Parallel Axis Theorem

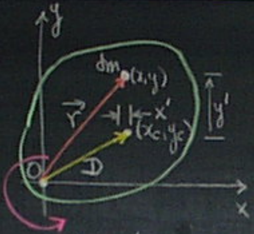
Rot. axis $\perp$ xy-plane at 'O'

CM: x_c, y_c : Distance D from 'O'

Element Δm: $r = \sqrt{x^2+y^2}$ to axis

$I_0 = \int r^2 dm = \int (x^2+y^2)\, dm$

$= \int_V \left[(x_c+x')^2 + (y_c+y')^2\right] dm$



$x = x_c + x'$

$y = y_c + y'$

$I_0 = \underbrace{\int_V (x'^2+y'^2)\, dm}_{I_{CM}} + \cancel{2x_c \int x' dm}$

$+ \cancel{2y_c \int y' dm} + \underbrace{(x_c^2+y_c^2)}_{D^2} \int dm$

$\int x' dm \equiv 0 \quad \int y' dm \equiv 0$ Def. of CM.

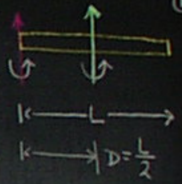
$$\boxed{I_0 = I_{CM} + D^2 M}$$

Example

$I_{End} = I_{CM} + MD^2$

$= \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2$

$I_{End} = \frac{ML^2}{3}$



①

Perpendicular Axes Theorem

Thin plates: t << lngth or width

Plate lies in xy-plane

$R^2 = x^2 + y^2$

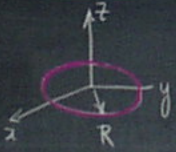
$I_z = \int \rho R^2 dV = \int \rho (x^2+y^2)\, dV$

$I_x = \int \rho y^2 dV$

$I_y = \int \rho x^2 dV$

$\therefore \boxed{I_z = I_x + I_y}$

a) Disk



$I_z = \frac{1}{2} MR^2$

$I_x = I_y$ Symmetry

$\therefore I_z = 2I_x = 2I_y$

$I_x = I_y = \frac{MR^2}{4}$

b) Square Plate

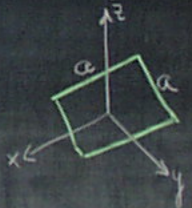


Plate sides = a

$I_x = I_y$: Symmetry

$I_x = I_y = \frac{I_z}{2} = \frac{Ma^2}{12}$

$I_z = \frac{M(a^2+a^2)}{12}$

②

Lecture 22 Blackboard #1

Perpendicular Axes Theorem

Thin plates: $t \ll$ length or width

Plate lies in xy-plane

$R^2 = x^2 + y^2$

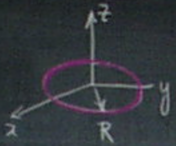
$I_z = \int \rho R^2 \, dV = \int \rho (x^2 + y^2)\, dV$

$I_x = \int \rho y^2 \, dV$

$I_y = \int \rho x^2 \, dV$

$\therefore \boxed{I_z = I_x + I_y}$

a) Disk



$I_z = \frac{1}{2} M R^2$

$I_x = I_y$ Symmetry

$\therefore I_z = 2I_x = 2I_y$

$I_x = I_y = \frac{MR^2}{4}$

b) Square Plate

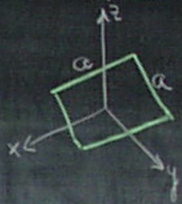


Plate sides $= a$

$I_z = \frac{M(a^2 + a^2)}{12}$

$I_x = I_y$: Symmetry

$I_x = I_y = \frac{I_z}{2} = \frac{Ma^2}{12}$

(2)

Example: Rotation and Energy

Mass m falls a distance H.

What is speed of M and ang. vel. ω of wheel?

$E_1 = K_1 + U_1 = 0 + mgH$

$E_2 = K_2 + U_2 = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 + 0$

$v = R\omega$ Geometry/Kinematics.

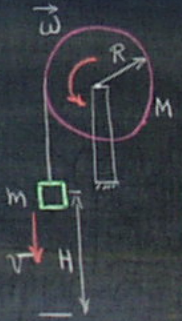
$I = \frac{1}{2} M R^2$ Solid Cylinder

$E_1 = E_2$ Cons. of Energy.

$mgH = \frac{1}{2} m v^2 + \frac{1}{2}\left(\frac{1}{2} M R^2\right)\left(\frac{v}{R}\right)^2$

$= \frac{1}{2}\left(m + \frac{M}{2}\right) v^2$

$v = \sqrt{\dfrac{2gH}{1 + \frac{M}{2m}}}$



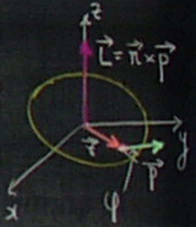
Angular Momentum of a Particle.

Linear Mom: $\frac{d\vec{p}}{dt} = \vec{F}$

If $\vec{F} \neq 0$ $\vec{p}$ not conserved

$\boxed{\vec{\tau} = \vec{r} \times \vec{p}}$

If $\vec{F} = f\hat{r} \Rightarrow \vec{L}$ is conserved.

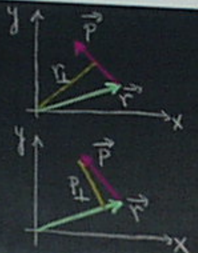


(3)

Lecture 22 Blackboard #2

$\vec{L} \perp$ to plane cont. $\vec{r}$ and $\vec{p}$
- lies on RH normal
- Cross product of $\vec{r}, \vec{p}$
- $\therefore \vec{L} = 0$ if $\vec{r}$ parallel to $\vec{p}$

$|\vec{L}| = |\vec{r} \times \vec{p}| = rp \sin\varphi$ (kg·m/s²)
$= r_\perp p = r p_\perp$



i) L: Particle Moving in a Straight Line

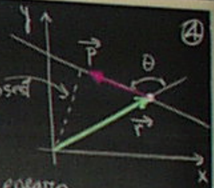
- Assume $\vec{F} = 0$; $\vec{v}$ = constant.
- Direction of $\vec{L}$ = constant.
- Magnitude of $\vec{L}$ = constant.

$\vec{L} = rp \sin\theta \,\hat{k}$

If $\vec{r}$ and $\vec{p}$ in xy-plane
$\vec{L}$ is along z!

④

$b = r \sin\theta$
= dist. of closest approach



L≠0 if position vector appears to rotate about origin.
L=0 if |r| just gets longer or shorter
no rotation.

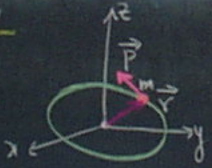
Particle in Uniform Circular Motion

- Origin at center.

$\vec{L} = \vec{r} \times \vec{p}$ $\vec{p} \neq$ constant.
$|\vec{v}|$ = constant.

$= rp\hat{k} = Mrv\,\hat{k}$
$= Mr^2\omega\,\hat{k}$

$\therefore \vec{L} = I\omega\hat{k}$



$\vec{L}$: const in mag. + dir.

Note: $\vec{a}_c = \frac{v^2}{R}$ $\vec{F}_c = \frac{Mv^2}{R} \neq 0$

Angular Momentum of a Conical Pendulum

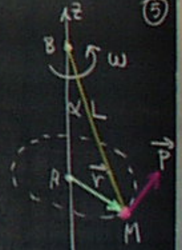
.Assume circ. motion, constant ω.

a) Origin at A:

$\vec{L}_A = \vec{r} \times \vec{p} = rp\,\hat{k}$ $\vec{r} \perp \vec{p}$ everywhere.
$p = mv = mr\omega$ r: radius of circle.
$\vec{L}_A = mr^2\omega\,\hat{k}$

L: Const in mag.
Const. in direc!

⑤



Lecture 22 Blackboard #3

Particle in Uniform Circular Motion

· Origin at center.

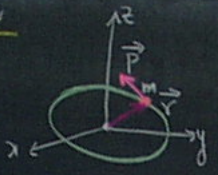
$\vec{L} = \vec{r} \times \vec{p}$ $\qquad \vec{p} \neq$ constant.

$\qquad |v| =$ constant.

$= rp\hat{k} = Mrv\hat{k}$

$= Mr^2\omega\hat{k}$

$\therefore \vec{L} = I\omega\hat{k}$



$\vec{L}$: const in mag. + dir.

Note: $\vec{a}_c = \frac{v^2}{R}$ $\quad F_c = \frac{Mv^2}{R} \neq 0$

Angular Momentum of a Conical Pendulum

· Assume circ. motion, constant ω.

a) Origin at A:

$\vec{L}_A = \vec{r} \times \vec{p} = rp\hat{k}$ $\qquad \vec{r} \perp \vec{p}$ everywhere.

$p = mv = mr\omega$ $\qquad r$: radius of circle.

$\vec{L}_A = mr^2\omega\hat{k}$

L: Const in mag.

Const. in direc!

b) Origin at B:

$|L_B| = |\vec{r}' \times \vec{p}|$

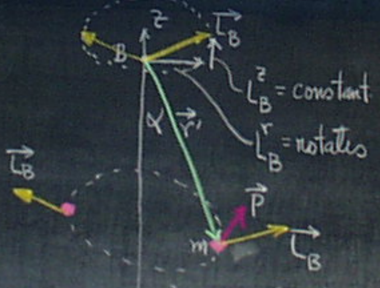
$= |\vec{r}'||\vec{p}|$ $\qquad \vec{r}' \perp \vec{p}$

$|L_B| = mvl = mrl\omega$

$|r'| = l$; length of string.

$\vec{L}_B$: magnitude not constant

Direction not constant.



$|L_B^z| = Mrl\omega/\sin\alpha$

Angular Momentum ⟷ Forces

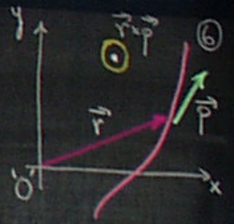
Is $\vec{L}$ conserved?

Test: If $\frac{d\vec{L}}{dt} = 0 \quad \therefore \vec{L}$ = conserved!

$\vec{L}_O = \vec{r} \times \vec{p}$ $\qquad$ (Origin at 'O')

$\frac{d\vec{L}_O}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$

$= \cancel{\vec{v} \times m\vec{v}} + \vec{r} \times \vec{F}$



Lecture 22 Blackboard #4

(7)

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

If $\vec{F} /\!/ \vec{r}$

$\frac{d\vec{L}}{dt} = 0$; $\therefore \vec{L}$ is conserved.

$\vec{F} /\!/ \vec{r} \Rightarrow$ Central Forces
Gravity, etc.

If $\vec{F}$ is not central
$\vec{L}$ is not conserved!!

$$\boxed{\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}} \leftarrow \text{Rot. Dynamics.}$$

$\vec{\tau} \equiv$ Torque

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}} \quad \text{Linear Dynamics.}$$

Rate of change of $\vec{L} \equiv$ Torque applied.
$\vec{\tau}$ and $\vec{L}$: Must have same coord. origin
: Inertial Coord. System.

(8)

2D: Rotations

- Uniform circular motion
  constant ω.
- Tension in string; T.
- Radius R_1: $L_1 = I_1\omega_1 =$
- Radius R_2: $L_2 = I_2\omega_2$.

$T \equiv$ central force
to or away from center!

$\therefore L_1 = L_2$ $\quad \vec{L}$ is conserved.

$$I_1\omega_1 = I_2\omega_2$$

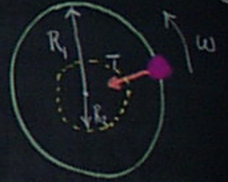
$$mR_1^2\omega_1 = mR_2^2\omega_2$$

$$\omega_2 = \frac{R_1^2}{R_2^2}\omega_1$$

If $R_2 < R_1$; $\omega_2 > \omega_1$

$$\left.\begin{aligned} K_1 &= \tfrac{1}{2} I_1 \omega_1^2 \\ K_2 &= \tfrac{1}{2} I_2 \omega_2^2 \end{aligned}\right\} \quad \frac{K_2}{K_1} = \frac{\omega_2^2}{\omega_1^2}\,\frac{I_2}{I_1} = \frac{R_1^2}{R_2^2} > 1\,?$$

$\Delta K \neq 0$ From where?



2D Rotations

- Uniform circular motion
  constant ω.
- Tension in string; T.
- Radius R_1: $L_1 = I_1 \omega_1 =$
- Radius R_2: $L_2 = I_2 \omega_2$.

$T \equiv$ central force
to or away from center!

$\therefore L_1 = L_2$ $\vec{L}$ is conserved.

$$I_1 \omega_1 = I_2 \omega_2$$

$$m R_1^2 \omega_1 = m R_2^2 \omega_2$$

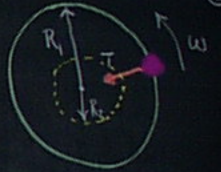
$$\omega_2 = \frac{R_1^2}{R_2^2} \omega_1$$

If $R_2 < R_1$; $\omega_2 > \omega_1$

$$\left.\begin{array}{l} K_1 = \frac{1}{2} I_1 \omega_1^2 \\ K_2 = \frac{1}{2} I_2 \omega_2^2 \end{array}\right\} \quad \frac{K_2}{K_1} = \frac{\omega_2^2}{\omega_1^2} \frac{I_2}{I_1} = \frac{R_1^2}{R_2^2} > 1 \, ?$$

$\Delta K \neq 0$ From where?

8



Example Comets

Apogee: Farthest $\vec{v}_2 \perp \vec{r}_2$

Perigee: Closest $\vec{v}_1 \perp \vec{r}_1$

Ang momentum about center of sun (mag.)

$\vec{L}_1 = \vec{r}_1 \times \vec{v}_1 = m r_1 v_1$ (perigee)

$\vec{L}_2 = \vec{r}_2 \times \vec{v}_2 = m r_2 v_2$ (apogee)

$\vec{L} = \vec{r} \times \vec{p}$ Direction out of bboard!

Gravity = central force

$\therefore L_1 = L_2$.

$$m r_1 v_1 = m r_2 v_2$$

$$v_2 = \left(\frac{r_1}{r_2}\right) v_1$$

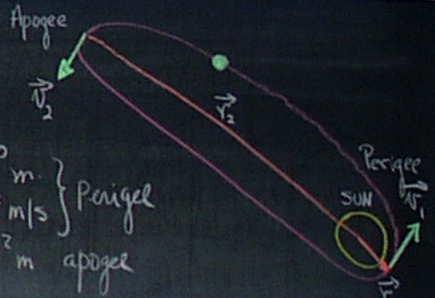
$$= \frac{8.75 \times 10^{10}}{5.26 \times 10^{12}} \times 5.46 \times 10^4$$

$$= 9.08 \times 10^2 \text{ m/s}.$$

Halley's Comet

$$\left.\begin{array}{l} r_1 = 8.75 \times 10^{10} \text{ m} \\ v_1 = 5.46 \times 10^4 \text{ m/s} \end{array}\right\} \text{Perigee}$$

$r_2 = 5.26 \times 10^{12}$ m apogee



Lecture 22 Blackboard #6