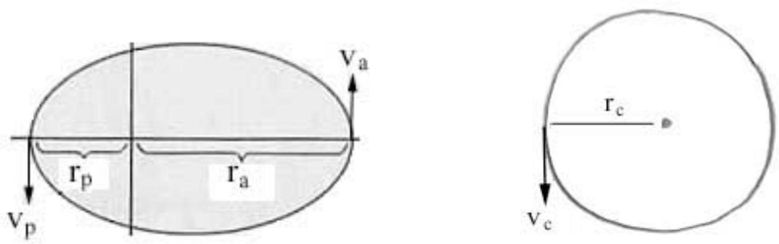
Central Force Motion
Challenge Problems

Problem 1: Elliptic Orbit

A satellite of mass m_s is in an elliptical orbit around a planet of mass m_p which is located at one focus of the ellipse. The satellite has a velocity v_a at the distance r_a when it is furthest from the planet.



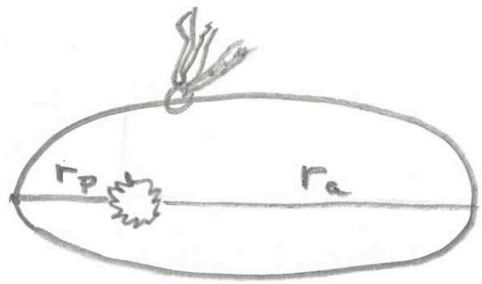
a) What is the magnitude of the velocity v_p of the satellite when it is closest to the planet? Express your answer in terms of m_s, m_p, G, v_a, and r_a as needed.

b) What is the distance of nearest approach r_p?

c) If the satellite were in a circular orbit of radius $r_0 = r_p$, is it's velocity v_0 greater than, equal to, or less than the velocity v_p of the original elliptic orbit? Justify your answer.

Problem 2: ***Planetary Orbits***

Comet Encke was discovered in 1786 by Pierre Mechain and in 1822 Johann Encke determined that its period was 3.3 years. It was photographed in 1913 at the aphelion distance, $r_a = 6.1\times10^{11}\,m$, (furthest distance from the sun) by the telescope at Mt. Wilson. The distance of closest approach to the sun, perihelion, is $r_p = 5.1\times10^{10}\,m$. The universal gravitation constant $G = 6.7\times10^{-11}\,N\cdot m^2\cdot kg^{-2}$. The mass of the sun is $m_s = 2.0\times10^{30}\,kg$.



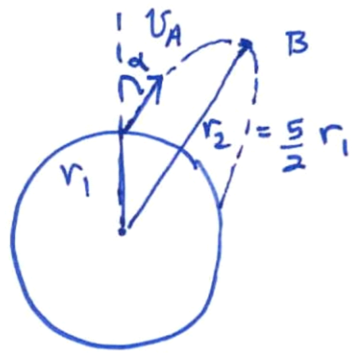
a) Explain why angular momentum is conserved about the focal point and then write down an equation for the conservation of angular momentum between aphelion and perihelion.

b) Explain why mechanical energy is conserved and then write down an equation for conservation of energy between aphelion and perihelion.

c) Use conservation of energy and angular momentum to find the speeds at perihelion and aphelion.

Problem 3: Satellite Motion

A spherical non-rotating planet (with no atmosphere) has mass m_1 and radius r_1. A projectile of mass $m_2 << m_1$ is fired from the surface of the planet at a point A with a speed v_A at an angle $\alpha = 30^\circ$ with respect to the radial direction. In its subsequent trajectory the projectile reaches a maximum altitude at point B on the sketch. The distance from the center of the planet to the point B is $r_2 = (5/2)r_1$.



In this problem you will find the initial speed v_A in terms of G, m_1 and r_1.

a) Is there a point about which the angular momentum of the projectile is constant? If so, use this point to determine a relation between the speed v_B of the projectile at the point B in terms of v_A and the angle $\alpha = 30^\circ$.

b) Now use conservation of mechanical energy constant to find an expression to find v_A in terms of G, m_1 and r_1.

Problem 4: Inverse Square Central Force: *Lowest Energy Solution*

The effective potential energy for an inverse-square restoring central force $\vec{\mathbf{F}}_{1,2} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$ is given by

$$U_{\text{effective}} = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

Make a graph of the effective potential energy $U_{\text{effective}}(r)$ as a function of the relative separation r. Find the radius and the energy for the motion with the lowest energy. What type of motion does this correspond to?

Problem 5

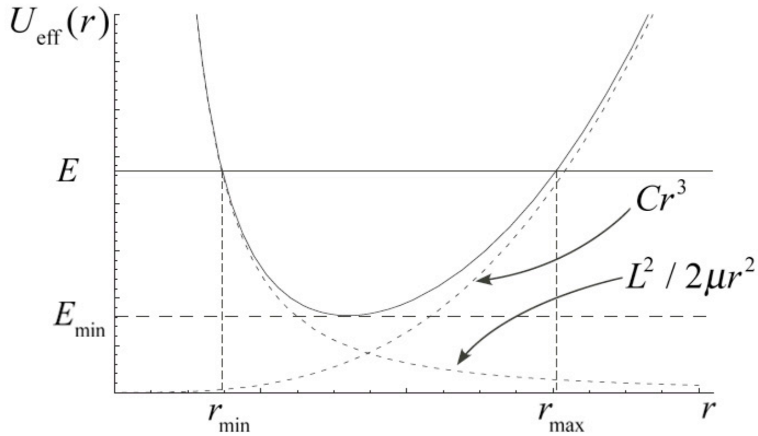
A particle of mass m moves under an attractive central force of magnitude $F = br^3$. The angular momentum is equal to L.

a) Find the effective potential energy and make sketch of effective potential energy as a function of r.

b) Indicate on a sketch of the effective potential the total energy for circular motion.

c) The radius of the particle's orbit varies between r_0 and $2r_0$. Find r_0.

Problem 6:



The effective potential corresponding to a pair of particles interacting through a central force is given by the expression

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + Cr^3$$

where L is the angular momentum, μ is the reduced mass and C is a constant. The total energy of the system is E. The relationship between $U_{\text{eff}}(r)$ and E is shown in the figure, along with an indication of the associated maximum and minimum values of r and the minimum allowed energy E_{min}. In what follows, assume that the center of mass of the two particles is at rest.

a) Find an expression for the radial component $f(r)$ of the force between the two particles. Is the force attractive or repulsive?

b) What is the radius r_0 of the circular orbit allowed in this potential? Express your answer as some combination of L, C, and μ.

c) When E has a value larger than E_{min}, find how rapidly the separation between the particles is changing, dr/dt, as the system passes through the point in the orbit where $r = r_0$. Give your answer in terms of some combination of E, E_{min}, L, C, μ and r_0.

d) Does the relative motion between the particles stop when $r = r_{\text{max}}$? If not, what is the total kinetic energy at that point in terms of some combination of E, L, C, μ, r_{max} and r_{min}?

Problem 7: Determining the Mass of a Neutron Star

A binary system known as 4U0900-40 consists of a "neutron star" and a normal "optical star." You are given two graphs of actual data obtained from observations of this system. The top graph shows the time delays (in seconds) of X-ray pulses detected from the neutron star as a function of time (in days) throughout its orbit. These delays indicate the time of flight for an X-ray pulse (traveling with the speed of light) to cross the orbit of the binary in its trip to the earth. (Ignore the heavy dots scattered about the "x"-axis.)

The bottom graph displays the velocity component of the optical star toward (or away from) the Earth as a function of its orbital phase. The velocities (in km per second) are determined from Doppler shifts in its spectral lines. Time on this plot is given in units of orbital phase, where the time between phases 0.0 and 1.0 corresponds to one orbital period.

Assume that the orbit of 4U0900-40 is circular and that we are viewing the system edge on, i.e., the Earth lies in the plane of the binary orbit. In each graph, the solid curve is a computed fit to the individual data points.

You are to find from the data the orbital period, orbital radii and the masses of the neutron star and the optical star. You need only give answers to two significant figures.

For this problem, use $c = 3.0 \times 10^{8}\,\mathrm{m \cdot s^{-1}}$ for the speed of light and $G = 6.7 \times 10^{-11}\,\mathrm{N \cdot m^{2} \cdot kg^{-2}}$ for the Universal Gravitation Constant.

Suggested procedure (you will need a ruler for the first three parts):

a. Determine the orbital period, T, in days.

b. Estimate the velocity, v_0 of the optical star in its orbit around the center of mass.

c. Use T and v_0 to find the radius of the orbit of the optical star, a_0, around the center of mass.

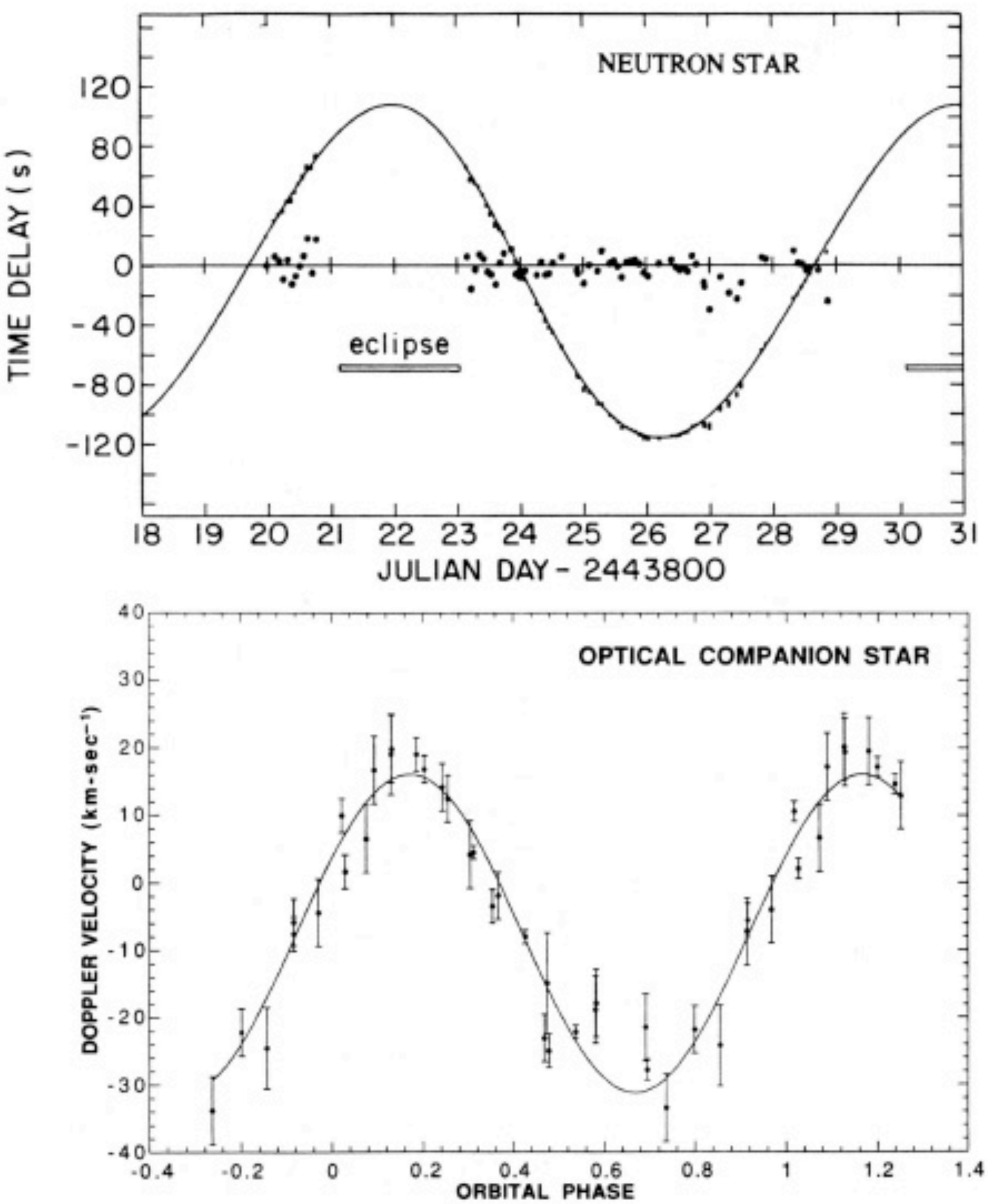
d. Estimate the size of the orbit, a_N, of the neutron star around the center of mass. First express your answer in light$\cdot$seconds, then in meters.

e. Find the ratio of the mass of the neutron star, m_N, to the mass of the optical star, m_O. [Hint: Recall that in any binary $m_1 r_1 = m_2 r_2$.]

f. Use Kepler's law to find the total mass of the binary system.

g. Find m_N and m_O; express your answers in units of the mass of the Sun (2×10^{30} kg).

BINARY X-RAY PULSAR SYSTEM



Doppler Curves for 4U0900-40.