Law of Universal Gravitation

1687: Newton "Math. Principles of Natural Philosophy"

"Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them."

History: Ancient astronomers attempted to explain orderliness of movement of sun + planets against star bknd.

- day-night cycle.
- annual progression of seasons.
- periodic motion of planets.

Observations:

- accurate calendars
- star maps
- wanderings of planets

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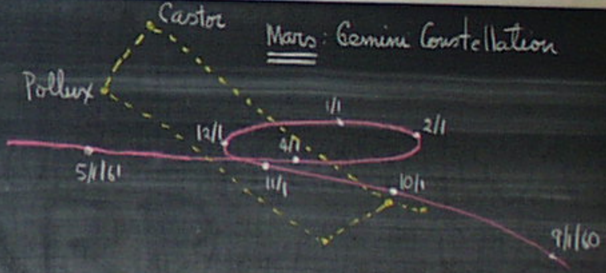
1. Pythagoras (582 – 497 BC)
Plato (427 – 347 BC)

- spherical - stationary earth
- heavenly bodies move in circles
- stars fixed on a distance shell
- Sun, moon, planets move on spheres to match their periods.
- Earth-Centered.

Problems with Model

- Sun and planets did not move uniformly
- Planets varied in brightness. Why?
- Some planets relative to stars showed retrograde motion!!

Mars: Gemini Constellation



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Lecture 27 Blackboard #1

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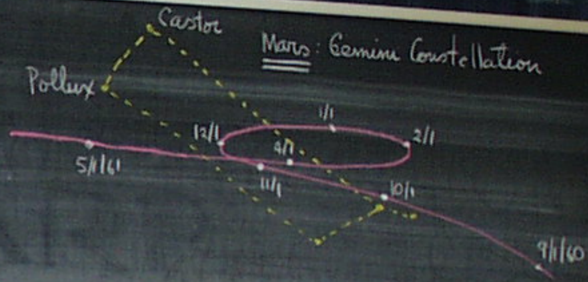

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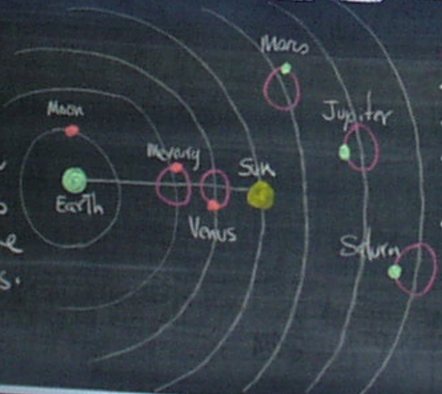
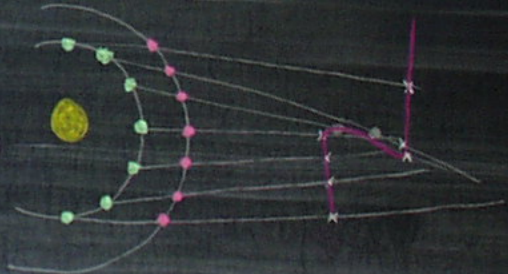
2. Ptolemy (150 AD)

- Generalized Greek work
- Earth - Centered System
- Epicycles
- Inner planets locked to Sun
- Explained brightness changes
- Sun, moon, planets revolve on spheres to match their periods.



3. Copernicus

- Radical new theory
- Sun is at center
- Earth + Planets circles about sun.
- Epicycles for uniform circular motion
- Retrograde totally explained.



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Lecture 27 Blackboard #2

4. Kepler (1571–1630)

- Elliptic Orbits
- Laws of Motion
- Quantified Data of Tycho Brahe (1546-1601)
  - All planets move on ellipses/sun at focal pt.
  - Radius vector sun→planet sweeps out equal areas in equal times: Cons. of $\vec{L}$
  - Square of orbital period is prop. to cube of semi-major axis of ellipse.

5. Newton's Law of Gravity

$$F = \frac{G m_1 m_2}{r^2}$$

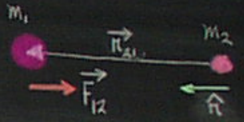
$$G = 6.672\times 10^{-11}\ N\cdot m^2/kg^2$$

Grav. Constant: Cavendish Exp.

- Inverse Square
- Always Attractive/All masses.
- Infinite Range.

$$\vec{F}_{12} = -\frac{G m_1 m_2}{r_{21}^2}\hat{r}$$

= Force on m_1 due to m_2.



$\vec{F}_{12} = -\vec{F}_{21}$ Newton's 3rd Law

Action-Reaction Pair.

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Gravitational Potential Energy

$$\Delta U = U_f - U_i = -\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r}$$

$$\Delta U = G M_e m \int_{r_i}^{r_f} \frac{dr}{r^2} = G M_e m\left[-\frac{1}{r}\right]_{r_i}^{r_f}$$

$$U_f - U_i = -G M_e m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

M_e: Mass of Earth

m: mass of object

Reference U_i is arbitrary.

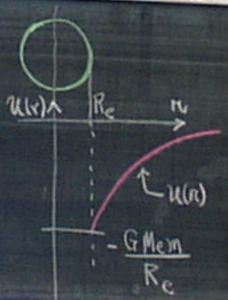
Let $U_i = 0$ at $r_i = \infty$

$F(\infty) = 0$

$$U(r) = -\frac{G M_e m}{r} \qquad r > R_E$$

For any pair of particles,

$$U(r) = -\frac{G m_1 m_2}{r}$$



Principle of Superposition

$$\vec{F}_{Net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots$$

$$U_{Total} = U_{12} + U_{23} + U_{13} + \dots$$

$$= -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} + \dots\right)$$

⑤

Lecture 27 Blackboard #3

Gravitational Potential Energy

$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} \vec{F}\cdot d\vec{r}$

$\Delta U = G M_e m \int_{r_i}^{r_f} \frac{dr}{r^2} = G M_e m \left[-\frac{1}{r}\right]_{r_i}^{r_f}$

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M_e : Mass of Earth

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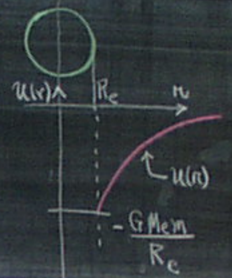

Principle of Superposition

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$\vec{F}_{Net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots$

$U_{Total} = U_{12} + U_{23} + U_{13} + \dots$

$= -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} + \dots\right)$

Potential Energy of a Spherical Shell and Particle

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- Point particle mass = m
- Shell: mass M density = ρ
  radius R
  thickness t
- Ring of Material:
  Width of ring = $R\,d\theta$
  Thickness = t
  Ring Radius = $R\sin\theta$

Mass of Ring:

$dM = \underbrace{\underbrace{(R\,d\theta)}_{\text{Width}} \cdot \underbrace{2\pi R\sin\theta}_{\text{Circumference}} \cdot t}_{\text{Volume of Ring}} \cdot \underbrace{\rho}_{\text{Density}}$

$dM = 2\pi R^2 t\rho \sin\theta\, d\theta$

Pot. energy of ring and part. m

$dU = -\frac{G m\, dM}{s} = -\frac{2\pi G m R^2 t\rho \sin\theta\, d\theta}{s}$

Change variables

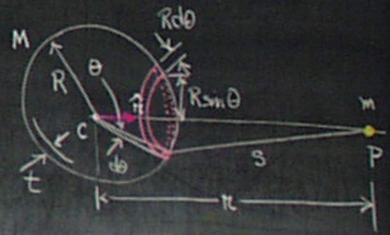
$s^2 = r^2 + R^2 - 2rR\cos\theta$

Law of Cosines

$2s\frac{ds}{d\theta} = 2rR\sin\theta$

$\frac{\sin\theta}{s}\,d\theta = \frac{ds}{rR}$

$\therefore dU = -\frac{2\pi G m R t\rho}{r}\, ds$



Lecture 27 Blackboard #4

Case I: $r > R$ Particle outside Shell

$\theta = 0 \quad s = r - R$

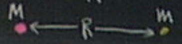
$\theta = \pi \quad s = r + R$

$$U(r>R) = \int dU = -\frac{2\pi G m R t \rho}{r} \int_{r-R}^{r+R} ds$$

$$U = -\frac{Gm}{r}(4\pi R^2 t \rho)$$

$$M = \text{Volume} \times \rho = (4\pi R^2 t)\rho$$

$$\therefore U(r) = -\frac{GmM}{R} \quad (r>R)$$



$$\vec{F}(r) = -\frac{\partial U}{\partial r}\hat{r} = -\frac{GMm}{r^2}\hat{r}$$
$(r>R)$

Case II: $r < R$ Particle Inside Shell

$\theta = 0 \quad s = R - r$

$\theta = \pi \quad s = R + r$

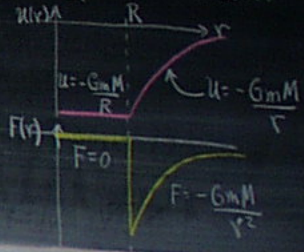
$$U(r<R) = -\frac{2\pi G m t \rho R}{r} \int_{R-r}^{R+r} ds$$

$$= -\frac{Gm}{R}(4\pi R^2 t \rho)$$

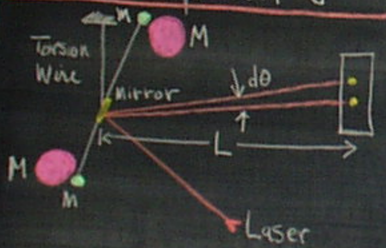
$$U(r<R) = -\frac{GmM}{R} = \text{Constant!}$$

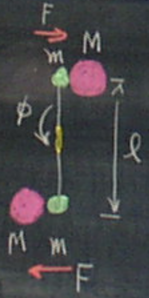
$$\vec{F}(r) = -\frac{\partial U}{\partial r}\hat{r} = 0 \quad \text{Vanishes!!!}$$

(7)



Cavendish Experiment: G





$$F = \frac{GMm}{r^2} \quad \text{Force } m \leftrightarrow M$$

$$\vec{\tau} = \vec{R} \times \vec{F} \quad \text{On Torsion Wire}$$

$$= \frac{\ell G m M}{r^2}$$

$$\vec{\tau} = -\kappa\phi \quad \text{Hooke's Law for torsion wire.}$$

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \quad \text{Period of Torsion Pendulum.}$$

$$I = \frac{m\ell^2}{2}$$ Moment of Inertia of masses, m on rod neglect rod

$$-\kappa\phi = \frac{\ell G M m}{r^2}$$

$$\phi = \frac{\ell G M m}{\kappa r^2}$$

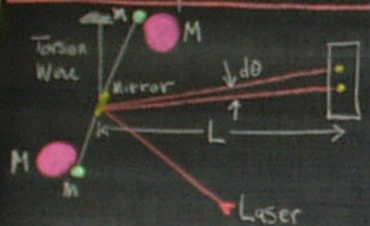
$$\phi' = 2\phi \quad \left[\begin{array}{l}\text{Total deflection}\\ \text{Reversed Torques}\end{array}\right]$$

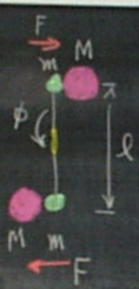
$d\theta = 2\phi$

Reflection of laser from plane mirror

Lecture 27 Blackboard #5

Cavendish Experiment : G





$F = \frac{GMm}{r^2}$ Force $m \longleftrightarrow M$

$\vec{\tau} = \vec{R} \times \vec{F}$ On Torsion Wire

$= \frac{\ell GmM}{r^2}$

$\vec{\tau} = -\kappa\phi$ Hooke's Law for torsion wire.

$T = 2\pi\sqrt{\frac{I}{\kappa}}$ Period of Torsion Pendulum.

$I = \frac{m\ell^2}{2}$ Moment of Inertia of masses, m on rod; neglect rod

$-\kappa\phi = \frac{\ell GMm}{r^2}$

$\phi = \frac{\ell GMm}{\kappa r^2}$

$\phi' = 2\phi$ [Total deflection Reversed Torques]

$d\theta = 2\phi$ Reflection of laser from plane mirror

Parameters

r: 4.65 cm [Balls center-to-center]

$M = 1.5$ kg

$m = 0.015$ kg

$\ell = 10$ cm

$L \sim 19.7$ m [Dist. mirror to wall]

$\kappa = 8.5 \times 10^{-9}$ N m/rad

$T \approx 10$ min.

$\Delta S_{\text{Image on Wall}} = L\, d\theta$

$= \frac{4\ell L\, GmM}{\kappa r^2}$

$\Delta S = \frac{4 \times 0.10 L \times 6.672 \times 10^{-11} \times 0.015 \times 1.50}{8.5 \times 10^{-9} \times (.0465)^2}$

$G: 6.672 \times 10^{-11}\ \text{N·m}^2/\text{kg}^2$ Best Value.

$(\Delta S)_{Th} = .0327 L$ (m)

$L = 19.7$ m

$(\Delta S)_{Th} = 64.4$ cm

$(\Delta S)_{Exp} = ??$