

General Planetary Motion, Cont.

Kinetic Energy and Momentum

The total kinetic energy in the inertial frame 'O' can be expressed in terms of $\vec{V} = d\vec{R}/dt$, the velocity of the CM and $\vec{v} = d\vec{r}/dt$, the relative velocity of the two particles.

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_1^2 = \frac{d\vec{r}_1}{dt} \cdot \frac{d\vec{r}_1}{dt}$$

$$v_2^2 = \frac{d\vec{r}_2}{dt} \cdot \frac{d\vec{r}_2}{dt}$$

Using Eq's (5) and (6) for \vec{r}_1 and \vec{r}_2

$$K = \frac{1}{2} M \left(\frac{d\vec{R}}{dt} \right) \cdot \left(\frac{d\vec{R}}{dt} \right) + \frac{1}{2} \mu \left(\frac{d\vec{r}}{dt} \right) \cdot \left(\frac{d\vec{r}}{dt} \right)$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2$$

↑ CM velocity.
↑ relative velocity.

Total KE in frame 'O' is the sum of a hypothetical particle of mass $M = m_1 + m_2$, moving with velocity V of CM plus the KE of a particle of mass μ moving with the relative velocity v .

Total Linear Momentum in frame 0

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = M \vec{V}$$

'Relative' linear momenta of the pair of particles is

$$\begin{aligned} \mu \vec{v} &= \frac{m_1 m_2}{m} (\vec{v}_2 - \vec{v}_1) \\ &= \frac{1}{m} (m_1 \vec{p}_2 - m_2 \vec{p}_1) \end{aligned}$$

If the system of particles is isolated (no external forces)

$$\vec{P} = \text{constant.}$$

Total \vec{P} in cm frame = 0.

Angular Momentum

- central forces
- pointlike particles
- no torques !!

$$\vec{L}_{\text{cm}} = \vec{l}_1 + \vec{l}_2 = m_1 \vec{r}_1' \times \vec{v}_1' + m_2 \vec{r}_2' \times \vec{v}_2' = \text{constant.}$$

$$\vec{L}_{\text{cm}} = \mu \vec{r} \times \vec{v} = \text{constant.}$$

$$\vec{L}_{\text{cm}} = \mu r^2 \omega = \text{constant.}$$

The value of the angular momentum of the pair of particles about their CM is equivalent to that of a single particle of mass μ .

Total Energy

Assume the force of interaction \vec{F} between the particles is a central force derivable from a potential $U(r)$, then the total energy of the system is a constant of the motion:

$$E = \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2 + U(r) = \text{constant.}$$

Note: For $m_1 \gg m_2$, $\mu \sim m_2$ and equations reduce to their single-particle equivalents.

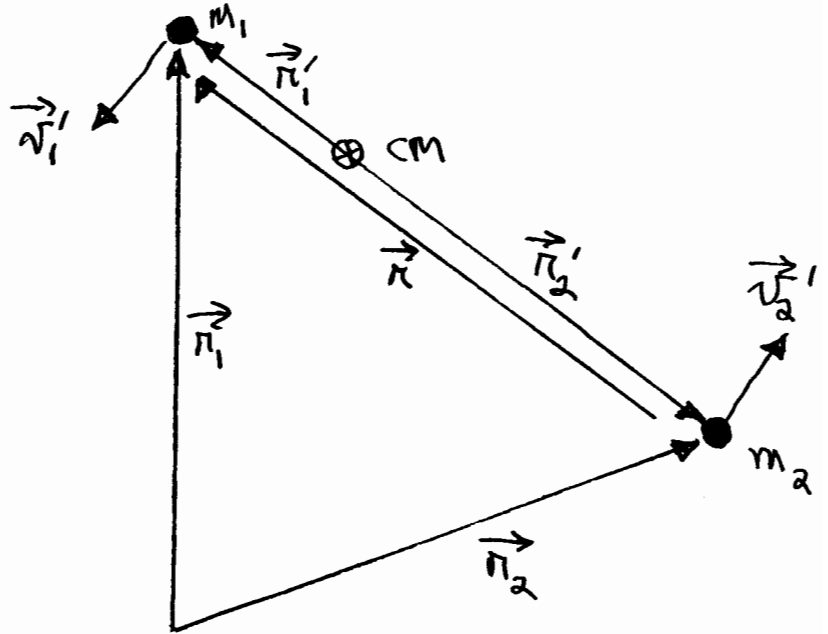
CM - Angular Momentum

$$\vec{L}_{cm} = \vec{l}_1 + \vec{l}_2 = m_1 \vec{r}'_1 \times \vec{v}'_1 + m_2 \vec{r}'_2 \times \vec{v}'_2$$

$$\vec{r} = \vec{r}'_1 - \vec{r}'_2$$

$$\vec{r}'_1 = \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}'_2 = \frac{-m_1}{m_1 + m_2} \vec{r}$$



$$\vec{L}_{cm} = \frac{m_1 m_2}{m_1 + m_2} \vec{r} \times \vec{v}'_1 + \frac{m_1 m_2}{m_1 + m_2} \vec{r} \times \vec{v}'_2$$

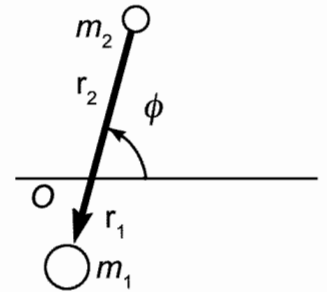
$$= \mu \vec{r} \times (\vec{v}'_1 - \vec{v}'_2)$$

$$= \mu \vec{r} \times \vec{v}$$

Planetary Motion - 2 Particle Systems

- Sun + Planet
- Planet + Moon
- Earth + Satellite

- choose CM system of coordinates.
 $\therefore \vec{R} \equiv 0$ and $\vec{V} \equiv 0$.



(a) Two particles orbit around their common center of mass at O.

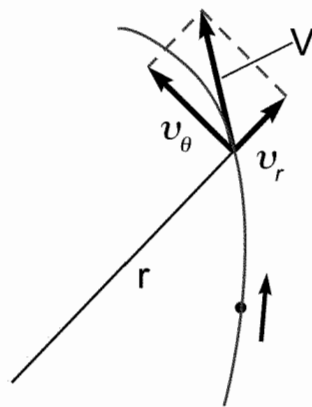
$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{M}$$

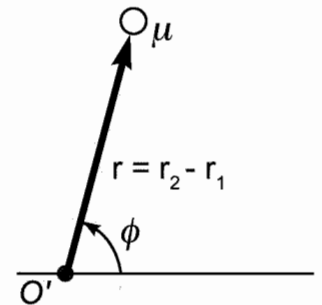
$$\vec{r}_1 = \vec{r}_2 - \vec{r}$$

$$\vec{r}_1 = \frac{-\mu}{m_1} \vec{r}$$

$$\vec{r}_2 = \frac{\mu}{m_2} \vec{r}$$



The polar components of velocity.



(b) The situation in (a) is described in terms of the equivalent single particle with mass $\mu = m_1 m_2 / (m_1 + m_2)$. The vector r is the relative coordinate of the particles, so, in magnitude, $r = r_1 + r_2$.

The total energy of the system is

$$E = K + U$$

$$= \frac{1}{2} \mu v^2 + U(r)$$

[$V \equiv 0$ CM system]

The velocity $\vec{v} = v_r \hat{r} + v_\phi \hat{\phi}$, has radial and azimuthal components.

⑨ $v_\phi = v_t = r\omega = r \left(\frac{d\phi}{dt} \right)$

⑩ $L = \mu r^2 \omega = \mu r^2 \left(\frac{d\phi}{dt} \right)$

(8)

$$E = \frac{1}{2} \mu (\dot{r}^2 + \dot{\phi}^2 r^2) + u(r)$$

(11)

$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{GM_1 m_2}{r}$$

\uparrow KE due to radial motions w.r.t cm.
 \uparrow KE due to ang. rot. about cm.
 \uparrow Gravitational PE

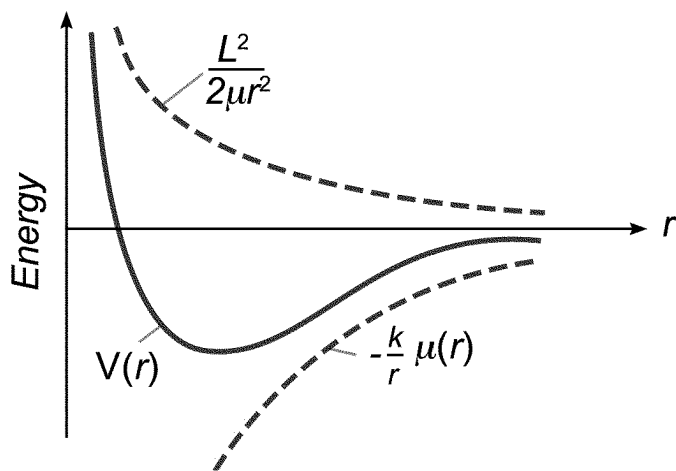
let $V(r) = \frac{L^2}{2\mu r^2} - \frac{GM_1 m_2}{r}$ [Effective PE Function]

let $k = GM_1 m_2 = G\mu M = G\mu(m_1 + m_2)$

$$V(r) = \frac{L^2}{2\mu r^2} - \frac{k}{r}$$

(12)

$$E = \frac{1}{2} \mu \dot{r}^2 + V(r)$$



The effective potential energy function $V(r)$ for the case of planetary motion. The ordinary potential energy function is $U(r) = -G\mu M/r = -k/r$.

Solutions to Eq. of Motion

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Want:

$$r = f(t)$$

$$\phi = g(t)$$

$$r = h(\phi) \Rightarrow \text{trajectory of orbit.}$$

$$\textcircled{13} \quad v_r = \frac{dr}{dt} = \sqrt{\frac{2}{\mu} (E - V(r))} \quad \text{From Eq. } \textcircled{12}$$

$$\textcircled{14} \quad \frac{d\phi}{dt} = \frac{L}{\mu r^2} \quad \text{From Eq. } \textcircled{10}$$

$$\therefore \frac{d\phi}{dr} = \frac{L}{\mu r^2} \frac{1}{\sqrt{\frac{2}{\mu} (E - V(r))}} \quad \text{Eq. } \textcircled{14} / \textcircled{13}$$

Evaluate integrals, solving:

$$r = \frac{L^2 / \mu k}{1 - \sqrt{1 + \left(\frac{2EL^2}{\mu k^2}\right)} \sin(\phi - \phi_0)}$$

$$r = \frac{r_0}{1 - \epsilon \cos \phi}$$

$$\phi_0 = -\pi/2 \text{ (convention)}$$

$$\frac{r_0}{r} = 1 - \epsilon \cos \phi = \frac{\epsilon \delta}{r} \quad \text{[Equation of a Conic Section]}$$

$$r_0 = \frac{L^2}{\mu k}$$

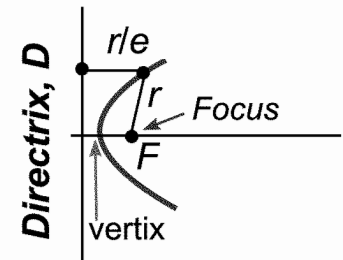
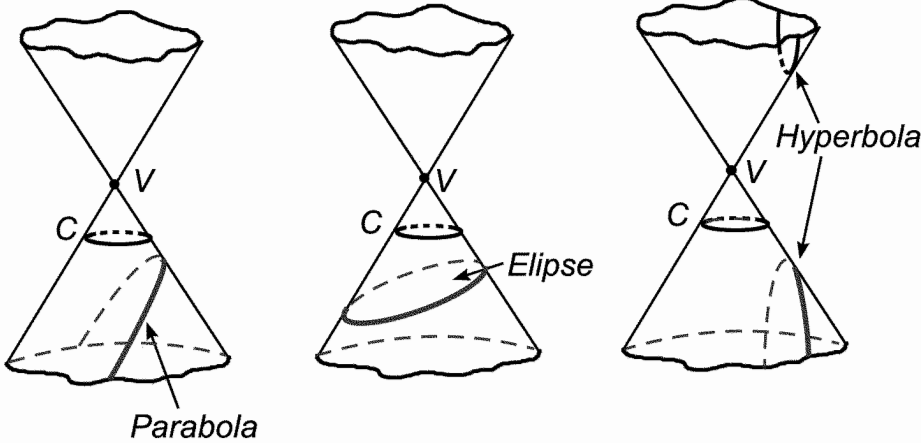
Radius of circular orbit corres.
to h, μ , and k .

$$\epsilon \equiv e = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

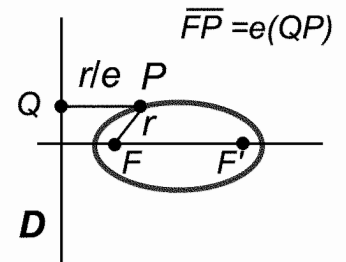
Orbit Characteristics

$E = 0$ Circular Orbits
 $E > 1$ Hyperbolic ✓
 $E = 1$ Parabolic ✓
 $0 < E < 1$ Elliptical ✓

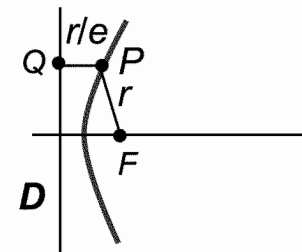
$E = E_{\min}$
 $E > 0$
 $E = 0$
 $E < 0$



(a) Parabola $e = 1$



(b) Ellipse $0 < e < 1$



(c) Hyperbola $e > 1$

The conic sections.

- (a) The parabola has $e = 1$.
 - (b) The ellipse has $0 < e < 1$.
 - (c) The hyperbola has $e > 1$.
- There is another branch of the hyperbola (not shown), which lies to the left of the branch illustrated and has the opposite curvature. The circle (not shown) has $e = 0$ and corresponds to an ellipse for which F and F' coincide.

$$E = \frac{1}{2} \mu v_r^2 + V(r)$$

$$V(r) = \frac{L^2}{2\mu r^2} - \frac{Gm_1 m_2}{r}$$

Sum of both terms contributing to $V(r)$ produces a minimum.

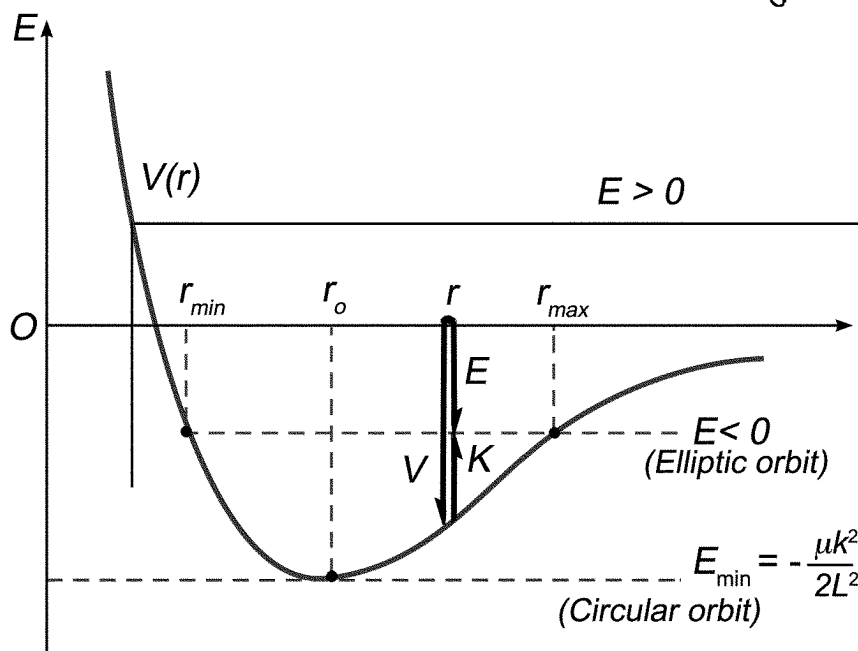
$$V(r) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty.$$

If $L \neq 0$, the repulsive centrifugal potential $L^2/2\mu r^2$ dominates at small r . The attractive gravitational potential $Gm_1 m_2/r$ dominates at large r .

KE for the radial motion is

$$K = E - V(r).$$

Motion is restricted to regions where $K \geq 0$.
Nature of motion determined by E .



The effective potential energy function for the case of planetary motion, showing radius values for two energies, corresponding to elliptic and circular orbits.

1. $E > 0$: [Hyperbola]

r is unbounded for large values but must exceed a certain minimum if $L \neq 0$. Particles are kept apart by centrifugal barrier.

2. $E = 0$: [Parabola]

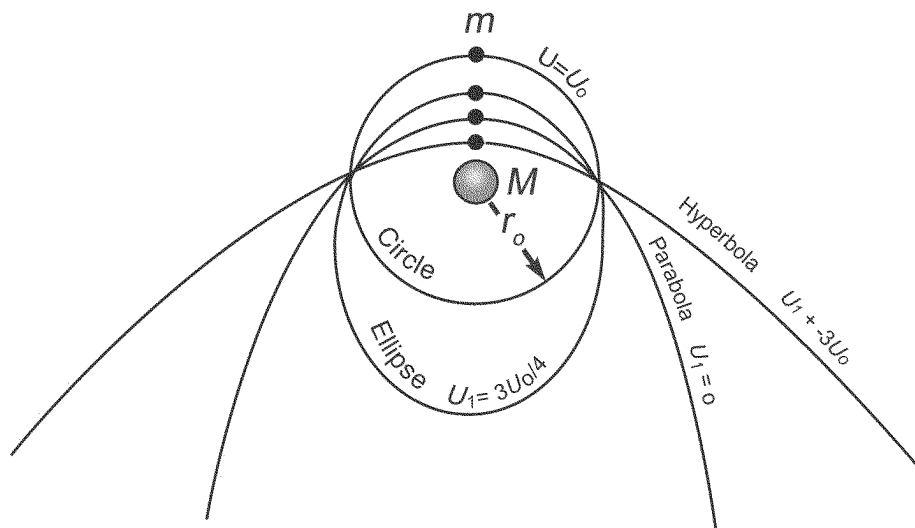
Exactly on the boundary between bounded and unbounded motion.

3. $E < 0$: [Ellipse]

Motion is bounded for large and small r . Two particles form a bound system.

4. $E = E_{\min}$: [Circle]

r has an exact fixed value. Particles orbit each other a constant distance apart.



Different orbital paths corresponding to the same value of angular momentum.

Circular Orbits

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E has a minimum value when $\dot{r}_n = 0$.
and $\frac{dV}{dr} = 0$.

$$V(r) = \frac{L^2}{2\mu r^2} - \frac{k}{r}$$

$$\frac{dV(r)}{dr} = \frac{-L^2}{\mu r^3} + \frac{k}{r^2} = 0$$

$$r_0 = \frac{L^2}{\mu k} \quad (\text{Radius})$$

$$E_{\min} = V(r_0) = \frac{-\mu k^2}{2L^2}$$

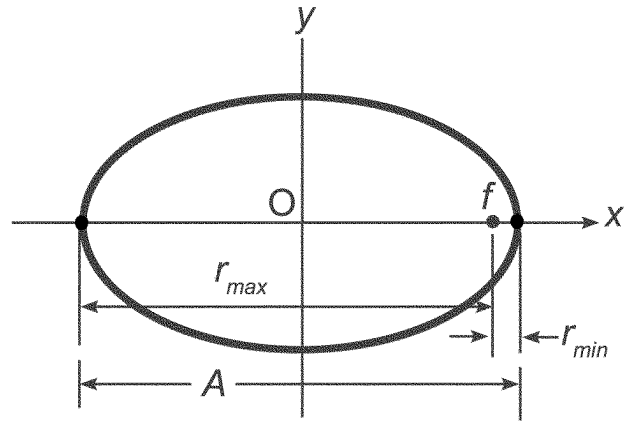
$$U(r_0) = \frac{-\mu k^2}{L^2} \quad (\text{Gravitational PE})$$

Total energy is negative and is exactly one-half of gravitational potential energy.

Elliptical Orbits

$E < 0, 0 \leq E < 1$

$$r = \frac{r_0}{1 - \epsilon \cos \phi}$$

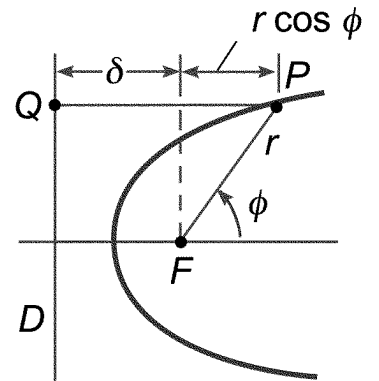


$$\left. \begin{aligned} r_{max} &= \frac{r_0}{1 - \epsilon} & (\phi = 0) \\ r_{min} &= \frac{r_0}{1 + \epsilon} & (\phi = \pi) \end{aligned} \right\} \text{turning points in motion}$$

$A = r_{min} + r_{max}$ (Length of major axis)

$$= r_0 \left[\frac{1}{1 + \epsilon} + \frac{1}{1 - \epsilon} \right]$$

$$A = \frac{2r_0}{1 - \epsilon^2}$$

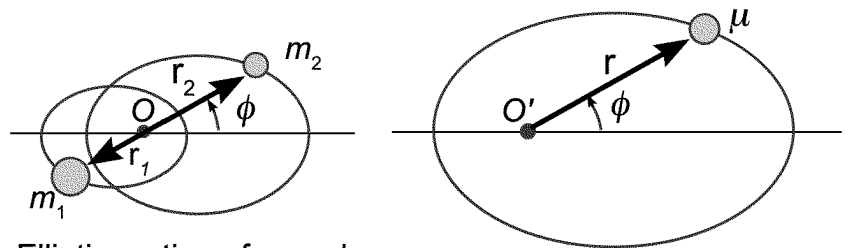


Geometry for obtaining the general equation for a conic.

	Planets	Earth
$r_{min} = r_p$	perihelion	perigee
$r_{max} = r_a$	aphelion	apogee

$$\frac{r_{max}}{r_{min}} = \frac{1 + \epsilon}{1 - \epsilon}$$

$$\epsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$



(a) Elliptic motion of m and m2 around their C.M. at O. The point O is the right-hand focus of the smaller ellipse and the left-hand focus of the larger ellipse.

(b) The corresponding elliptic motion of mu around O'.

$$A = \frac{2r_0}{1-\epsilon^2} = \frac{2L^2/\mu k}{1 - \left(1 + \frac{2EL^2}{\mu k^2}\right)}$$

$$A = \frac{k}{(-E)}$$

Length of major axis is independent of L .
Orbits with same major axis have the same total energy.

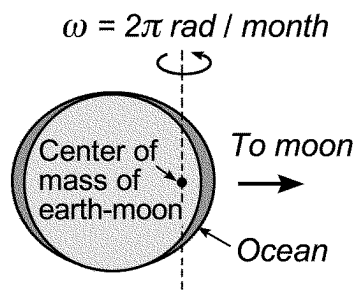
$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

TABLE Planetary Data*

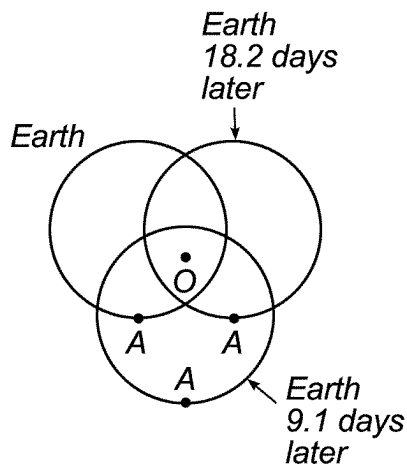
PLANET	DIAMETER (EARTH = 1)	MASS (EARTH = 1)	SEMIMAJOR AXIS (A.U.)	SIDEREAL PERIOD (YEARS)	ECCENTRICITY OF ORBIT
Mercury	0.3824	0.0553	0.3871	0.24084	0.2056
Venus	0.9489	0.8150	0.7233	0.61515	0.0068
Earth	1	1	1	1.00004	0.0167
Mars	0.5326	0.1074	1.5237	1.8808	0.0934
Jupiter	11.194	317.89	5.2028	11.862	0.0483
Saturn	9.41	95.17	9.5388	29.456	0.0560
Uranus	4.4	14.56	19.1914	84.07	0.0461
Neptune	3.8	17.24	30.0611	164.81	0.0100
Pluto†	0.4	0.02	39.5294	248.53	0.2484

Kepler's Laws

- Approximate.
- Objects have comparable masses (binary-stars).
- Circle around common c.m.
- Total E conserved
- Total L conserved.
- Treat as a reduced mass system. See notes.
- Perturbation due to other planets.



(a) The ocean bulges on opposite sides of the earth because of the interaction with the moon.



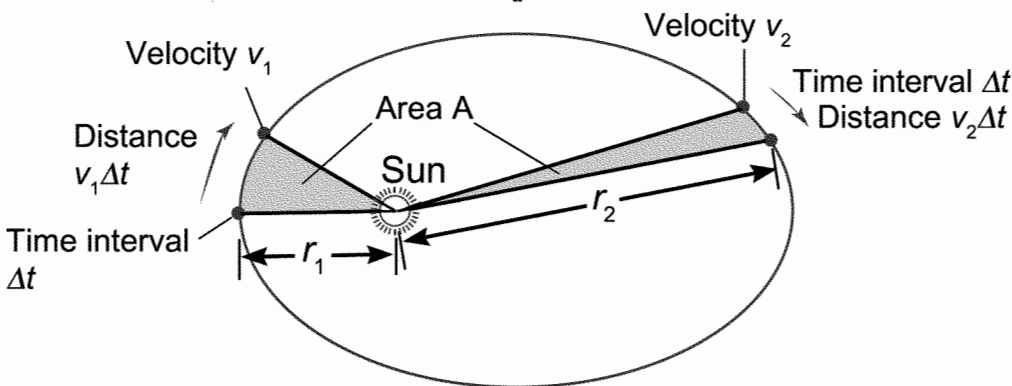
(b) The earth orbits about the center of mass O of the earth-moon system. We show the earth at three times during this motion. Note that the earth's rotation is not included in this analysis: point A on the earth has not rotated.

Kepler's Laws - Revisited

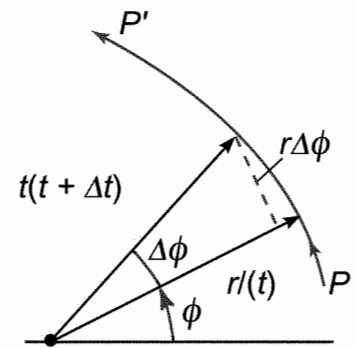
I. A planet moves in an elliptical path with the focus at the position of the cm of the planet-sun system.

(Earth-Sun system has its cm only 450 km from the center of the sun).

II. The position vector for a planet (measured from the cm of the planet-sun system) sweeps out equal areas in equal time intervals; that is, $dA/dt = \text{constant}$.



Geometric representation of Kepler's second law.



Illustrating Kepler's second law

Angular momentum is conserved for motion due to a central force — eg. gravitational force.

Consider path PP' . During time Δt , position line $r(t)$ sweeps out the area

$$\Delta A = \frac{1}{2} r^2 \Delta \phi$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \lim_{\Delta t \rightarrow 0} \frac{\Delta \phi}{\Delta t} \quad ; \quad \frac{dA}{dt} = \frac{1}{2} r^2 \omega = \frac{L}{2\mu} = \text{constant}$$

Second Law is valid for any central force.

III. The ratio of the square of the period (T^2) to the cube of the semi-major axis (a^3) is approximately the same for all planets.

$$dt = \frac{2\mu}{L} dA$$

$$\int dt = \frac{2\mu}{L} \int dA$$

For a complete revolution we have

$$T = \frac{2\mu}{L} A = \frac{2\mu}{L} \pi a b$$

For an elliptical orbit

$$b = a \sqrt{1 - \epsilon^2}$$

$$\text{and } a(1 - \epsilon^2) = \frac{L^2}{\mu k}$$

$$\begin{aligned} T^2 &= \frac{4\mu^2}{L^2} \pi^2 a^2 b^2 = \frac{4\mu^2}{L^2} \pi^2 a^2 a^2 (1 - \epsilon^2) \\ &= \frac{4\mu^2}{L^2} \pi^2 a^3 \frac{L^2}{\mu k} = \frac{4\mu \pi^2 a^3}{GMm} = \frac{4\pi^2 a^3}{GMm} \frac{mM}{(m+M)} \end{aligned}$$

$$T^2 = \frac{4\pi^2 a^3}{G(M+m)}$$

Combined mass (Sun + Planet)

Function of planet.

Example - Satellite Orbit

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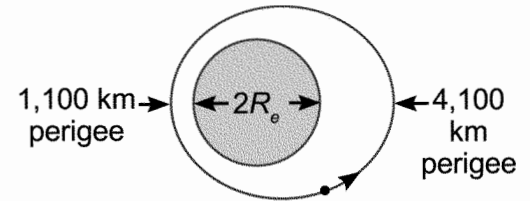
- Elliptic orbit around the earth.

$$R_E = 6400 \text{ km}$$

$$m = 2000 \text{ kg}$$

$$\text{perigee} = 1100 \text{ km}$$

$$\text{apogee} = 4100 \text{ km.}$$



$$m \ll M_E$$

$$\therefore \mu \sim m$$

Major axis of ellipse

$$A = [r_p + r_a + 2R_E]$$

$$= [1100 + 4100 + 2 \times 6400] \text{ km}$$

$$= 1.8 \times 10^7 \text{ m.}$$

$$A = \frac{k}{(-E)} = \frac{G\mu M_E}{(-E)} = \frac{GmM_E}{(-E)} = \frac{(mg)R_E^3}{(-E)}$$

$$E = -\frac{mgR_E^2}{A} = -\frac{2 \times 10^3 \times 9.8 \times (6.4 \times 10^6)^2}{1.8 \times 10^7} = -4.5 \times 10^{10} \text{ J}$$

(Energy of satellite in orbit)

Initial energy of satellite just prior to launch.

$$E_i = -\frac{GmM_E}{R_E} = -mgR_E = -12.5 \times 10^{10} \text{ J}$$

Energy required to place satellite in orbit, no friction
is $E - E_i = 8 \times 10^{10} \text{ J}$

Angular momentum

$$v_{\min} = \frac{r_0}{1+\epsilon}$$

$$v_{\max} = \frac{r_0}{1-\epsilon}$$

$$\epsilon = \frac{v_{\max} - v_{\min}}{v_{\max} + v_{\min}} = \frac{v_{\max} - v_{\min}}{A} = \frac{3 \times 10^3}{1.8 \times 10^4} = \frac{1}{6}$$

$$\epsilon^2 = 1 + \frac{2EL^2}{mk^2}$$

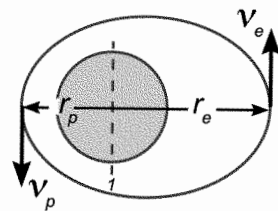
$$L = 1.2 \times 10^{14} \text{ kg} \cdot \text{m}^2/\text{s}$$

Speed v of the satellite at any r is given by the total energy equation.

$$E = \frac{1}{2} m v^2 - \frac{k}{r}$$

$$r_p = (1100 + 6400) \text{ km} = 7.5 \times 10^6 \text{ m}$$

$$v_p = 7900 \text{ m/s}$$



Conservation of L : $\cancel{m} v_p r_p = \cancel{m} v_a r_a$

$$v_a = \frac{v_p r_p}{r_a} = 5600 \text{ m/s}$$

Celebrating Newton

The legacy and legend of Isaac Newton live on 300 years after the publication of his masterpiece, the Principia

By STEFI WEISBURD

*Then ye who now on heavenly nectar fare,
Come celebrate with me in song the name
Of Newton, to the Muses dear; for he
Unlocked the hidden treasures of Truth:
So richly through his mind had Phoebus cast
The radiance of his own divinity.
Nearer the gods no mortal may approach*

— Edmund Halley's preface to
Newton's *Principia*

Science is a search for the essence of everything, for the fundamental laws that govern the universe. If there is one person whose work embodies the spirit and remarkable products of this pursuit, it is Isaac Newton. His *Philosophiæ Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), commonly known as the *Principia*, may well be the most important document in the history of science.

In many ways, the *Principia* is a blueprint for modern physical science. With it, Newton created a mathematical framework for physics and conceived basic laws of motion and of universal gravitation that unify a diverse array of phenomena both in the heavens and on earth. The revolutionary power of the *Principia* and other Newtonian works is felt to this day: His celestial mechanics guide the paths of satellites and spacecraft, his reflecting telescope is enabling astronomers to study recently discovered supernovas, his numerical methods are used in computers and his mathematics and approach to solving many physical problems remain as vital today as in his time.

And the *Principia* has influenced not only science but Western culture in general. Newton's ideas fostered the development of social sciences, they played cen-

ter stage during the Age of Reason and they inspired the French and American authors of new governments. "The Newtonian revolution . . . remains one of the most profound revolutions in the history of human thought," writes I. Bernard Cohen in *Revolution in Science* (1985, The Belknap Press of the Harvard University Press).

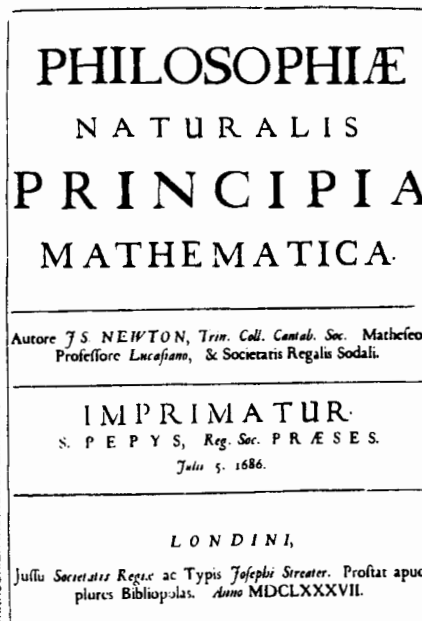
This year marks the 300th anniversary of the *Principia's* publication. While Einstein's relativity theories and quantum mechanics have shown the limits of Newton's work (applicable only to the macroscopic, slowly moving physical world), scientists today are as much in awe of

Newton's accomplishments as Edmund Halley and others were while Newton lived. To celebrate his genius, scientists and historians are gathering at a number of commemorative symposia planned for this year in Washington, D.C., Tel Aviv, Oxford, Holland and elsewhere. In addition, the Smithsonian's National Museum of American History in Washington, D.C., is hosting a special exhibit on Newton and the *Principia*. And in Britain, four commemorative stamps have been issued in Newton's honor.

These activities, says physicist Frank A. Wilczek at the Institute for Theoretical Physics in Santa Barbara, Calif., are "not only a celebration of Newton, but a celebration of [his] whole scientific world view and method that has led to such enormous insights" long after his death.

Historians are fond of saying that Newton was the culmination of the 17th-century scientific revolution. Newton's predecessors, such as Galileo, Kepler and Hooke, were moving away from the Aristotelian world view, in which the behavior of objects is dictated by the "qualities" they possess; Aristotelians believed, for example, that a stone falls because its "nature" necessitates that it move toward the center of the universe, or that planets travel in circular orbits because the circle is a heavenly form.

In contrast, the emerging view during the scientific revolution was more clearly rooted in the underlying forces or laws that can be expressed mathematically. Newton acknowledged that he stood "on the shoulders of Giants" who developed this approach. But, writes Paul Theerman, curator of the Smithsonian exhibit, "Newton was no mere disciple; his genius



The frontispiece of Newton's *Principia*.