Lecture 24 Blackboard #1

Spinning Object: $\vec{L}$ Conserved

$L_1 = I_1\omega_1 = m r_1^2 \omega_1$

$L_2 = I_2\omega_2 = m r_2^2 \omega_2$

$L_1 = L_2$

$\omega_2 = \left(\frac{r_1}{r_2}\right)^2 \omega_1$

$E_1 = \frac{1}{2} I_1 \omega_1^2 = \frac{1}{2} m_1 r_1^2 \omega_1^2$

$E_2 = \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} m_1 r_2^2 \omega_1^2$

$E_2/E_1 = \frac{\omega_2^2}{\omega_1^2}\frac{r_2^2}{r_1^2} = \left(\frac{r_1}{r_2}\right)^2 > 1$

Example: Stool + Bicycle Wheel

- Stool at rest
- Initial Ang Mom: $\vec{L}_0$: Up

System = Person + Wheel + Stool

Initial: $\vec{L}_{System} = \vec{L}_0 = I_0\omega_0 \hat{k}$

Final: $\vec{L}_{System} = \vec{L}_{Psn+Stool} + \vec{L}_{Wheel}$

$L_{Sys} = \vec{L}_{Psn+Wheel} - \vec{L}_0$

$L_i = L_f$

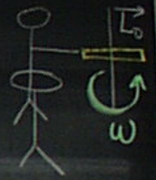
$\vec{L}_0 = \vec{L}_{Psn+Wheel} - \vec{L}_0$

$\vec{L}_{Psn+Stool} = 2\vec{L}_0$

$I_p\omega_p = 2I_0\omega_0$

$\omega_p = \frac{2I_0\omega_0}{I_p}$

(1)



Work and Energy in Rotational Motion

- Force acts on body
- Rotates object through angle θ

$ds = r\,d\theta$ in a time (dt)

$dW = \vec{F}\cdot d\vec{s} = (F\sin\varphi)\,r\,d\theta$

$F\sin\varphi = F_t$ (tangential comp. of $\vec{F}$)

$F\cos\varphi = F_\perp$ ($\perp ds$, radial no work)

$(F\sin\varphi)\,r = \tau_z$ (Torque about axis)

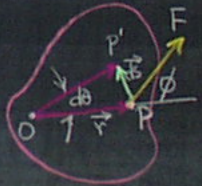
$dW = \tau_z\,d\theta$

$W = \int_{\theta_1}^{\theta_2} \tau_z\,d\theta$

If $\tau_z = $ const.; $W = \tau_z(\theta_2 - \theta_1)$

Power: $P = \frac{dW}{dt} = \tau_z\frac{d\theta}{dt} = \tau_z\,\omega$

Const. τ_z



Work-Energy Theorem: Rotations

(2)

Work by torque $\tau \rightarrow$ Changes KE

$\tau \rightarrow \alpha \quad \omega_1 \rightarrow \omega_2$

$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\cdot\omega$

$\tau\,d\theta = dW = I\omega\,d\omega$

$W = \int_{\theta_1}^{\theta_2} \tau\,d\theta' = \int_{\omega_1}^{\omega_2} I\omega'\,d\omega'$

$v_c = r_p\omega = L\sqrt{\frac{3g}{L}}$ End

Work and Energy in Rotational Motion

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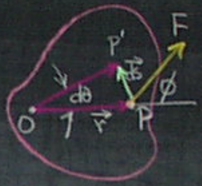
If $\tau_z = $ const.; $W = \tau_z(\theta_2 - \theta_1)$

Power: $P = \dfrac{dW}{dt} = \tau_z \dfrac{d\theta}{dt} = \tau_z\,\omega$ ← Cons. τ_z



Work-Energy Theorem: Rotations

②

Work by torque $\tau \to$ Changes KE

$\tau \to \alpha \qquad \omega_1 \to \omega_2$

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\cdot\omega$$

$\tau\,d\theta = dW = I\omega\,d\omega$

$$W = \int_{\theta_1}^{\theta_2} \tau\,d\theta' = \int_{\omega_1}^{\omega_2} I\omega'\,d\omega'$$

$$W_{Total} = \frac{I}{2}(\omega_2^2 - \omega_1^2)$$

Assume $I = $ const.

$\omega_1 \to \omega_2$

$\theta_1 \to \theta_2$

$W = \Delta K = K_f - K_i = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$

If $\vec{F}$ is conservative (spring/gravity)

$W = -\Delta U = -U_2 + U_1 = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$

$\frac{1}{2}I\omega_1^2 + U_1 = \frac{1}{2}I\omega_2^2 + U_2 = E$

$E_1 = E_2 = E$

Cons. of Mechanical Energy in Rotational Motion.

If there is Friction,

$E_2 - E_1 = W_{friction}$!!

Example: Falling Rod

③

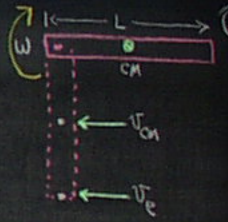
- Rod Falls $[\Delta_{cm} = \frac{L}{2}]$
- What is ω at bottom?

$E_i = E_f$ Cons. of M Energy.

$U_1 + \frac{1}{2}I\omega_1^2 = U_2 + \frac{1}{2}I\omega_2^2$

$\frac{L}{2}(Mg) + 0 = 0 + \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega_2^2$

$$\omega_2 = \sqrt{\frac{3g}{L}}$$



$v_c = r_c\,\omega = \frac{L}{2}\sqrt{\frac{3g}{L}}$ Center

$v_e = r_e\,\omega = L\sqrt{\frac{3g}{L}}$ End

Lecture 24 Blackboard #2

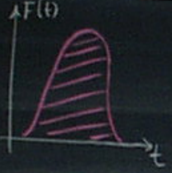
Angular Impulse

$\vec{I} = \int_{t_1}^{t_2} \vec{F}(t)\,dt$

$\frac{d\vec{p}}{dt} = \vec{F}$

$\Delta\vec{p} = \vec{P}_f - \vec{P}_i = \vec{F}\Delta t = \vec{I}$

= Area under F-t curve.



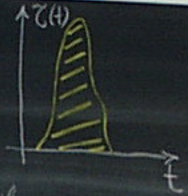
$\frac{d\vec{L}}{dt} = \vec{\tau}$

$\vec{L}_f - \vec{L}_i = \Delta\vec{L} = \vec{\tau}\,dt$

Let $\vec{J} = \int_{t_1}^{t_2} \vec{\tau}(t)\,dt$

= Ang. Impulse.

Change in $\vec{L}$ is equal to the angular impulse! Area under curve.



Example: Spinning Disks.

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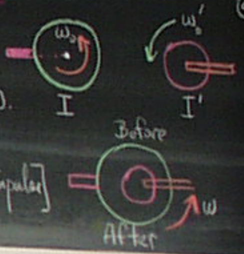
- Two disks I, and I'
- Spinning ω_0 and ω_0'
- Push together: common ω.
- No external torques!!

$\vec{J}_\theta = -\vec{J}'_\theta$ [Eq. Opp. Int. Impulse]



$J_\theta = (I\omega - I\omega_0)$ Large Disk

$J'_\theta = (I'\omega - I'\omega_0')$ Small Disk

$I\omega - I\omega_0 = -[I'\omega - I'\omega_0']$

$\omega = \frac{I\omega_0 + I'\omega_0'}{I+I'}$

$L_i = I\omega_0 + I'\omega_0'$

$L_f = (I+I')\omega$

$L_i = L_f$

$I\omega_0 + I'\omega_0' = I\omega + I'\omega$

$\vec{L}$ Conserved

K Not Conserved.

Example: Projectile-Cylinder Collision

- Particle m collides ⊥ axis of rotation
- Distance 'd' from axis
- What is ω after collision?

$\sum \tau_{ext} \equiv 0$

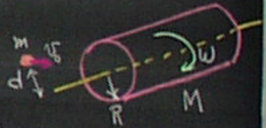
$\therefore \vec{L}_i = \vec{L}_f$ $\quad \vec{L}$ is conserved.

Calculate about axis of rotation.

$L_i = (L_{cyl} + L_{proj})_f$

$m v_0 d = (\frac{1}{2} MR^2 + m d^2)\omega$

$\omega = \frac{m v_0 d}{\frac{1}{2} MR^2 + m d^2}$



$\frac{1}{2} I_f \omega^2 < \frac{1}{2} m v_0^2$

KE not conserved/Inelastic

Lecture 24 Blackboard #3

$J_\theta = (I\omega - I\omega_0)$ Large Disk

$J'_\theta = (I'\omega - I'\omega'_0)$ Small Disk

$\therefore I\omega - I\omega_0 = -[I'\omega - I'\omega'_0]$

$$\omega = \frac{I\omega_0 + I'\omega'_0}{I+I'}$$

$L_i = I\omega_0 + I'\omega'_0$

$L_f = (I+I')\omega$

$L_i = L_f$

$I\omega_0 + I'\omega'_0 = I\omega + I'\omega$

$\vec{L}$ Conserved

K Not Conserved.

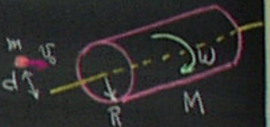
Example Projectile-Cylinder Collision

- Particle m collides $\perp$ axes of rotation
- Distance 'd' off axis
- what is ω after collision?

$\sum \vec{\tau}_{ext} \equiv 0$

$\therefore \vec{L}_i = \vec{L}_f$ $\quad \vec{L}$ is conserved.

Calculate about axis of rotation.



$L_i = (L_{cyl} + L_{proj})_f$

$m v_0 d = (\frac{1}{2}MR^2 + md^2)\omega$

$$\omega = \frac{m v_0 d}{\frac{1}{2}MR^2 + md^2}$$

$\frac{1}{2}I_T\omega^2 < \frac{1}{2}m v_0^2$

KE not conserved / Inelastic

Rotation and Translation

. Generalized Fixed Axis Rotation

1) Axis of rotation fixed in inertial frame.

OR

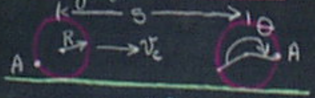
2) Axis of rotation is through CM of body and maintains fixed direction in space.

$$\frac{d\vec{P}}{dt} = \vec{F}_{ext}$$

$$K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I_c\omega^2$$

$$\vec{L} = \vec{L}_{cm} + \vec{L}_{spin} = \vec{R}\times\vec{P} + \int (\vec{r}_c\times\vec{v}_c)\,dm$$

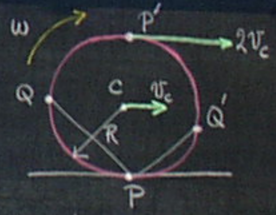
Rolling Cylinders/Spheres



$S = R\theta$

$$v_c = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$a_c = \frac{d^2s}{dt^2} = \frac{dv_c}{dt} = R\frac{d\omega}{dt} = R\alpha$$



All points common ω.

Contact point P at rest. Inst.

P: $v_P = 0$

C: $v_c = R\omega$

P': $v_{P'} = 2R\omega$

Another View

Trans. + Rotation about CM is equivalent to pure rotation about an axes through point of contact of rolling body.

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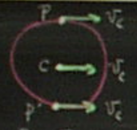
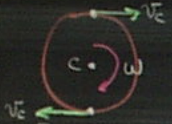
Lecture 24 Blackboard #4



Pure Trans



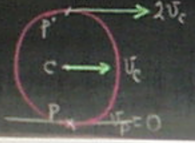
Pure Rotation

Trans + Rot

$v_P = v_c - v_c = 0$

$v_c = v_c + 0 = v_c$

$v_{P'} = v_c + v_c = 2v_c$



Kinetic Energy

$K = \frac{1}{2} I_P \omega^2$

$I_P = I_c + MR^2$ Parallel-Axis

$K = \frac{1}{2} I_c \omega^2 + \frac{1}{2} M R^2 \omega^2$

$= \frac{1}{2} I_c \omega^2 + \frac{1}{2} M v_{cm}^2$

$= K_{cm} + K_{Translation}$

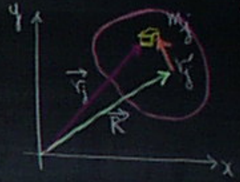
$\vec{L}$: Translation and Rotation

Want to show that $\vec{L}$ has two parts:

$L_z = I_0 \omega + (\vec{R} \times M\vec{V})_z$

$\vec{R}$: Position vector of CM

$\vec{V} = \dot{\vec{R}}$; velocity of CM



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m_j: mass element

$\vec{r}_j$: position vector

$\vec{L} = \sum_j \vec{r}_j \times m_j \dot{\vec{r}}_j$

$\vec{R} = \sum m_j \vec{r}_j$

M = Total Mass

$\vec{r}_j = \vec{R} + \vec{r}_j'$

$\vec{L} = \sum (\vec{R} + \vec{r}_j') \times m_j (\dot{\vec{R}} + \dot{\vec{r}}_j')$

$= \underset{1}{\vec{R} \times \sum m_j \dot{\vec{R}}} + \underset{2}{\cancel{\sum m_j \vec{r}_j' \times \dot{\vec{R}}}} + \underset{3}{\cancel{R \times \sum m_j \dot{\vec{r}}_j'}} + \underset{4}{\sum m_j \vec{r}_j' \times \dot{\vec{r}}_j'}$

Term-2: $\sum m_j \vec{r}_j' = \sum m_j (\vec{r}_j - \vec{R})$

$= \sum m_j \vec{r}_j - MR$

$\equiv 0$ Def of CM

Term-3: $\sum m_j \dot{\vec{r}}_j' \equiv 0$ Def of CM Velocity

Term-1: $\vec{R} \times \sum m_j \dot{\vec{R}} = \vec{R} \times M\dot{\vec{R}} = \vec{R} \times M\vec{V}_{cm}$

$\therefore \vec{L} = \vec{R} \times M\vec{V}_{cm} + \sum \vec{r}_j' \times m_j \dot{\vec{r}}_j'$

↳ $\vec{L}_0$: Ang mom due to CM motion. Called Orbital Ang. Mom.

8

Lecture 24 Blackboard #5

m_j : mass element
r_j : position vector
$\vec{L} = \sum_j \vec{r}_j \times m_j \dot{\vec{r}}_j$
$\vec{R} = \sum m_j \vec{r}_j$
M = Total Mass
$\vec{r}_j = \vec{R} + \vec{r}_j'$

$\vec{L} = \sum (\vec{R} + \vec{r}_j') \times m_j (\dot{\vec{R}} + \dot{\vec{r}}_j')$

$= \underset{①}{\vec{R} \times \sum m_j \dot{\vec{R}}} + \underset{②}{\cancel{\sum m_j \vec{r}_j' \times \dot{\vec{R}}}} + \underset{③}{\cancel{R \times \sum m_j \dot{\vec{r}}_j'}} + \underset{④}{\sum m_j \vec{r}_j' \times \dot{\vec{r}}_j'}$

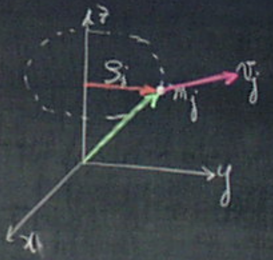
Term-2: $\sum m_j r_j' = \sum m_j (\vec{r}_j - \vec{R})$
$= \sum m_j \vec{r}_j - MR$
$\equiv 0$ Def of CM

Term-3: $\sum m_j \dot{\vec{r}}_j' \equiv 0$ Def of CM Velocity
Term-1: $\vec{R} \times \sum m_j \dot{\vec{R}} = \vec{R} \times M \dot{\vec{R}} = \vec{R} \times M \vec{V}_{cm}$
$\therefore \vec{L} = \vec{R} \times M \vec{V}_{cm} + \sum \vec{r}_j' \times m_j \dot{\vec{r}}_j'$
↳ $\vec{L}_0$: Ang mom due to CM motion.
Called Orbital Ang. Mom.

(8)

Consider fixed z axis through CM.
$L_z = (\sum \vec{r}_j' \times m_j \dot{\vec{r}}_j')_z$: Spin Ang. Mom
$\vec{v}_j \perp \vec{r}_j$
$\perp$ dis to axis = S
$L_z(j) = m_j v_j S_j$
$v_j = \omega S_j$

$\therefore L_z(j) = m_j S_j^2 \omega$
$L_z = \sum L_z(j) = \sum m_j S_j^2 \omega$
$\boxed{L_z = I_z \omega}$ Spin Ang. Momentum.



$\boxed{\vec{L} = \vec{R} \times M \vec{V}_{cm} + I_z \omega}$
Orbit + Spin

(9)

Torques: Rot + Trans.
Can show: $\tau_z = I_z \alpha$
Rot. motion about CM depends only on torque about CM.
Indep of Transl. Motion.
CM axis can accelerate — Result is the same.

Lecture 24 Blackboard #6