Oscillations

Periodic phenomena are those that are repeated at regular intervals of some independent variable — time, space or combination.

- orbital motion of the planets
- pendulum clock
- mass-spring system
- atomic vibrations

Oscillatory motion is one that is periodic in time. Motion in space of disturbances caused by oscillation represents wave-motion.

e.g. Electromagnetic Waves (8.02) – Maxwell's Equations.

The simplest and most important oscillatory motion to understand is the vibration of a particle about a stable equilibrium position – under the influence of a Hooke's Law restoring force.

"Force which is proportional to the displacement of a particle from its equilibrium and is restoring".

A back-and-forth motion about the equilibrium takes place. This simple type of unique frequency vibration is called Simple Harmonic Motion (SHM). In the most basic case an object oscillates between two spatial positions for an indefinite period of time, with no loss in mechanical energy. In real systems, retarding forces (friction) reduce the mechanical energy and the oscillations are said to be damped. If an external force is applied such that the energy loss is balanced by an energy input, the motion is a forced oscillation.

All oscillations arising from elasticity of matter are SHM or a composite of SHM.

- pendulum
- tuning fork
- stringed instrument
- buildings swaying, etc.

Dissipative forces, such as friction, cause recurring motions to be quasi-periodic. Oscillations decay with time.

Fourier: "Any finite periodic motion can be represented as a summation of a number of suitably chosen SHM's" <u>Fourier Series</u>

Another important principle is that in any <u>physical situation</u>, sufficiently <u>small displacements</u> from a condition of <u>stable equilibrium</u> result in approximately <u>simple harmonic motion</u> about the <u>potential minimum</u>.

A particle moving along the x-axis exhibits SHM when its displacement from equilibrium x, varies in time according to

$$x = A\cos(\omega t + \delta)$$

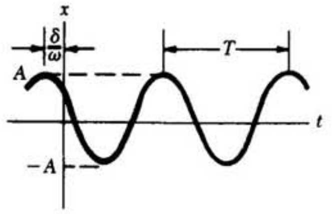


Figure Displacement versus time for a particle undergoing simple harmonic motion. The amplitude of the motion is A and the period is T.

A, ω and δ are constants.

A: amplitude of the motion, is just the maximum displacement of the particle in either direction.

ω: angular frequency of the motion.

δ: phase constant ('phase angle')

A and δ are determined by the initial displacement and velocity of the particle. They give us the displacement at $t=0$.

$(\omega t+\delta) \rightarrow$ phase of the motion.

$x(t)$ is a periodic function which repeats itself when ωt increases by 2π.

The period, T, is the time for the particle to go through one full cycle of its motion.

i.e. $\omega t + \delta + 2\pi = \omega(t+T) + \delta$

$$T = \frac{2\pi}{\omega} \qquad (s)$$

$$f = \nu = \frac{1}{T} = \frac{\omega}{2\pi} \qquad (Hz)$$

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad s^{-1}$$

Velocity of the particle undergoing SHM

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \delta)$$

Acceleration of the particle

$$a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \delta)$$

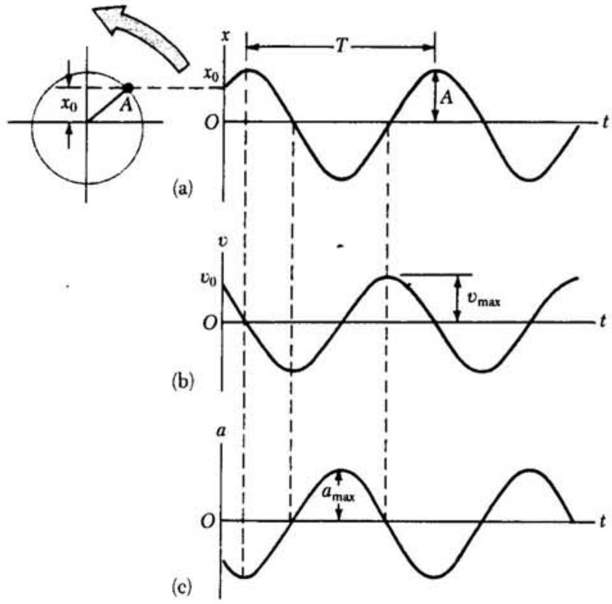
$$a = -\omega^2 x(t)$$

Extreme values of v are $\pm \omega A$

Extreme values of a are $\pm \omega^2 A$

$$\therefore \; v_{max} = \omega A$$

$$a_{max} = \omega^2 A$$



Graphical representation of simple harmonic motion: (a) the displacement versus time, (b) the velocity versus time, and (c) the acceleration versus time. Note that the velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement.

$x = A\cos(\omega t + \delta)$

Amplitude A and phase constant δ must be chosen to meet the initial conditions of the motion.

Suppose at $t = 0$, $x = x_0$ and $v = v_0$.

$$x = A\cos(\omega t + \delta) \longrightarrow x_0 = A\cos\delta$$

$$v = -\omega A \sin(\omega t + \delta) \longrightarrow v_0 = -\omega A \sin\delta$$

$$\therefore \tan\delta = \frac{-v_0}{\omega x_0}$$

$$x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = A^2\cos^2\delta + A^2\sin^2\delta = A^2$$

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

Properties of SHM

i) The displacement, velocity and acceleration all vary sinusoidally with time but are not in phase.

ii) The acceleration of the particle is proportional to the displacement, but in the opposite direction.

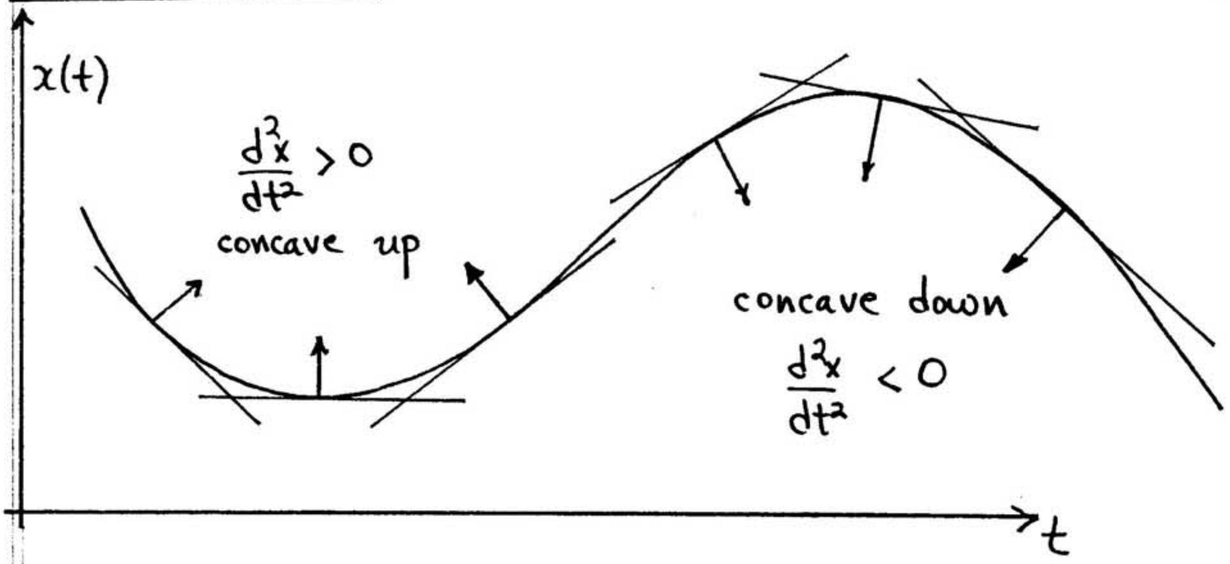
iii) The frequency and the period of motion are independent of the amplitude.

If $x(t)$ represents the motion of a particle of mass m, then

$$F = ma(t)$$
$$= -m\omega^2 x$$

↑ Linear restoring force.

2ND Derivatives



$\frac{d^2x}{dt^2} > 0$

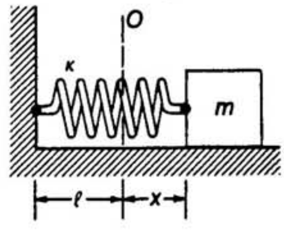
- slope dx/dt is an <u>increasing</u> function of x.
- tangent turns <u>ccw</u> ax x-increases
- curve is <u>concave up</u>

$\frac{d^2x}{dt^2} < 0$

- slope dx/dt is a <u>decreasing</u> function of x.
- tangent turns <u>cw</u> as x-inereases
- curve is <u>concave down.</u>

Mass - Spring System

A mass m is connected to a spring with force constant k. It moves on a horizontal frictionless surface.



If the spring is stretched an amount x from its equilibrium position the mass feels a linear restoring force (Hooke's Law) which is proportional to the displacement and is always directed towards the equilibrium position, opposite the displacement.

$$F(x) = -kx$$

Under this force the eq. of motion of m becomes

$$m\frac{d^2x}{dt^2} = -kx$$

or $$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Let $\omega^2 = k/m$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{[d.e. describing motion]}$$

Assume a solution of the form:

$$x = A\cos(\omega t + \delta)$$

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \delta)$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \delta) = -\omega^2 x$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$ [Eq. is satisfied by sol.]

$$\omega = \sqrt{k/m}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

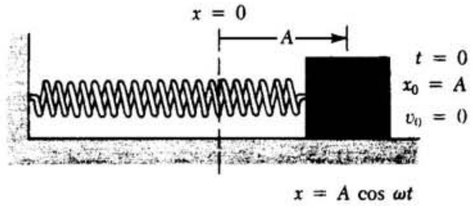
$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{k/m}$$

Equivalent Solutions:

$$x(t) = A\sin(\omega t + \varphi)$$

$$x(t) = A\cos\omega t + B\sin\omega t$$

Special Case I



A mass-spring system that starts from rest at $x_0 = A$. In this case, $\delta = 0$, and so $x = A\cos\omega t$.

The mass is extended from equilibrium a distance A and released from rest from this stretched position.

$$\left.\begin{array}{l} x = A \\ v = 0 \end{array}\right\} t = 0$$

$$\therefore \boxed{x = A\cos\omega t}$$

$$v = \frac{dx}{dt} = -\omega A \sin\omega t$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A\cos\omega t$$

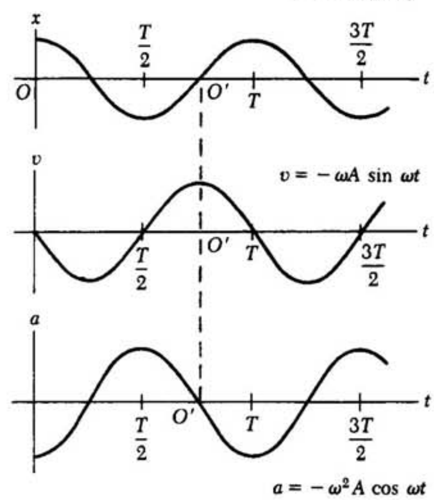
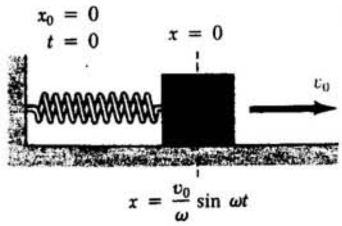


Figure Displacement, velocity, and acceleration versus time for a particle undergoing simple harmonic motion under the initial conditions that at $t = 0$, $x_0 = A$ and $v_0 = 0$.

Special Case II



The mass-spring system starts its motion at the equilibrium position, $x_0 = 0$ at $t = 0$. If its initial velocity is v_0 to the right, its x coordinate varies as $x = \frac{v_0}{\omega} \sin \omega t$.

Mass starts from its unstretched position moving to the right with a velocity v_0.

$$\left.\begin{aligned} x &= 0 \\ v &= v_0 \end{aligned}\right\} \quad t = 0.$$

$$0 = A \cos(\omega t + \delta) \qquad (1)$$
$$v_0 = -\omega A \sin(\omega t + \delta) \qquad (2)$$

From (1) $\delta = \pm \pi/2$.
From (2) $\delta = -\pi/2$ and $A = v_0/\omega$

$$\therefore \quad x = \frac{v_0}{\omega} \cos(\omega t - \pi/2)$$

$$\boxed{x = \frac{v_0}{\omega} \sin \omega t}$$

$$v = v_0 \cos \omega t$$

$$a = -\omega v_0 \sin \omega t$$

Energy in SHM

-No friction, we expect total mechanical energy to be conserved.

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \delta).$$

The elastic potential energy stored in the spring for any elongation x is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \delta)$$

Using $\omega^2 = \frac{k}{m}$, the total energy is given by

$$E = K + U = \frac{1}{2}kA^2\left[\sin^2(\omega t+\delta) + \cos^2(\omega t + \delta)\right]$$

$$\boxed{E = \frac{1}{2}kA^2}$$

The energy of a SHM is a constant of the motion and proportional to the square of the amplitude.

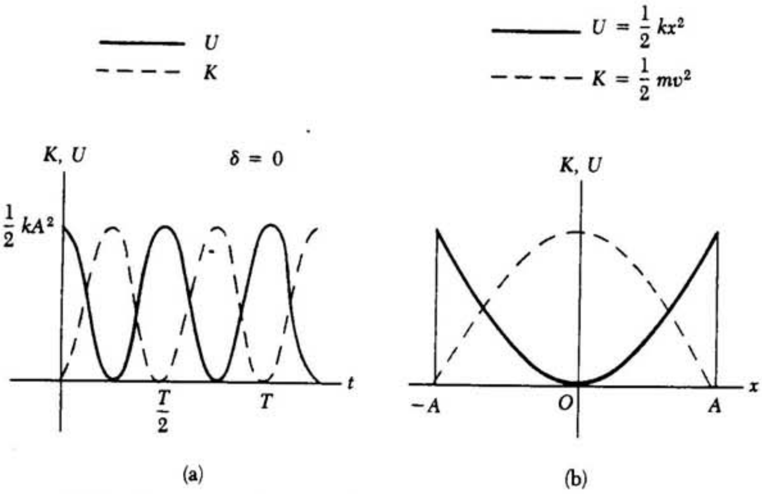


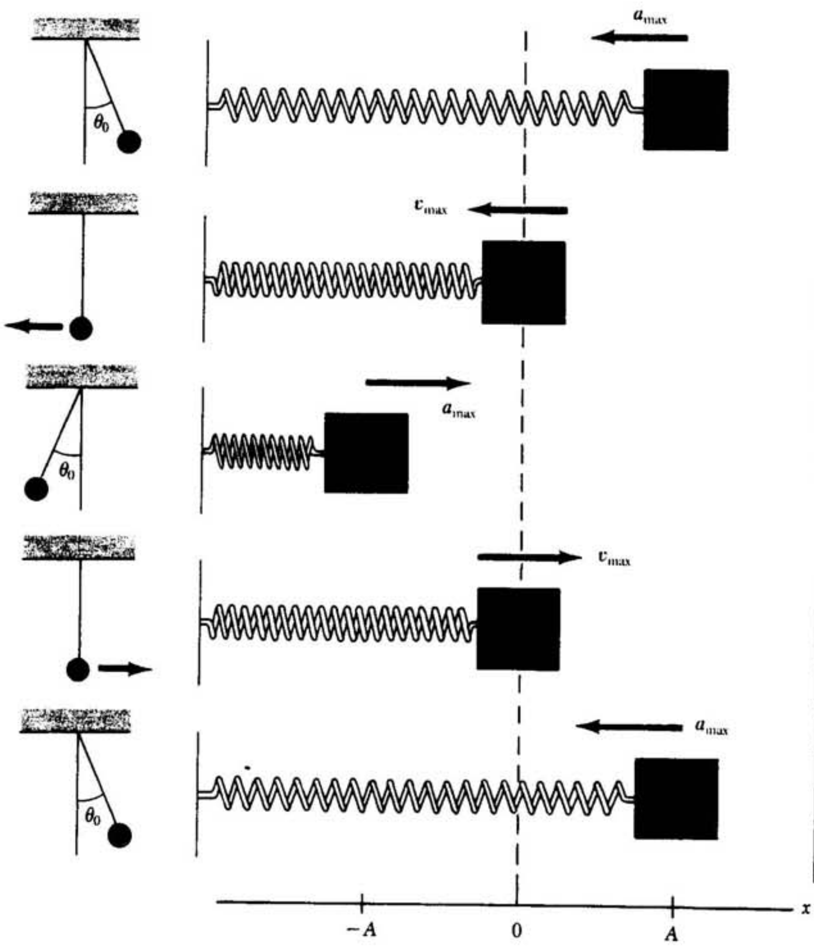
Figure (a) Kinetic energy and potential energy versus time for a simple harmonic oscillator with $\delta = 0$. (b) Kinetic energy and potential energy versus displacement for a simple harmonic oscillator. In either plot, note that $K + U =$ constant.

Definitions and equations of simple harmonic motion

equation of motion	$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$
period	$T = 2\pi\sqrt{\frac{m}{k}}$
natural frequency	$\nu_n = \frac{1}{T} = \frac{1}{2}\pi\sqrt{\frac{k}{m}}$
natural angular frequency	$\omega_n = 2\pi\nu_n = \sqrt{\frac{k}{m}}$
restoring force	$F = -kx = -m\omega_n^2 x$
displacement	$x = A\sin(\omega_n t + \phi) = \alpha\sin\omega_n t + \beta\cos\omega_n t$
	$\alpha = A\cos\phi,\ \beta = A\sin\phi$
	$A^2 = \alpha^2 + \beta^2,\ \phi = \tan^{-1}\frac{\beta}{\alpha}$
velocity	$v = A\omega_n\cos(\omega_n t + \phi)$
	$= \alpha\omega_n\cos\omega_n t - \beta\omega_n\sin\omega_n t$
acceleration	$a = -A\omega_n^2\sin(\omega_n t + \phi) = -\omega_n^2 x$
speed	$\lvert v\rvert = \omega_n\sqrt{A^2 - x^2}$
potential energy	$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2(\omega_n t + \phi)$
	$= \frac{1}{4}kA^2[1 - \cos 2(\omega_n t + \phi)]$
kinetic energy	$K = \frac{1}{2}kA^2 - U = \frac{1}{4}kA^2[1 + \cos 2(\omega_n t + \phi)]$
total energy	$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$
initial conditions: amplitude	$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}$
phase constant	$\phi = \sin^{-1}\frac{x_0}{A} = \cos^{-1}\frac{v_0}{\omega_n A}$
	$= \tan^{-1}\frac{\omega_n x_0}{v_0}$
	$\alpha = \frac{v_0}{\omega_n} \quad \beta = x_0$
time-average energies	$\overline{U} = \overline{K} = \frac{1}{2}E$

$$E = K + U = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2$$

$$v = \pm \sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$$



t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^1$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Simple harmonic motion for a mass-spring system and its analogy to the motion of a simple pendulum. The parameters in the table at the right refer to the mass-spring system, assuming that at $t = 0$, $x = A$ so that $x = A \cos \omega t$ (Case I).

Example

Horizontal Mass-Spring System.
- No friction

$m = 0.50\ kg$
$F = 7.5N$ stretches spring 3.0 cm from relaxed length.

a) what is spring constant k?

$$k = -\frac{F}{x} = -\frac{-7.5\ N}{.03} = 250\ N/m.$$

b) what is angular frequency?

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{250}{.50}} = 22.36\ r/s$$

$$\nu = \frac{\omega}{2\pi} = 3.56\ Hz$$

$$T = 1/\nu = 0.281\ s.$$

Suppose at $t=0$ the spring-mass is compressed 5.0 cm and released.

c) What is eq. of motion?

Assume: $x(t) = A\cos(\omega t + \varphi)$

$t=0$, $x(0) = .05\,m$
$v(0) = 0.$

$\therefore\ x(t) = .05\cos(\omega t)$ (m)

d) What is maximum $v(t)$?

$$v(t) = \frac{dx}{dt} = -\omega(.05)\sin(\omega t)$$

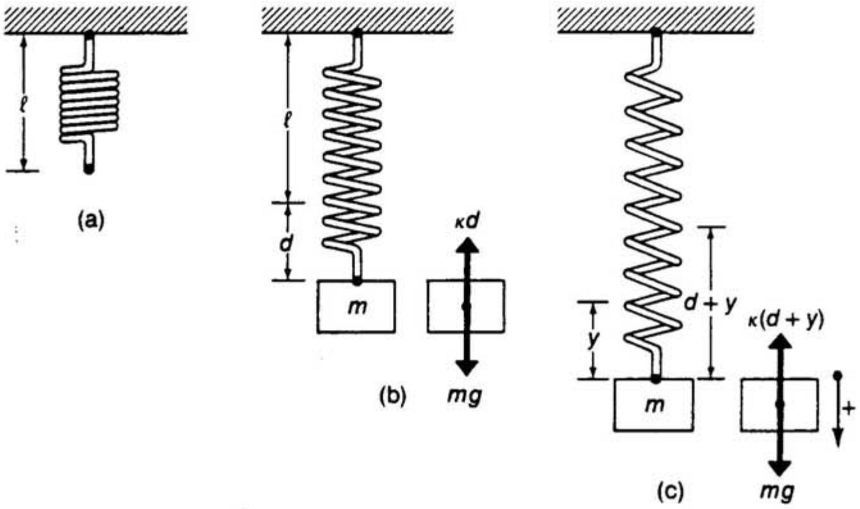
$v_m = .05\omega = .05 \times 22.36 = 1.12\ m/s$
t_m when $\omega t = \pi/2,\ 3\pi/2,$ etc.

e) What is maximum $a(t)$?

$$a(t) = \frac{d^2x}{dt^2} = -(.05)\omega^2\cos(\omega t)$$

$a_m = .05\omega^2 = 25.0\ m/s^2$
t_m when $\omega t = 0,\ 2\pi,$ etc.

Example: Mass – (Spring + Gravity) System



$$m \frac{d^2y}{dt^2} = -k(d+y) + mg$$

At equilibrium, $y = 0$, the net force is zero, so

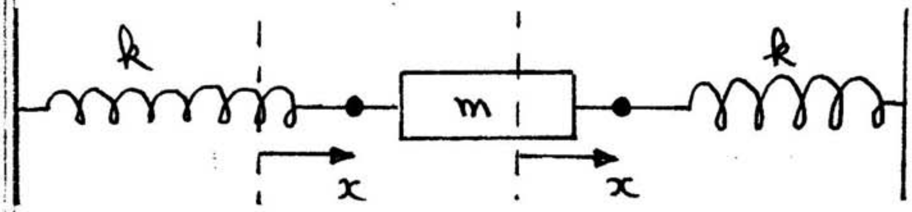
$$0 = mg - kd$$
$$\therefore kd = mg.$$

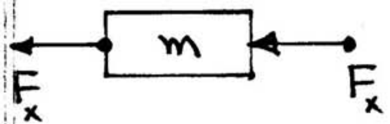
Finally we get $m\dfrac{d^2y}{dt^2} = -ky$.

SHM where the displacement, y, is measured wrt the equilibrium position with the spring stretched an amount $d = mg/k$.
The frequency of oscillation is the same as before.

Example:

Mass between springs.





Mass displaced to the right a distance x.

$$-F_x - F_x = ma_x$$

$$-kx - kx = ma$$

$$-2kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{2kx}{m} = 0$$

$$x = A\cos(\omega t + \varphi) \qquad \text{Eq. of Motion}$$

$$\omega^2 = \left(\frac{2k}{m}\right)$$

$$\omega = \sqrt{\frac{2k}{m}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} \quad \text{Hz} \qquad \text{Frequency}$$

$$T = 2\pi\sqrt{\frac{m}{2k}} \quad \text{s} \qquad \text{Period}$$