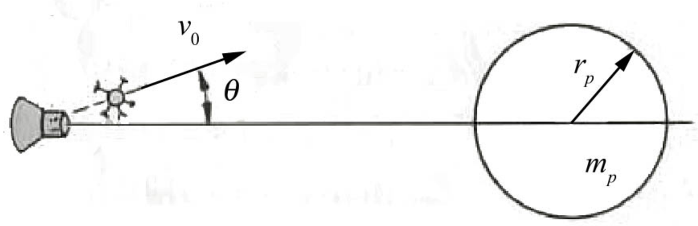
Angular Momentum Problems
Challenge Problems

Problem 1: A spaceship is sent to investigate a planet of mass m_p and radius r_p. While hanging motionless in space at a distance $5r_p$ from the center of the planet, the ship fires an instrument package with speed v_0. The package has mass m_i which is much smaller than the mass of the spacecraft. The package is launched at an angle θ with respect to a radial line between the center of the planet and the spacecraft. For what angle θ will the package just graze the surface of the planet?

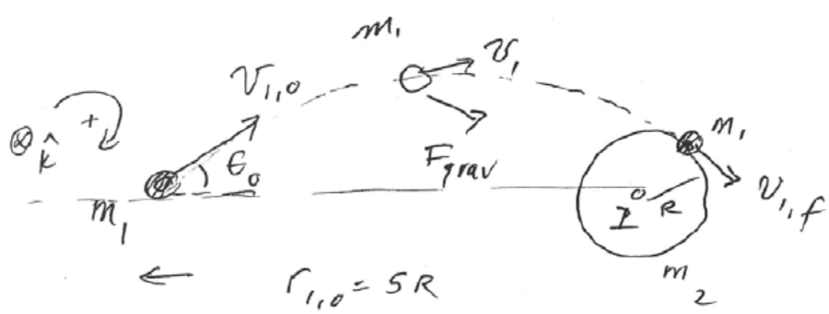


Problem 1 Solution:

The gravitational force by the planet on the spaceship $\vec{F}^G_{pm}$ always points towards the center of the planet. The torque about the center of the planet (point labeled O) due to the gravitational force is given by the expression

$$\vec{\tau}_O = \vec{r}_{O,m} \times \vec{F}^G_{pm}. \tag{1.1}$$

The vector $\vec{r}_{O,m}$ points from the center of the planet to the spaceship so $\vec{r}_{O,m}$ and $\vec{F}^G_{pm}$ are antiparallel hence the torque $\vec{\tau}_O$ is zero. Therefore the angular momentum of the spaceship about the center of the planet is constant. In the figure below the several positions of the spaceship with the associated velocities are shown. (Think of this as an angular momentum diagram.) In the figure denote $v_0 \equiv v_{1,0}$



Then the initial angular momentum of the spaceship about the center of the planet is

$$\vec{L}_{O,i} = \vec{r}_{O,i} \times m_1 \vec{v}_{1,0} = 5R\hat{r} \times (-m_1 v_0 \cos\theta \hat{r} + m_1 v_0 \sin\theta \hat{\theta}) = 5R m_1 v_0 \sin\theta \hat{k}. \quad \textbf{(1.2)}$$

The angular momentum about the center of the planet when the spaceship just grazes the planet is

$$\vec{L}_{O,f} = \vec{r}_{O,f} \times m_1 \vec{v}_{1,f} = R m_1 v_{1,f} \hat{k}. \quad \textbf{(1.3)}$$

Because angular momentum about the center of the planet is constant,

$$\vec{L}_{O,i} = \vec{L}_{O,f}. \quad \textbf{(1.4)}$$

Substituting Eqs. (1.2) and (1.3) into Eq. (1.4) and taking the z-component on both sides yields

$$5R m_1 v_0 \sin\theta = R m_1 v_{1,f}. \quad \textbf{(1.5)}$$

Thus we can solve for the final speed

$$v_{1,f} = 5 v_0 \sin\theta. \quad \textbf{(1.6)}$$

There is no non-conservative work done on the spacecraft so the mechanical energy is constant,

$$E_i = E_f. \quad \textbf{(1.7)}$$

Choose infinity as the zero point for potential energy then the energy equation becomes

$$\frac{1}{2} m_1 v_0{}^2 - \frac{G m_1 m_p}{5R} = \frac{1}{2} m_1 v_{1,f}{}^2 - \frac{G m_1 m_p}{R}. \quad \textbf{(1.8)}$$

Substitute in Eq. (1.6) for the final speed yielding

$$\frac{1}{2} m_1 v_0{}^2 - \frac{G m_1 m_p}{5R} = \frac{1}{2} m_1 (5 v_0 \sin\theta)^2 - \frac{G m_1 m_p}{R}. \quad \textbf{(1.9)}$$

Collecting terms we can rewrite Eq. (1.9) as

$$\frac{4 G m_1 m_p}{5R} = \frac{1}{2} m_1 v_0{}^2 (25 \sin^2\theta - 1) \quad \textbf{(1.10)}$$

We can now solve for the angle θ:

$$\theta = \sin^{-1}\left(\frac{1}{5}\sqrt{\frac{8Gm_p}{5R{v_0}^2}+1} \right). \tag{1.11}$$

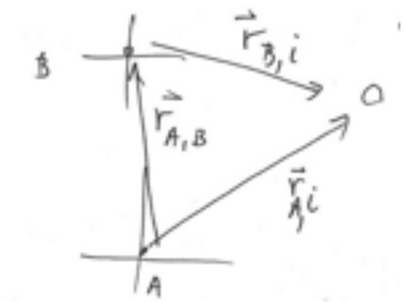
Problem 2: Conservation Laws

a) Show that if the total linear momentum of a system of particles is zero, the angular momentum of the system is the same about all origins. Explain how you may apply this result involving an elastic collision of two rigid bodies.

b) Show that if the total force on a system of particles is zero, the torque on the system is the same about all origins. Explain how you can use this result for static equilibrium problems.

Problem 2 Solutions:

a) Assume $\bar{p}T = \sum_i m_i v_i = 0$. Choosetwo points A and B with $\vec{r}_{A,B}$ = constant vector



$$\vec{r}_{A,i} = \vec{r}_{A,B} + \vec{r}_{B,i}$$

$$\bar{L}_A = \sum_i r_{A,i} \cdot m_i v_i$$

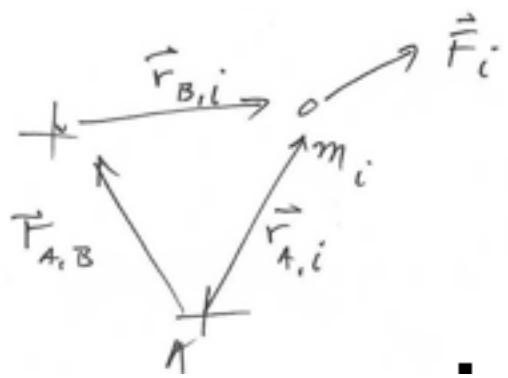
$$= \sum_i \left(\vec{r}_{A,B} + \vec{r}_{B,i}\right) \cdot m_i \vec{v}_i$$

$$= \underbrace{\vec{r}_{A,B} \cdot \sum_i m_i \vec{v}_i}_{\sin\theta \ \vec{r}_{A,B} \text{ is the same for each } m_i} + \sum_i \vec{r}_{B,i} \cdot m_i \vec{v}_i$$

$$\sin\theta \sum m_i \vec{v}_i = 0 \text{ and } \vec{L}_B = \sum \vec{r}_{B,i} \cdot m_i \vec{v}_i$$

$$\vec{L}_A = \vec{L}_B$$

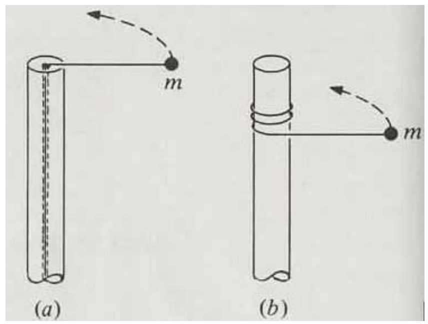
b) This argument will be identical, just substitute $\bar{F}_i$ for $m_i \vec{v}_i$



$\vec{r}_{B,i}$
$\vec{F}_i$
m_i
$\vec{r}_{A,B}$
$\vec{r}_{A,i}$
A

Problem 3: Conservation Laws Post

A body of particle of mass m (treat it as a point like particle) is attached to a post of radius R by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0. In case (a) the string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. In case (b) the string wraps around the outside of the post. What quantities remain constant in each case? Find the final speed of the body when it hits the post for each case.

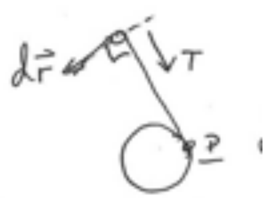


Problem 3 Solutions:

a) Since $\bar{T}\cdot dr = dW_{n.c.} \neq 0$, energy is not conserved.

$\vec{\tau}_{cm} = \vec{F}_{cm,m}\cdot\vec{T} = 0$, because $\bar{T}$ is a radial force so $\dfrac{d\bar{L}_{cm}}{dt} = 0$, $\bar{L}_{cm}$ is conserved. Momentum is not conserved because $\bar{F}_{ext} = \bar{T} \neq 0$.

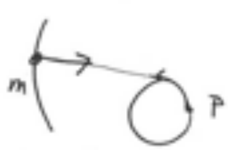
b)



$d\omega_{n.c.} = \bar{T}\cdot dr = 0$ because about the point p, the mess is instantaneously moving in a circular orbit so

$$\bar{T} \perp dr$$

This energy is conserved.



As the mass moves, $\bar{T}$ does not point towards one specific point so no point in which $E_p \neq 0$ for all p.

This $\frac{d\bar{L}_p}{dt} \neq -$ and angular momentum is not conserved. As in part (a) momentum is not conserved since $\bar{F}_{ext} = \bar{T} \neq 0$.

Since $\bar{L}_{cm}$ is conserved:

$$\bar{L}_{cm,0} = r_{cm,0} \times mv_0$$

$$\vec{v}_0 = mv_0\hat{\theta}\ ,\ \vec{r}_{cm,0} = r_0\hat{r}$$

$$\bar{L}_{cm,0} = r_0\hat{r} \cdot mv_0\hat{\theta} = mr_0v_0\hat{k}$$

$$\bar{L}_{cm,f} = r_0\hat{r} \cdot mv_f\hat{\theta} = mr_fv_f\hat{k}$$

since $v_{f,\ radial}$,=0 (hits post)

Thus $\bar{L}_{cm,0} = \bar{L}_{cm,f} \Rightarrow v_f = \frac{r_0v_0}{r_f}$

$$E_0 = E_f \Rightarrow$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 \Rightarrow v_0 = v_f$$