Relativistic Velocity Transformations

①

- Particle velocity $\vec{u}\,(u_x, u_y)$
- What is $\vec{u}'\,(u_x', u_y')$
- S' has velocity v rel to S

$$u_x = \lim_{\Delta t\to 0} \frac{\Delta x}{\Delta t} \qquad u_y = \lim_{\Delta t\to 0} \frac{\Delta y}{\Delta t}$$

$$u_x' = \lim_{\Delta t\to 0} \frac{\Delta x'}{\Delta t'} \qquad u_y' = \lim_{\Delta t\to 0} \frac{\Delta y'}{\Delta t'}$$

Lorentz-Transform differentials:

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

$$\Delta y' = \Delta y \qquad \Delta z' = \Delta z$$

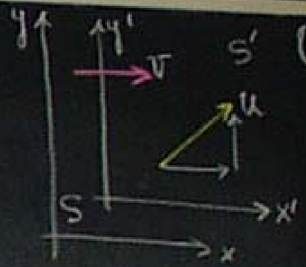
$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

$$\therefore \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - v\Delta t)}{\gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)} = \frac{\frac{\Delta x}{\Delta t} - v}{1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}}$$

Let $\Delta t \to 0$

$$u_x' = \frac{u_x - v}{1 - v/c^2\, u_x}$$

$$u_y' = \frac{u_y}{\gamma(1 - v/c^2\, u_x)} \qquad u_z' = \frac{u_z}{\gamma(1 - v/c^2\, u_x)}$$



②

Invert to get:

$$u_x = \frac{u_x' + v}{1 + v/c^2\, u_x'}$$

$$u_y = \frac{u_y'}{\gamma[1 + v/c^2\, u_x']}$$

$$u_z = \frac{u_z'}{\gamma[1 + v/c^2\, u_x']}$$

If $v \ll c$

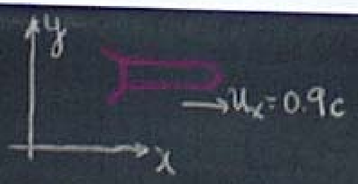
$$u_x' = u_x - v$$

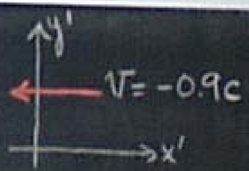
$$u_y' = u_y$$

$$u_z' = u_z$$

"Galilean Transf"

Example:





$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} = \frac{0.9c - (-0.9c)}{1 - \frac{(-.9c)(0.9c)}{c^2}} = \frac{1.8c}{1.81} = 0.99c$$

Example

Let $u_x = c$

$$u_x' = \frac{c - v}{1 - vc/c^2} = \frac{c^2(c-v)}{(c - vc)} \equiv c$$

Independent of v!!

Limiting velocity $\equiv c$.

Lecture 35 Blackboard #1

Invert to get:

$$u_x = \frac{u_x' + v}{1+\frac{v}{c^2}u_x'}$$

$$u_y = \frac{u_y'}{\gamma\left[1+\frac{v}{c^2}u_x'\right]}$$

$$u_z = \frac{u_z'}{\gamma\left[1+\frac{v}{c^2}u_x'\right]}$$

If $v \ll c$

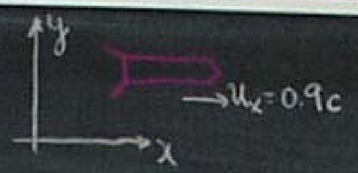
$$u_x' = u_x - v$$

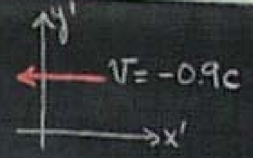
$$u_y' = u_y$$

$$u_z' = u_z$$

"Galilean Transf"

Example:





$$u_x' = \frac{u_x - v}{1-\frac{v}{c^2}u_x} = \frac{0.9c-(-0.9c)}{1-\frac{(-.9c)(0.9c)}{c^2}} = \frac{1.8c}{1.81} = 0.99c$$

Example

Let $u_x = c$

$$u_x' = \frac{c-v}{1-vc/c^2} = \frac{c^2(c-v)}{(c-vc)} \equiv c$$

Independant of v!!

Limiting velocity $\equiv c$.

② ③

Doppler Effect: Longitudinal

- Sound pitch increases for approaching source
- Sound pitch decreases for receding source
- What about light?

Source produces flashes with period $\tau_0 = 1/\nu_0$ in rest Frame-S'.
S' moving with velocity v rel. to S.

Time dilation: $\tau = \gamma\tau_0$

Pulses travel with speed c.

Observed frequency in S

$\nu_D = \frac{c}{L}$ L: dist between two pulses.

Source is moving with v

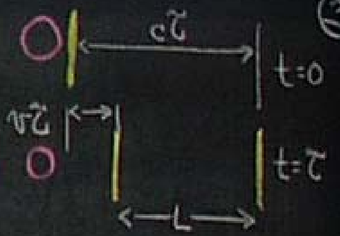
$$L = c\tau - v\tau = (c-v)\tau$$

$$\nu_D = \frac{c}{(c-v)}\frac{1}{\tau}$$

$$\nu_D = \frac{1}{1-v/c}\,\frac{1}{\gamma\tau_0}$$

$$\nu_D = \nu_0\,\frac{\sqrt{1-v^2/c^2}}{1-v/c}$$

$\nu_D = \nu_0\sqrt{\frac{1+v/c}{1-v/c}}$ (source approaching)



Frame-S

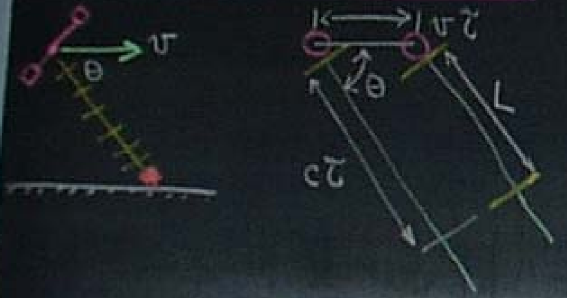
ν_D: frequency in observer's rest frame

ν_0: frequency in source rest frame.

$\nu_D = \nu_0\sqrt{\frac{1-v/c}{1+v/c}}$ (source receding) $\Rightarrow \nu_D < \nu_0$

Lecture 35 Blackboard #2

Doppler Effect Off-Line (Transverse)



Period of flashes $\tau = \gamma\tau_0$

Observed freq $\nu_D = \frac{c}{L}$

L = dist. between flashes

$v\tau$: dist. source moves bet. flashes

$L = c\tau - v\tau\cos\theta$

$= (c - v\cos\theta)\,\tau$

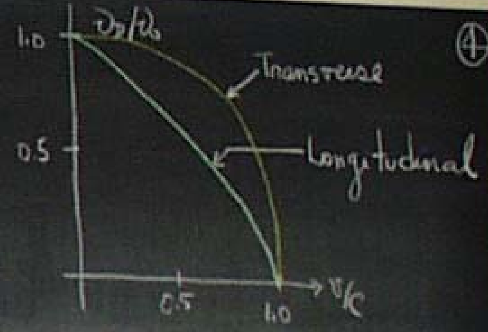
$$\nu_D = \frac{c}{L} = \frac{c}{(c - v\cos\theta)\,\tau_0\gamma}$$

$$\nu_D = \nu_0\,\frac{\sqrt{1 - v^2/c^2}}{1 - v/c\,\cos\theta}$$

$$\nu_D(\theta = 90^\circ) = \nu_0\sqrt{1 - v^2/c^2}$$

ν_D (classically) $\equiv 0$!!

Purely Relativistic Effect.



Example: Two Observers: Doppler Shift

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- Rest frame frequency: ν_0
- S' move with v_1 (rel to source) measures ν_1
- S'' move with v_2 (rel to S') measures ν_2
- Assume also S'' moves with V rel. to source.

$$\nu_1 = \nu_0\sqrt{\frac{1 - v_1/c}{1 + v_1/c}} \quad \text{[Observer S']}$$

$$\nu_2 = \nu_1\sqrt{\frac{1 - v_2/c}{1 + v_2/c}} \quad \text{(Observer S'')}$$

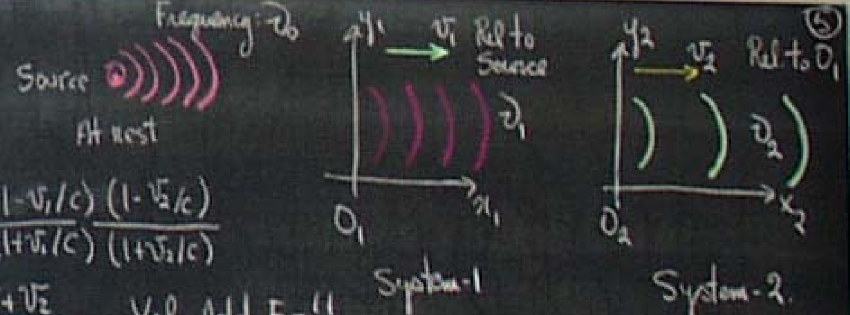
$$\nu_2 = \nu_0\sqrt{\frac{1 - v_1/c}{1 + v_1/c}}\sqrt{\frac{1 - v_2/c}{1 + v_2/c}}$$

Must also have:

$$\nu_2 = \nu_0\sqrt{\frac{1 - V/c}{1 + V/c}}$$

$$\therefore \frac{1 - V/c}{1 + V/c} = \frac{(1 - v_1/c)}{(1 + v_1/c)}\frac{(1 - v_2/c)}{(1 + v_2/c)}$$

$$V = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}$$ Vel. Add. Eq!!



Lecture 35 Blackboard #3

Example: Two Observers: Doppler Shift

· Rest frame frequency: ν_0

· S' moves with v_1 (rel to source) measures ν_1

· S'' moves with v_2 (rel to S') measures ν_2

· Assume also S'' moves with V rel. to source.

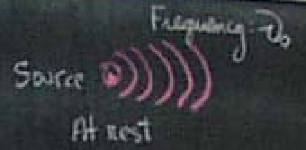
$$\nu_1 = \nu_0\sqrt{\frac{1-v_1/c}{1+v_1/c}} \quad [\text{Observer } S']$$

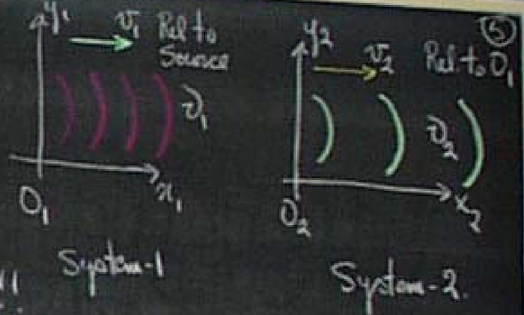
$$\nu_2 = \nu_1\sqrt{\frac{1-v_2/c}{1+v_2/c}} \quad (\text{Observer } S'')$$

$$\nu_2 = \nu_0\sqrt{\frac{1-v_1/c}{1+v_1/c}}\sqrt{\frac{1-v_2/c}{1+v_2/c}}$$

Must also have:

$$\nu_2 = \nu_0\sqrt{\frac{1-V/c}{1+V/c}}$$





⑤

$$\frac{1-V/c}{1+V/c} = \frac{(1-v_1/c)}{(1+v_1/c)}\frac{(1-v_2/c)}{(1+v_2/c)}$$

$$V = \frac{v_1+v_2}{1+v_1v_2/c^2} \quad \text{Vel. Add. Eq!!}$$

Twin Paradox

A: Bob

B: Dave

· A sees B travel distance L with velocity v

$T = L/v$

· B reverses and returns in time T.

· A observes time T_B' on B's clock.

$$T_B' = T_A/\gamma$$

$$\frac{\text{Age of B}}{\text{Age of A}} = \frac{T_B'}{T_A} = \frac{1}{\gamma} \quad (\text{seen by A})$$

· B sees A travel with vel $-v$.

$$T_B = 2T$$

$$T_A' = T_B/\gamma$$

$$\frac{\text{Age of B}}{\text{Age of A}} = \frac{T_B}{T_A'} = \gamma$$

A: B is younger!

B: A is younger!

Paradox!!!

Resolution

· Bob travels to a star

$v = 0.8c \quad \gamma = 5/3$

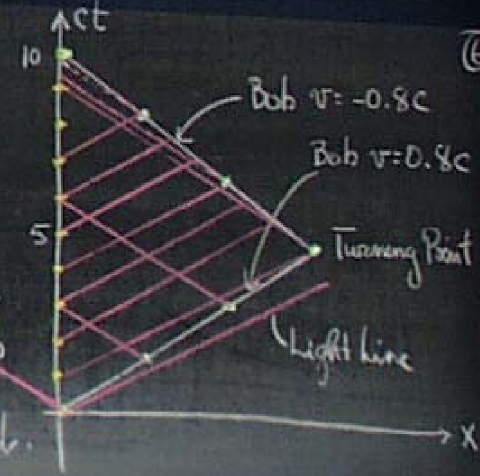
· Travel Time T_B = 3 yrs.

· Returns: Total T_B = 6 yrs.

· Dave's clock reads γT_B = 10 yrs.

· Both A+B send out signals of light once a year.

· Record sent and received signals.



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Lecture 35 Blackboard #4

D: World line = ct-axes; $x_D = 0$.
Mark 10yrs on ct-axes.
B: Inclined ct': $v = 0.8c$.
Stays at $x' = 0$.
Mark off 3-yrs and turn back.
Light signals // to light line.
B: Sends 6 signals; last on arrival
D: Sends 10 signals: last on arrival.

Outbound: Clocks recede
$$\nu = \nu_0\sqrt{\frac{1-\beta}{1+\beta}} = \frac{\nu_0}{3}$$
B receives first signal after 3yrs.
D receives signals once every 3 of his yrs.
Inbound: Clocks approach
$$\nu = \nu_0\sqrt{\frac{1+\beta}{1-\beta}} = 3\nu_0$$

B: receives 9 signals on return
Total = 10!
D: Receives 3 signals in last yr.
Total = 6!
B sends 6; Dave receives 6
D sends 10, Bob receives 10

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D: Sees B recede for 9 yrs and approach for 1 yr.
B: Sees himself recede 3yrs and approach 3yrs.
D: Slow rate 9 yrs.
fast rate 1 yr.
B: Slow rate 3 yrs.
Fast rate 3 yrs.

Doppler Effect
⇒ asymmetry
Bob is younger.

Appearance of Moving Objects

Board: length L_0, width W_0 } Rest Frame
Moving velocity v.
Picture: Instantaneous collection of light from all points
Points: A_0, B_0, C_0

B_0, C_0 light at film simult.
Light from A_0 must leave earlier
$\Delta t = W_0/c$
Board moves: $\Delta x = v\,\Delta t = \frac{v W_0}{c}$
At Δt: $B_0 \to B_1$ $C_0 \to C_1$
Board along-x is Lorentz cont.
$\therefore B_1C_1 = L_0/\gamma$

$B_0B_1 = \Delta x = vW_0/c$
Consider board at rest but rotated by θ:
$B_0B_1 = W_0 \sin\theta$
$B_1C_1 = B_1C_1 = L_0\cos\theta$
Appears as if board is rotated!
$\sin\theta = v/c$
$\cos\theta = \sqrt{1 - v^2/c^2} = \frac{1}{\gamma}$

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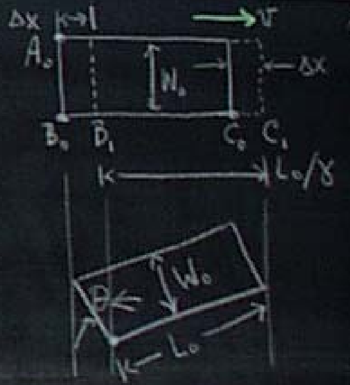

Lecture 35 Blackboard #5