8.012 Physics I: Classical Mechanics
Fall 2008

Problem 1: Masses and Rod [15 pts]
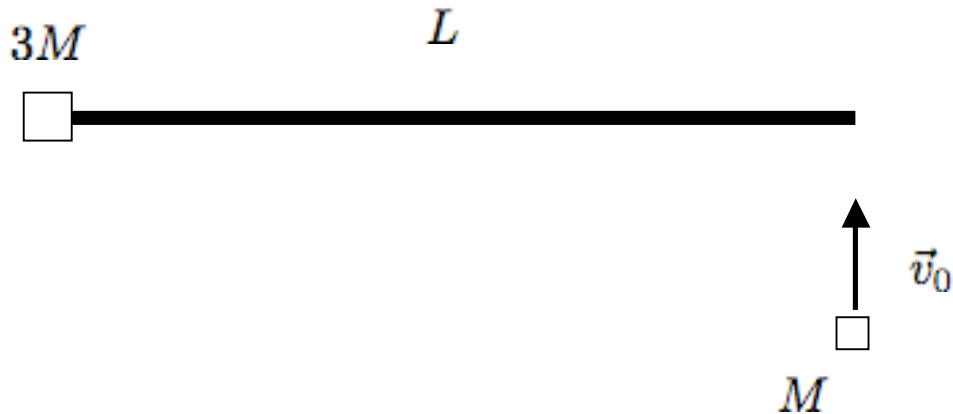
A block with mass 3M connected to a massless rod of length L lies at rest on a fixed frictionless table. A second block of mass M impinges on the system with speed v_0, strikes the opposite end of the rod at a right angle, and sticks. For this problem, assume that the dimensions of the blocks are much smaller than the length of the rod.

(a) [5 pts] What is the final velocity of the center of mass of the block-rod-block system?

(b) [5 pts] What is the final angular velocity of the block-rod-block system about its center of mass?

(d) [3 pts] What is the initial velocity of the more massive block after the collision? Be sure to indicate its direction and speed.

SOLUTIONS

(a) Linear momentum of system is conserved (no external forces), so

$$Mv_0 = 4Mv \Rightarrow \boxed{v = \frac{1}{4}v_0}$$

(b) Angular momentum of system is conserved (no external torques) so

$$L = Ml^2\omega_0 = Mlv_0 = I\omega$$

where l is the distance from the center of mass (COM) to the end of the bar:

$$R_{COM} = \frac{ML}{4M} = \frac{L}{4} \Rightarrow l = \frac{3}{4}L$$

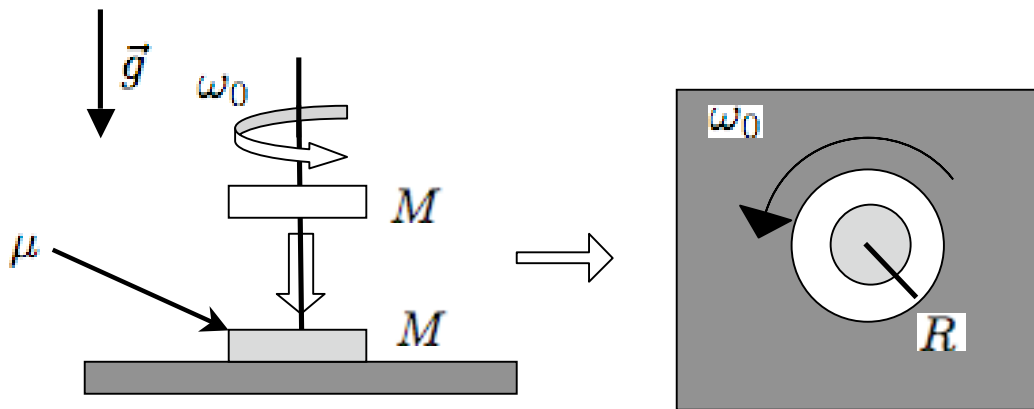and I is the moment of inertia of two point masses about the COM:

$$I = 3M\left(\frac{L}{4}\right)^2 + M\left(\frac{3L}{4}\right)^2 = \frac{3}{4}ML^2$$

and hence

$$\omega = \frac{Mlv_0}{I} = \frac{(3/4)MLv_0}{(3/4)ML^2} = \boxed{\frac{v_0}{L}}$$

(c) at the position of 3M, total velocity is combination of linear and rotation motion, so

$$v = \frac{1}{4}v_0 - R_{COM}\omega = \frac{1}{4}v_0 - \frac{L}{4}\frac{v_0}{L} = \boxed{0}$$

A uniform hoop with mass M, outer radius R and inner radius R/2 is initially spinning with angular velocity ω_0. The hoop is placed onto a uniform disk, also of mass M and outer radius R, that is initially at rest on a fixed frictionless table. The hoop and disk are aligned along their centers of mass. There is a coefficient of friction μ between the hoop and disk where their surfaces make contact. Gravitational acceleration is assumed to be constant and points downward.

(a) [10 pts] What is the initial torque (about its center of mass) acting on the disk? What is the initial torque (about its center of mass) acting on the hoop?

(b) [5 pts] The friction force between the hoop and disk cause them to eventually spin at the same angular velocity. What is this angular velocity?

(c) [5 pts] How much time does it take for the disk and hoop to reach this common angular velocity?

SOLUTIONS

(a) The torque on the hoop from the disk arises from friction $\vec{F} = -\mu N \hat{\theta}$, where $N = Mg$ is the normal contact force between the disk and hoop. First consider a small section where the hoop and disk make contact.

$$d\vec{\tau} = \vec{r} \times d\vec{F}$$

$$= r\hat{r} \times -\mu(dm)g\hat{\theta}$$

$$= -r\mu\sigma g(dA)\hat{z}$$

where $dm = \sigma dA = \sigma r dr d\theta$ is a small mass segment on the hoop with uniform areal density σ. The total torque is an integral over the area of the loop:

$$\vec{\tau} = \int d\vec{\tau} = -\int_0^{2\pi} \int_{R/2}^R r\mu\sigma g r dr d\theta \hat{z}$$

$$= -\frac{7\pi}{12}\mu\sigma g R^3 \hat{z}$$

but the mass of the hoop is:

$$M = \int_0^{2\pi} \int_{R/2}^R \sigma r dr d\theta = \frac{3\pi}{4}\sigma R^2$$

Hence the torque on the hoop as the result of frictional forces from the disk is

$$\boxed{\vec{\tau}_{hoop} = \frac{7}{9}\mu M g R \hat{z}}$$

Following Newton's third law, the frictional force on the disk from the hoop is equal and opposite to that of the hoop from the disk, and the force acts over the same surface area. Hence, the torques are equal and opposite:

$$\boxed{\vec{\tau}_{disk} = \frac{7}{9}\mu M g R \hat{z}}$$

(b) There are no external torques on the disk/hoop system, so total angular momentum is conserved, i.e.,

$$I_h \omega_0 = (I_h + I_d)\omega$$

The moment of inertia of the hoop is:

$$I_h = \int_0^{2\pi} \int_{R/2}^R r^2 \sigma r \, dr \, d\theta = \frac{15\pi}{32} \sigma R^4 = \frac{5}{8} M R^2$$

while

$$I_d = \frac{1}{2} M R^2$$

Hence,

$$\omega = \omega_0 \frac{(5/8)MR^2}{(1/2)MR^2 + (5/8)MR^2} = \boxed{\frac{5}{9}\omega_0}$$

(c) The time it takes for the hoop to spin down is given by the torque:

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I_h \dot{\vec{\omega}} \Rightarrow \frac{5}{8} M R^2 \dot{\omega} = -\frac{7}{9}\mu M g R$$

$$\Rightarrow \dot{\omega} = -\frac{56}{45}\mu \frac{g}{R}$$

$$\Rightarrow \omega = \omega_0 - \frac{56}{45}\mu \frac{g}{R} t$$

setting ω to the value from part (b) gives:

$$\boxed{t_{slow} = \frac{5}{14}\frac{R\omega_0}{\mu g}}$$

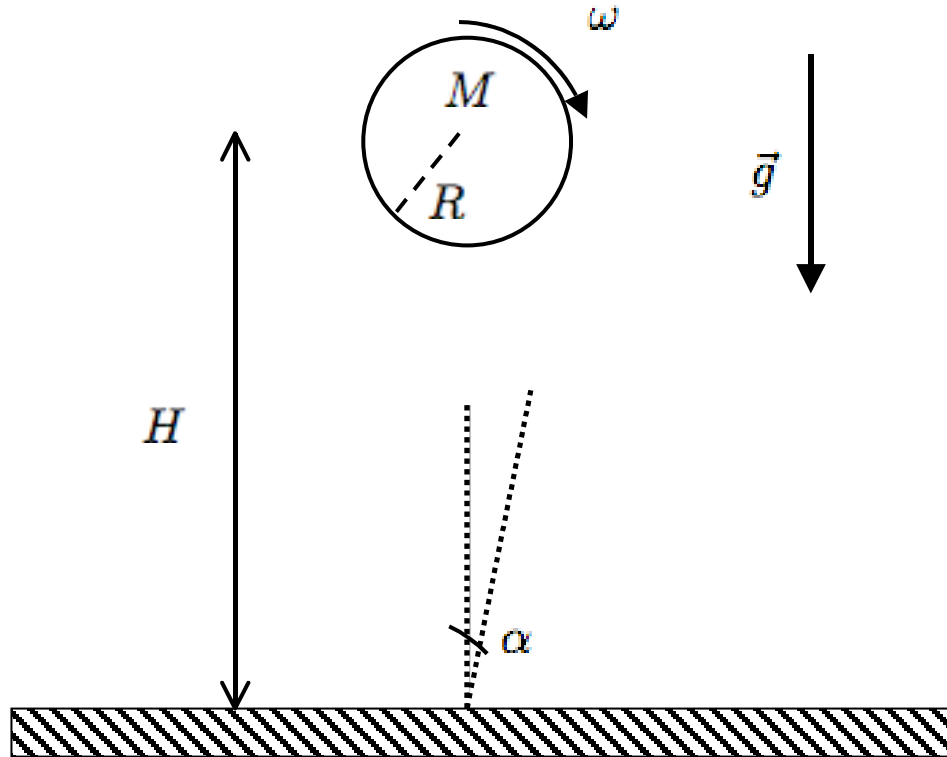as a check we can compute the speed up time for the disk:

$$\frac{1}{2}MR^2\dot{\omega} = \frac{7}{9}\mu MgR$$

$$\Rightarrow \omega = \frac{14}{9}\mu\frac{g}{R}t$$

again, substituting the value from part (b) gives:

$$t_{fast} = \frac{5}{14}\frac{R\omega_0}{\mu g}$$

Problem 7: Spinning Bouncing Ball [15 pts]

A uniform sphere of mass M and radius R spinning with angular velocity ω is dropped from a height H. It bounces on the floor and recoils with the same <u>vertical</u> velocity. During the bounce, the surface of the ball slips relative to the surface of the floor (i.e., it does not roll) and in the process the ball is acted upon by a friction force with magnitude $f = \mu N$, where N is the normal contact force between the ball and the floor and μ is a constant. Hence, the ball experiences impulses in both vertical and horizontal directions. Assume that the duration of contact, Δt, is very short.

(a) What angle α with respect to vertical does the ball recoil?

(b) What is the final rotation velocity of the ball?

(c) What value of H results in the ball bouncing off with no spin?

(a) The angle α is related to the horizontal and vertical components of the ball's velocity after the bounce. The vertical component is straightforward, as it is stated in the problem that it rebounds with the same velocity as it struck the ground, which is:

$$v_y = \sqrt{2gH}$$

The horizontal component arise from the impulse due to friction acting at the contact surface:

$$v_x = \frac{\Delta p_x}{m} \approx \frac{\bar{F}_x \Delta t}{m} = \frac{\mu \bar{N} \Delta t}{m}$$

This is related to the vertical impulse by:

$$\Delta p_y = \bar{N} \Delta t = 2mv_y = 2m\sqrt{2gH}$$

Hence:

$$v_x = 2\mu\sqrt{2gH}$$

and:

$$\boxed{\tan \alpha = \frac{v_x}{v_y} = 2\mu}$$

(b) The frictional force produces a torque that acts in the opposite direction as the spinning of the ball. This torque exerts a rotational impulse (change in angular momentum):

$$\Delta L = I\Delta \omega \approx -\bar{\tau} \Delta t = -R\mu \bar{N} \Delta t = -2R\mu m\sqrt{2gH}$$

Hence,

$$\omega - \omega_0 = \Delta \omega = -\frac{2R\mu m\sqrt{2gH}}{I}$$

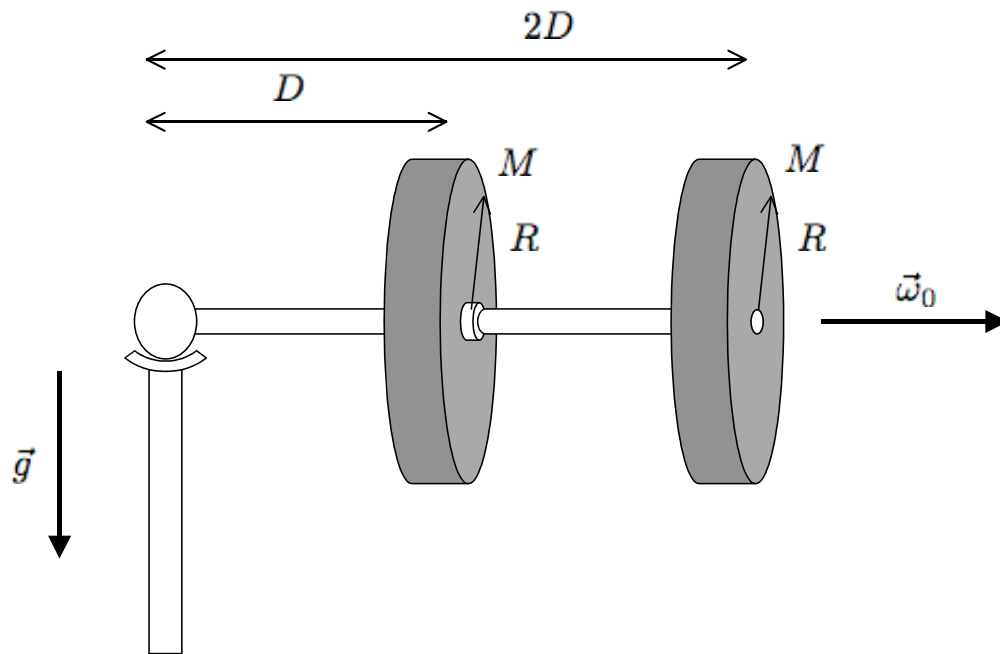Using the momentum of inertia for a sphere:

$$I = \frac{2}{5}MR^2$$

yields

$$\omega = \omega_0 - \frac{5\mu\sqrt{2gH}}{R}$$

(c) For no spin after the bounce, $\omega = 0$, so

$$\omega_0 = \frac{5\mu\sqrt{2gH}}{R}$$

$$\Rightarrow H = \frac{R^2\omega_0^2}{50\mu^2 g}$$

Problem 2: The Double Gyroscope [15 pts]

A gyroscope consists of two identical uniform disks with mass M and radius R mounted on a rigid axle with length 2D. The axle is fixed to the outer disk, while the inner disk is allowed to spin freely but is constrained to remain at a distance D from the pivot (at left) by a collar. The axle spins about the pivot freely on a frictionless mount. The outer disk and axle are initially spun up to an angular frequency ω_0. Assume that the mass of the axle and pivot mount are negligible. Gravity points downward. Ignore nutation.

(a) Calculate the precession rate Ω of the gyroscope assuming that the inner disk is not spinning. Assume that ω_0 is very large and the gyroscopic approximation applies.

(b) Friction between the axle and the disk closest to the pivot causes the latter to spin up. Assuming that a constant torque τ acts at the interface between the axle and inner hole of this disk, calculate the final spin rate ω of both disks and the final precession rate in terms of the initial precession rate.

(c) How much energy is lost from the gyroscope system during the spinning up of the inner disk?

(a) The net torque on the gyroscope with respect to the pivot point is

$$\vec{\tau} = \vec{r} \times \vec{F} = (MgD + 2MgD)\hat{\otimes} = 3MgD\hat{\otimes}$$

The gyroscope approximation is:

$$|\vec{\tau}| = \left| \frac{d\vec{L}}{dt} \right| = |\vec{L}|\,\Omega$$

The angular momentum of the first disk is

$$|\vec{L}| = \frac{1}{2}MR^2\omega_0$$

Hence

$$3MgD = \frac{1}{2}MR^2\omega_0\Omega$$

$$\Rightarrow \boxed{\Omega = \frac{6gD}{R^2\omega_0}}$$

(b) The constant frictional torque is

$$\tau_f = \frac{dL}{dt} = I\frac{d\omega}{dt} \Rightarrow \frac{d\omega}{dt} = constant$$

Hence:

$$\omega_{in} = \frac{\tau_f}{I}t$$

This is the result of the force from the axle acting on the inner wheel; Newton's third law tells us that an equal and opposite force acts on the axle + outer wheel and slows it down. Hence

$$\omega_{out} = \omega_0 - \frac{\tau_f}{I}t$$

The frictional force ceases when the surfaces do not slide relative to one another, i.e., when $\omega_{in} = \omega_{out} = \omega$, so

$$\boxed{\omega = \frac{1}{2}\omega_0}$$

To calculate the precession, note that the torque hasn't changed, while the magnitude of the angular momentum vector is

$$|\vec{L}| = 2 \times \frac{1}{2}MR^2\omega = \frac{1}{2}MR^2\omega_0$$

and is therefore also unchanged. So the precession rate is the same:

$$\boxed{\Omega = \frac{6gD}{R^2\omega_0}}$$

(c) Only the rotational kinetic energy of the system changes as a result of the inner wheel spinning up. The initial rotational kinetic energy is

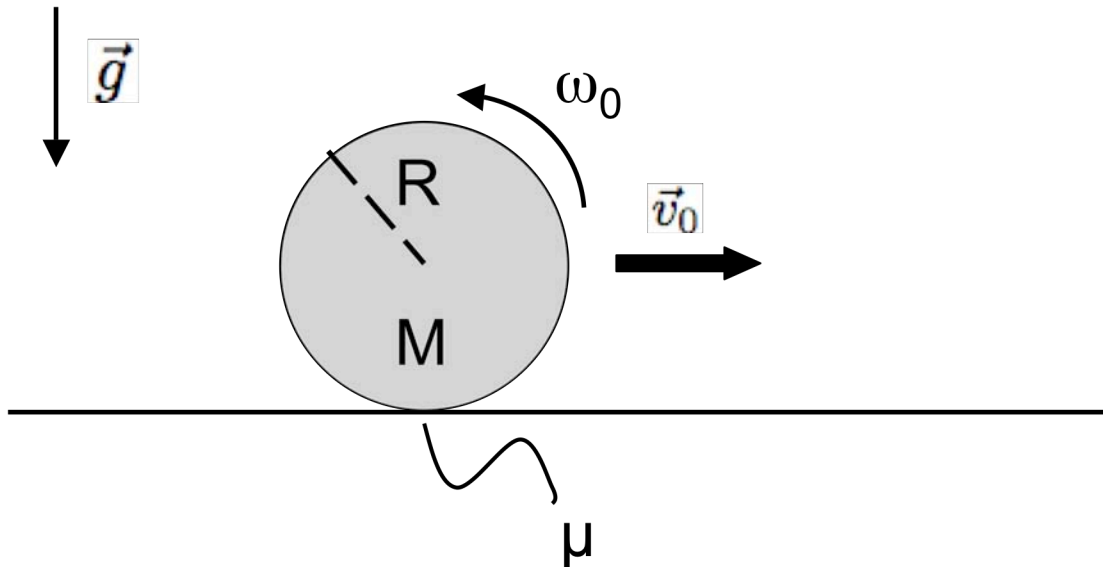$$E_i = \frac{1}{2}I_\perp\Omega^2 + \frac{1}{2}I\omega_0^2$$

where $I_\perp$ is the moment of inertia of the entire gyroscope rotating about the pivot. The final rotational kinetic energy is

$$E_f = \frac{1}{2}I_\perp\Omega^2 + 2 \times \frac{1}{2}I\omega^2 = \frac{1}{2}I_\perp\Omega^2 + \frac{1}{4}I\omega_0^2$$

Hence the total energy lost is:

$$\Delta E = E_i - E_f = \frac{1}{4}I\omega_0^2 = \boxed{\frac{1}{8}MR^2\omega_0^2}$$

Problem 2: The Bowling Ball [25 pts]

A bowling ball of radius R, mass M and uniform mass density is thrown down a lane with initial horizontal speed v_0. The ball is given some backspin – it is spun in the opposite direction of motion – with initial angular rate ω_0 as shown. The maximum coefficient of friction between the ball and lane surfaces is μ. Assume constant gravitational acceleration $\vec{g}$ points downward.

(a) [10 pts] Find the velocity $\vec{v}$ and angular rotation rate ω of the ball as a function of time after it makes contact with the lane, and up to the point when it starts to roll without slipping. Note: directions are important here!

(b) [10 pts] Find the "final" velocity and angular rotation rate of the ball when it starts to roll without slipping.

(c) [5 pts] How must the initial conditions, v_0 and ω_0, be related for the ball to come to rest?

Solutions

(a) This problem can solved by considering the translation of the center of mass and rotation about the center of mass of the ball separately. Referring to the force diagram to the right, the equations of motion are:
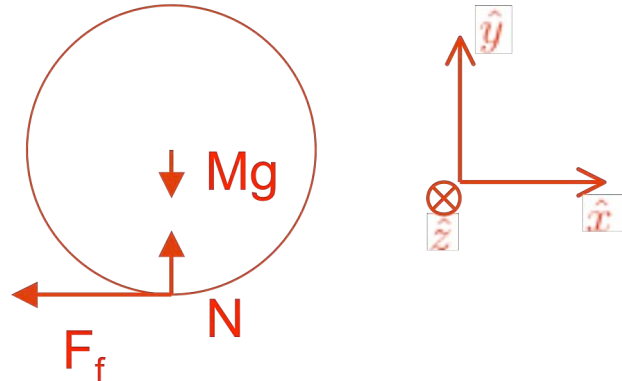
$$x : M\ddot{x} = -F_f \Rightarrow \ddot{x} = \frac{F_f}{M}$$

$$y : M\ddot{y} = -Mg + N = 0 \Rightarrow N = Mg$$

$$\theta : I\ddot{\theta} = -RF_f \Rightarrow \ddot{\theta} = -\frac{RF_f}{I}$$

Note that torque is working to slow down the rotation of the ball in this case, so there must be a minus sign in the angle equation if we set $\omega_0 > 0$. A ball is a sphere, so

$$I = \frac{2}{5}MR^2$$

While the ball is slipping, the friction force is at its maximum (constant) value of $F_f = \mu N = \mu Mg$, so the equations are easily solved. Using the initial conditions:

$$\dot{x}(0) = v_0$$
$$\dot{\theta}(0) = \omega(0) = \omega_0$$

we find (note the specific direction of velocity):

$$\vec{v}(t) = v_0\hat{x} - \mu g t \hat{x}$$
$$\omega(t) = \dot{\theta}(t) = \omega_0 - \frac{5}{2}\frac{\mu g}{R}t$$

(b) The condition of rolling without slipping occurs when

$$\omega(t) = -\frac{v(t)}{R}$$

Note the minus sign – this is because we defined positive rotation in the initial conditions such that the object has backspin, but when it is not slipping it must have forward spin. This condition can be used to find the time t_f when the ball starts to roll without slipping:

$$\omega_0 - \frac{5}{2}\frac{\mu g}{R}t_f = -\frac{v_0}{R} + \frac{\mu g}{R}t_f$$

$$\Rightarrow t_f = \frac{2}{7}\frac{R\omega_0 + v_0}{\mu g}$$

Plugging this back into the expressions from (a) we find:

$$\vec{v}_f = \frac{5}{7}v_0\hat{x} - \frac{2}{7}R\omega_0\hat{x}$$
$$\omega_f = \frac{2}{7}\omega_0 - \frac{5}{7}\frac{v_0}{R}$$

Note that after this point, the friction force ceases to act as the surfaces are no longer slipping relative to each other, and so the ball continues to roll/spin at these rates.

(c) The object comes to rest when its final velocity goes to 0. From the expression from (b) we find that this occurs if:

$$v_0 = \frac{2}{5} R \omega_0$$

Problem 4: A Ballistic Rotator [20 pts]

X

M

2R

Forward view

X

R

m

2R
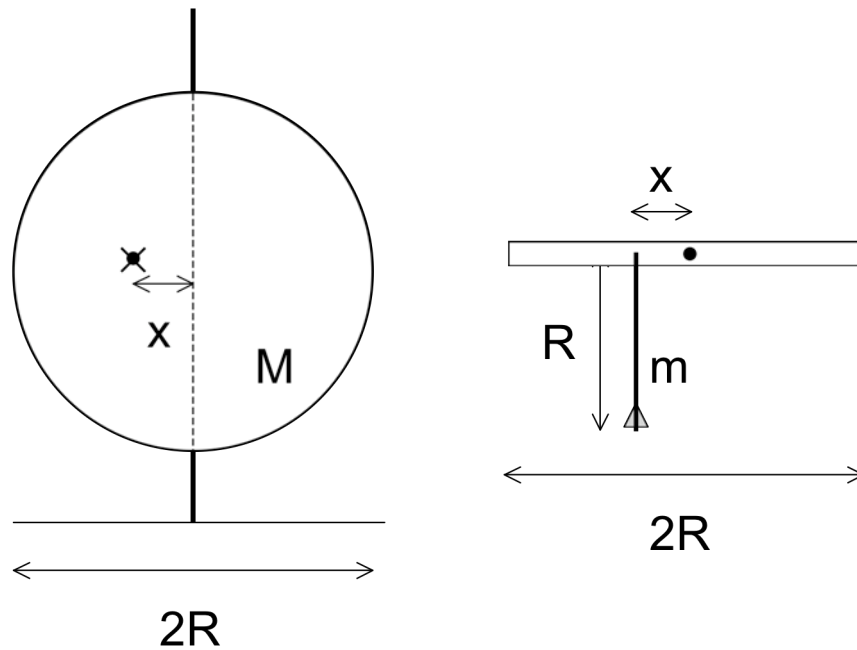
Top view

A thin arrow with length R, mass m and uniform linear mass density $\lambda = M/R$ is shot with velocity v into a circular target of radius R, mass M, uniform surface mass density $\sigma = M/\pi R^2$, and negligible thickness (i.e., it is essentially a thin disk). The arrow head sticks into the target just off center by a distance $x < R$ as shown above. The target is mounted onto a central frictionless shaft, about which it can spin freely. The target is at rest before it is struck by the arrow.

(a) [10 pts] Find the angular rotation rate of the target (with arrow embedded) after the arrow strikes it. You can express you answer in terms of I_T, the moment of inertia of the circular target *alone* about the rotation axis shown.

(b) [5 pts] Show that if m << M then to first order the rotation rate depends linearly on both x and v.

(c) [5 pts] Derive an expression for I_T in terms of M and R. **Warning!** This is a difficult problem, so save it for last!

Solution

(a) The arrow hitting the target is an example of an inelastic collision, since the arrow does not rebound off of the target. Hence energy s not conserved. Translational momentum is also not conserved, as the vertical shaft provides an impulse to prevent the target from moving. However, none of these forces present exert a torque on the arrow-target system during the collision (we can ignore gravity since it acts to slowly), so angular momentum is conserved. Let's measure angular momentum from the center of mass of the target. Then the angular momentum prior to the collision arises solely from the translational motion of the arrow relative to this point, i.e.,

$$\vec{L} = \vec{r} \times \vec{p} = mvx\hat{z} \Rightarrow L = mvx$$

where $\hat{z}$ points downward. After the collision all of the angular momentum is in rotation of the target and arrow about the shaft

$$L = (I_t + I_{arrow})\omega$$

(the direction of angular rotation should be such that L points downward) . The moment of inertia of the arrow can be found by successive applications of the parallax axis theorem, shifting from the center of mass of the arrow to its end, and then shifting from where the arrow sticks into the target to the rotation axis. Putting this together:

$$I_{arrow} = \frac{1}{12}mR^2 + m(R/2)^2 + mx^2 = \frac{1}{3}mR^2 + mx^2$$

So the angular rotation rate is:

$$\omega = \frac{mvx}{I_t + \frac{1}{3}mR^2 + mx^2}$$

(b) $I_t \approx MR^2$, $I_{arrow} \approx mR^2 + mx^2$ so if m << M, then $I_t >> I_{arrow}$, and the solution to (a) reduces to:

$$\omega \approx \frac{mvx}{I_t}$$

This depends linearly on both v and x, since I_t depends on neither of these quanitites.

(c) There are a number of ways of solving this, including using a fun trick the TA demonstrated in the review session. However, the most direct way is to use the standard integral expression for I:

$$I = \int_V \rho x^2 dV$$

where the factors in the integral, in polar coordinates, are:

$$x^2 = (r \cos\theta)^2$$

$$\rho = \frac{M}{\pi R^2}$$

$$dV = r\,dr\,d\theta$$

The volume integral is then:

$$I = \int_0^R \int_0^{2\pi} \frac{M}{\pi R^2} r^3 \cos^2 \theta \, dr \, d\theta$$

$$= \frac{1}{4\pi} M R^2 \int_o^{2\pi} \cos^2 \theta \, d\theta$$

The integral over θ can be computed by substituting the appropriate half-angle trig identity; an easier trick is to note that over 2π the integrals of $\cos^2$ and $\sin^2$ are the same (in both cases they are over a full period). Hence:

$$\int_o^{2\pi} \cos^2 \theta \, d\theta + \int_o^{2\pi} \sin^2 \theta \, d\theta = 2 \int_o^{2\pi} \cos^2 \theta \, d\theta = \int_o^{2\pi} d\theta = 2\pi$$

$$\Rightarrow \int_o^{2\pi} \cos^2 \theta \, d\theta = \pi$$

so

$$\boxed{I = \frac{1}{4} M R^2}$$