which the stone is being whirled at a speed v_0 when at position 2. Once the string is cut, the stone is moving vertically upwards, interacting gravitationally with the earth. For the vertical motion, set $t = 0$ when the string is just cut. Let's choose a coordinate system with the origin at point 2 in the figure and the positive y-direction upwards. Thus $y_0 = 0$. The key point to note is that for the vertical motion the initial speed when the string is cut is the speed of the uniform circular motion,

$$v_{y,0} = v_0 \tag{7.1}$$

Newton's Second Law becomes

$$-m\,g = m\,a_y\,. \tag{7.2}$$

The y- component of the acceleration is then

$$a_y = -g\,. \tag{7.3}$$

The y-component of the velocity is given by

$$v_y = v_0 - g\,t\,. \tag{7.4}$$

The stone reaches its highest point at time $t = t_1$ when the y-component of the velocity is zero,

$$v_y(t = t_1) = v_0 - g\,t_1 = 0 \tag{7.5}$$

We can solve Equation (7.5) for the time it takes for the stone to reach its highest point,

$$t_1 = \frac{v_0}{g}. \tag{7.6}$$

The y-component of the position at the highest point is

$$y(t = t_1) = h = v_0\,t_1 - \frac{1}{2}g\,t_1^2. \tag{7.7}$$

Substitute Equation (7.6) into Equation (7.7) to obtain

$$h = \frac{1}{2}\frac{v_0^2}{g}. \tag{7.8}$$

We can solve Equation (7.8) for the speed of the circular motion,

$$v_0 = \sqrt{2gh}. \tag{7.9}$$

The angular velocity ω is related to the speed by $v_0 = \omega R$, so

$$\omega = \frac{v_0}{R} = \frac{\sqrt{2g\,h}}{R} = \sqrt{\frac{2g\,h}{R^2}}. \tag{7.10}$$

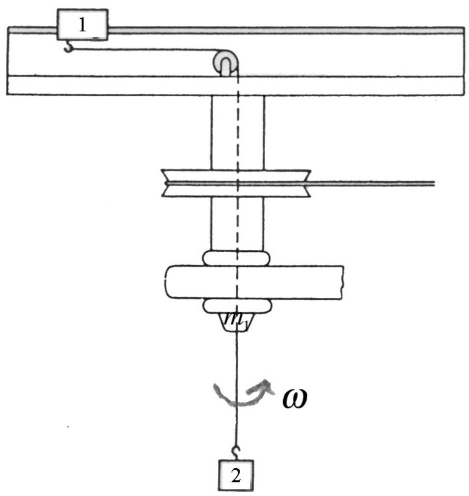
The last expression in Equation (7.10) is included to make checking the dimensions slightly easier;

$$\dim\left[\frac{2g\,h}{R^2}\right] = \frac{\left[\mathrm{L}\cdot\mathrm{T}^{-2}\right]\mathrm{L}}{\mathrm{L}^2} = \mathrm{T}^{-2}, \tag{7.11}$$

and so the expression for ω in Equation (7.10) has dimensions of inverse time, as it should.

Problem 8 Uniform Circular Motion Rotating Device

In the device shown below, a horizontal rod rotates with an angular velocity ω about a vertical axis. In the diagram, the horizontal string extending to the right at the middle of the apparatus represents the driving torque that maintains constant angular velocity ω.
An object 1 with mass m_1 is constrained to slide along the horizontal rod. A massless inextensible string of length s is attached to one end of object 1, passes over a massless pulley, and attaches to a suspended object 2 of mass m_2. Object 2 hangs along the central vertical axis of the device. Assume the coefficient of static friction between the object 1 and the rod is μ_s, and use g as the gravitational constant. Object 1 moves in a circle of radius r.



a) With what angular velocity can the device spin such that the static friction force is zero?

b) What is the minimum angular velocity with which the device can spin so that object 1 does not move radially inward?

c) What is the maximum angular velocity with which the device can spin so that the object 1 does not move radially outward?

Solutions:

(New figure needed – make sure positive radial direction is consistent.)
For all parts, use coordinates with the radially outward direction being positive $\hat{\mathbf{r}}$ and the upward vertical direction being positive $\hat{\mathbf{k}}$. The forces on object 1 are: gravity, $m_1\vec{\mathbf{g}} = -m_1 g\,\hat{\mathbf{k}}$; the force due to the tension in the connecting string, $\vec{\mathbf{T}} = -T\,\hat{\mathbf{r}}$; the normal force $\vec{\mathbf{N}} = N\,\hat{\mathbf{k}}$; and the frictional force $\vec{\mathbf{f}} = f_r\,\hat{\mathbf{r}}$, where f_r is the radial component of the frictional force, and may be positive, negative or zero.

The problem statement refers to "static friction," so we should take this to mean that object 1 is not moving with respect to the horizontal rod. There must be an inward force to keep object 1 moving in a circle. A contribution to this force will be the tension T in the string. The string also supports the hanging mass against gravity, so $T = m_2 g$ in all parts.

a) Given that there is no friction force, the tension must supply the inward force;

$$-T = -m_2 g = -m_1\omega^2 r$$
$$\omega = \sqrt{\frac{m_2}{m_1}\frac{g}{r}}. \tag{8.1}$$

b) If the angular speed ω is less than that found in part a), the tension in the string would be greater than that needed to maintain circular motion, and so some other force would be needed in addition to the tension. This must be the friction force, directed outward; f_r is positive. The above equation becomes

$$-T + f_r = -m_2 g + f_r = -m_1\omega^2 r. \tag{8.2}$$

Equation (8.2) is indeterminate, as it contains three variables. However, we are given that ω has its minimum value, which means that f_r has its maximum value, given by

$$\left(f_r\right)_{\max} = \mu_s N = \mu_s m_1 g. \tag{8.3}$$

Using this in Equation (8.2) and solving for $\omega_{\min}$ yields

$$\omega_{\min} = \sqrt{\left(\frac{m_2}{m_1} - \mu_s\right)\frac{g}{r}}. \tag{8.4}$$

Note that this implies that if the coefficient of static friction is sufficiently high, $\omega_{\min}$ could be as low as zero.

c) Repeat the above analysis, with the appropriate sign and direction changes:
If the angular speed ω is greater than that found in part a), the tension in the string would be less than that needed to maintain circular motion, and so some other force would be needed in addition to the tension. This must be the friction force, directed inward; f_r is negative. Equation (8.2) is still valid,

$$-T + f_r = -m_2 g + f_r = -m_1\omega^2 r. \tag{8.5}$$

Equation (8.5) is indeterminate, as it contains three variables. However, we are given that ω has its maximum value, which means that f_r has its minimum (that is, most negative) value, given by

$$\left(f_r\right)_{\min} = -\mu_s N = -\mu_s m_1 g \,. \tag{8.6}$$

Using this in Equation (8.6) and solving for $\omega_{\max}$ yields

$$\omega_{\max} = \sqrt{\left(\frac{m_2}{m_1} + \mu_s\right)\frac{g}{r}} \,. \tag{8.7}$$

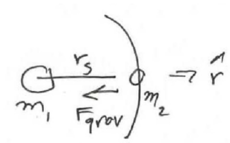
Problem 9 Universal Law of Gravitation and Orbital Uniform Circular Motion

A person, abandoned on a small spherical asteroid of mass m_1 and radius R, sees a satellite orbiting the asteroid in a circular orbit of period T.

a) What is the radius r_{sat} of the satellite's orbit?
b) What is the magnitude of the velocity of the satellite?
c) If the asteroid rotates with a period T_{a}, at what radius must the satellite orbit the asteroid so that the satellite appears stationary to a person on the asteroid?

Solution:

a) The only force on the satellite is the gravitation force pointing radially inward. The force diagram on the satellite is given in the figure below, with the satellite's mass denoted m_2.



Newton's Second Law in the radial direction is given by

$$\hat{\mathbf{r}}: \quad -G\frac{m_1 m_2}{r_{\text{sat}}^2} = -m_2 r_{\text{sat}} (2\pi / T)^2 . \tag{9.1}$$

We can solve Equation (9.1) for the radius of orbit of the satellite,

$$r_{\text{sat}} = \left(G\frac{m_1 T^2}{4\pi^2} \right)^{1/3} . \tag{9.2}$$

b) The magnitude of the velocity of the satellite is

$$v = r_{\text{sat}}\omega = r_{\text{sat}}(2\pi / T) = \left(G\frac{m_1 T^2}{4\pi^2} \right)^{1/3} \frac{2\pi}{T} = \left(G\frac{m_1 2\pi}{T} \right)^{1/3} . \tag{9.3}$$

c) In order for the satellite to appear stationary to an observer on the satellite, the satellite must orbit with the same rotational asteroid as the asteroid. Thus we can substitute $T = T_{\text{a}}$ in Equation (9.2) and find that the radius of the orbit must be

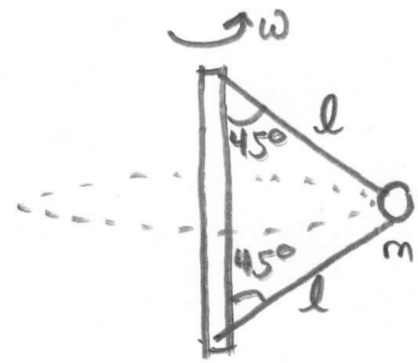
$$r_{\text{sat}} = \left(G \frac{m_1 T_{\text{a}}^{\,2}}{4\pi^2} \right)^{1/3}. \tag{9.4}$$

Problem 10 Uniform Circular Motion: Two strings

An object with mass m is connected to a vertical revolving axle by two massless inextensible strings of length l, each making an angle of 45° with the axle. Both the axle and the mass are revolving with angular velocity ω. Gravity is directed downwards.

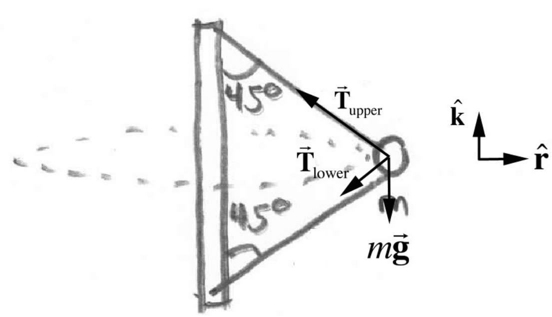
a) Draw a clear force diagram for the object.

b) Find the tensions T_{upper} in the upper string and T_{lower} in the lower string,.



Solution:

a) The forces acting on the whirling object are the tension of the upper string, $\vec{\mathbf{T}}_{\text{upper}}$; the tension in the lower string, $\vec{\mathbf{T}}_{\text{lower}}$; and the gravitation force $m\vec{\mathbf{g}}$. Choose units vectors pointing radially outward, $\hat{\mathbf{r}}$, and vertically upward, $\hat{\mathbf{k}}$. The force diagram on the object is shown in the figure below.



Applying Newton's Second Law, we have that

$$\hat{\mathbf{r}}: \quad -T_{\text{upper}}\sin 45^\circ - T_{\text{lower}}\sin 45^\circ = -ml\sin 45^\circ \omega^2 \qquad (10.1)$$

$$\hat{\mathbf{k}}: \quad T_{\text{upper}}\cos 45^\circ - T_{\text{lower}}\cos 45^\circ - mg = 0\,. \qquad (10.2)$$

Equation (10.1) simplifies to

$$T_{\text{upper}} + T_{\text{lower}} = ml\omega^2 \,. \tag{10.3}$$

Using $\cos 45^\circ = \sqrt{2}\,/\,2 = 1\,/\,\sqrt{2}$, Equation (10.2) becomes

$$T_{\text{upper}} - T_{\text{lower}} = \sqrt{2}mg \,. \tag{10.4}$$

We can solve for T_{upper} by adding Equations (10.3) and (10.4) and then dividing by two, yielding

$$T_{\text{upper}} = m\left(l\omega^2 + \sqrt{2}g\right)/\,2 \,. \tag{10.5}$$

We can solve for T_{lower} by subtracting Equation (10.4) from Equation (10.3) yielding

$$T_{\text{lower}} = m\left(l\omega^2 - \sqrt{2}g\right)/\,2 \,. \tag{10.6}$$

It's important to note that the result of Equation (10.6) cannot be valid unless the angular frequency ω is sufficiently large; if the frequency is smaller than the limiting value, the lower string would sag, and the 45° could not be maintained.