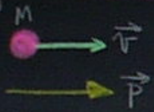
Momentum of a Particle

$\vec{P} = M\vec{v}$

$[P] = \frac{kg \cdot m}{s}$



Newton's Laws: More precise in terms of $\vec{P}$

$\vec{P}$: Vector; Direction of Velocity

Law-I:

If no forces act, $\vec{F}_R \equiv 0$ momentum is constant.

$\vec{F}_R \equiv 0 \Rightarrow \vec{a}=0 \Rightarrow \vec{V}=\text{constant}$

$\boxed{\vec{P} = \text{constant}} \Rightarrow$ Conserved Quantity.

Energy
Momentum
Angular Momentum } Conservation Laws

Law-II:

Rate of change of linear momentum = applied force.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = \left(\frac{dm}{dt}\right)\vec{v} + \underbrace{m\frac{d\vec{v}}{dt}}_{m\vec{a}}$$

$\vec{F} = m\vec{a}$ if $\left(\frac{dm}{dt}\right) = 0$ ↳ mass cons.

$\vec{F} = \frac{d\vec{P}}{dt}$ Most General Statement

Handles problems where mass changes: Rockets

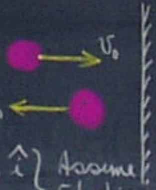
(1)

Law-III:

Rate of change of momentum generated by an action force on one body is exactly opposite to the rate of change of momentum generated by the reaction force on the other body.

Example:

Before: After:



$\vec{P}_i = m v_0 \hat{i}$
$\vec{P}_f = -m v_0 \hat{i}$ } Assume Elastic Velocities Equal

$\Delta\vec{p} = \vec{P}_f - \vec{P}_i = -2mv_0\hat{i}$

$\longleftarrow \Delta\vec{p}$

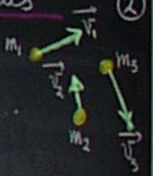
Where did missing $\Delta\vec{p}$ go? To the Wall!!!

$M_w \ggg M$ $\quad \vec{a}_w = \frac{\Delta P/\Delta t}{M_w} = -\frac{2m v_0}{M_w}\frac{1}{\Delta t}$ very small!

System of Particles

(2)

Total $\vec{P} \equiv$ Vector Sum



$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots$

$\vec{P}_1 = m_1\vec{v}_1$
$\vec{P}_2 = m_2\vec{v}_2$
$\vec{P}_3 = m_3\vec{v}_3$

Total $\vec{P}$ is Invariant] Total Conserved!

$\Rightarrow P \equiv$ Const. [Comp by Comp]

Lecture 17, Blackboard #1

Law-III: Rate of change of momentum generated by an action force on one body is exactly opposite to the rate of change of momentum generated by the reaction force on the other body.

Example:

Before

After:

$\vec{P}_i = m v_0 \hat{i}$

$\vec{P}_f = -m v_0 \hat{i}$

} Assume Elastic Velocities Equal

$\vec{\Delta p} = \vec{P}_f - \vec{P}_i = -2mv_0 \hat{i}$

$\longleftarrow \vec{\Delta p}$

Where did missing $\vec{\Delta p}$ go? To the Wall!!!

$M_w \ggg M$ $\quad \vec{a}_w = \frac{\Delta P/\Delta t}{M_w} = -\frac{2m}{M_w} v_0 \frac{1}{\Delta t}$ very small!

System of Particles ②

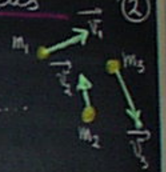
Total $\vec{P} \equiv$ Vector Sum

$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots$

$\vec{P}_1 = m_1 \vec{v}_1$

$\vec{P}_2 = m_2 \vec{v}_2$

$\vec{P}_3 = m_3 \vec{v}_3$



③

$\vec{P} = \vec{P}_1 + \vec{P}_2 + \cdots \vec{P}_n = \sum_{1}^{n} \vec{P}_i$

$P_x = P_{1x} + P_{2x} + \cdots P_{nx} = \sum P_{ix}$

$P_y = P_{1y} + P_{2y} + \cdots P_{ny} = \sum P_{iy}$

$P_z = P_{1z} + P_{2z} + \cdots P_{nz} = \sum P_{iz}$

2-Particle System: $F_{ext} \equiv 0$

$\frac{d\vec{P}_1}{dt} = \vec{F}_1 \quad \frac{d\vec{P}_2}{dt} = \vec{F}_2$ Newton's 2-nd Law Eq. of Motion

$\frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = \vec{F}_1 + \vec{F}_2 \quad \vec{F}_1 = -\vec{F}_2$ 3rd Law

$\therefore \frac{d(P_1 + P_2)}{dt} = 0$

$\boxed{\vec{P} = \vec{P}_1 + \vec{P}_2 = \text{Constant}}$

Total $\vec{P}$ is Invariant

} Particles collide Exchange $\vec{P}_i$ Total Conserved!!

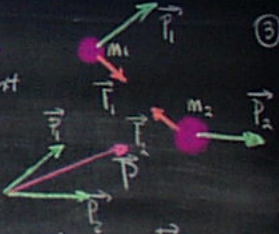
2-Particle System: $F_{ext} \neq 0$

$\frac{d\vec{P}_1}{dt} = \vec{F}_1 + \vec{F}_{1,ext} \quad \frac{d\vec{P}_2}{dt} = \vec{F}_2 + \vec{F}_{2,ext}$

$\frac{d}{dt}(\vec{P}_1 + \vec{P}_2) = \vec{F}_{1,ext} + \vec{F}_{2,ext}$

$\therefore \frac{d\vec{P}}{dt} = \vec{F}_{ext} = \vec{F}_{1,ext} + \vec{F}_{2,ext}$

$\vec{P} = \vec{P}_1 + \vec{P}_2$: Eq. of Motion



If $\vec{F}_{ext} \equiv 0$: $\frac{d\vec{P}}{dt} = 0$

$\Rightarrow \vec{P} \equiv$ Const. [Comp by Comp]

Lecture 17, Blackboard #2

System of Particles

Internally: collisions
interactions/complicated
Externally: No net force.
Total linear momentum
is conserved!!

Example: Inelastic Collision

What is v'?
Initial Momentum: $m_1 v_1 \hat{i}$
Final Momentum: $(m_1+m_2) v' \hat{i}$
Cons. of $\vec{P}$: $m_1 v_1 \hat{i} = (m_1+m_2) v' \hat{i}$

$$\therefore v' = \frac{m_1}{m_1+m_2} v_1 = \frac{1}{2} v_1 \text{ if } m_1 = m_2$$

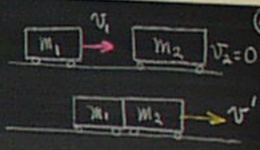
Energies??
Initial KE $\frac{1}{2} m_1 v_1^2 + 0$
Final KE $\frac{1}{2} (m_1+m_2) v'^2$

$$= \frac{1}{2}(2m_1)\frac{1}{4} v_1^2 \quad (m_1 = m_2)$$

$$= \frac{1}{4} m_1 v_1^2$$

Half E is lost

④



$K_i \neq K_f$ Energy not Conserved in Inelastic Collision

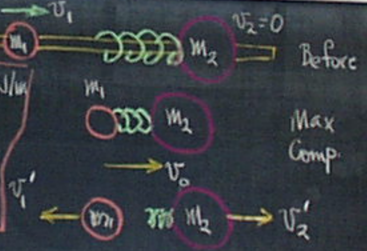
Example: Elastic Collision

- What is max. spring compression?
- What are final velocities of $m_1 + m_2$?
- At max. comp. rel. velocity = 0.
- Assume system has common velocity = v_0

Cons. of $\vec{P}$: $m_1 v_1 + m_2 \times 0 = (m_1+m_2) v_0$

$$v_0 = \frac{m_1}{m_1+m_2} v_1 = \frac{2}{2+5} \times 14 = 4 \text{ m/s}$$

$k = 1000$ N/m
$m_1 = 2$ kg
$m_2 = 5$ kg
$v_1 = 14$ m/s



⑤

Initial KE before collision

$$K_0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 \times 0 = \frac{1}{2} \times 2 \times 14^2 = 196 \text{ J}$$

KE left at max. compression:

$$K_{max} = \frac{1}{2}(m_1+m_2) v_0^2 = \frac{1}{2}(2+5)\, 4^2 = 56 \text{ J}$$

$$\Delta K = K_0 - K_{max} = \frac{1}{2} k x^2 \quad \text{Stored in Spring Comp.}$$

$$x = \sqrt{\frac{2}{k}(K_0 - K)} = \sqrt{\frac{2}{1000}(196-56)} = 0.53 \text{ m}$$

Lecture 17, Blackboard #3

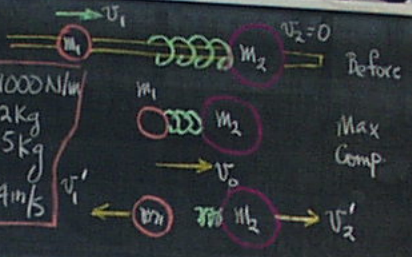
Example: Elastic Collision

- What is max. spring compression?
- What are final velocities of $m_1 + m_2$?
- At max. comp. rel. velocity = 0.
- Assume system has common velocity $= v_0$

Cons. of $\vec{P}$: $m_1 v_1 + m_2 \times 0 = (m_1 + m_2) v_0$

$$v_0 = \frac{m_1}{m_1+m_2} v_1 = \frac{2}{2+5} \times 14 = 4\,m/s$$

$k = 1000\,N/m$
$m_1 = 2\,kg$
$m_2 = 5\,kg$
$v_1 = 14\,m/s$



Initial KE before collision

$$K_0 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 \times 0 = \tfrac{1}{2} \times 2 \times 14^2 = 196\,J$$

KE left at max. compression:

$$K_{max} = \tfrac{1}{2}(m_1 + m_2) v_0^2 = \tfrac{1}{2}(2+5)\,4^2 = 56\,J$$

$\Delta K = K_0 - K_{max} = \tfrac{1}{2} k x^2$ Stored in Spring Comp.

$$x = \sqrt{\frac{2}{k}(K_0 - K)} = \sqrt{\frac{2}{1000}(196-56)} = 0.53\,m$$

5

When particles separate, energy stored in spring is returned to particles.
Total E and $\vec{P}$ are conserved.

$\vec{P}$: $m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 \times 0$ (1)

E: $\tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 \times 0$ (2)

$2v_1' + 5v_2' = 28$ (3)

$2v_1'^2 + 5v_2'^2 = 392$ (4)

$$v_2' = \frac{28 - 2v_1'}{5}$$

$$2v_1'^2 + 5\left[\frac{28 - 2v_1'}{5}\right]^2 = 392$$

Solve quadratic

	v_1'	v_2'
NG →	14	0
	−6	8

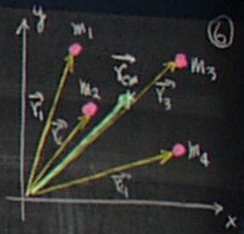
Center-of-Mass

: Average position of all masses in system.

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots m_n \vec{r}_n}{m_1 + m_2 + \cdots m_n}$$

$$= \frac{1}{M}\left[m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots m_n \vec{r}_n\right]$$

$$\vec{r}_{cm} = \sum m_i \vec{r}_i \Big/ \sum m_i$$



$M = \sum m_i$

$$x_{cm} = \frac{1}{M}(m_1 x_1 + m_2 x_2 + \cdots m_n x_n) = \frac{1}{M} \sum m_i x_i$$

$$y_{cm} = \frac{1}{M}(m_1 y_1 + m_2 y_2 + \cdots m_n y_n) = \frac{1}{M} \sum m_i y_i$$

$z_{cm} \equiv$ same!

6

Center-of-Mass

$\vec{r}_{cm} = \sum m_i \vec{r}_i / M$

$x_{cm} = \sum m_i x_i / M$

$y_{cm} = \sum m_i y_i / M$

$z_{cm} = \sum m_i z_i / M$

$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$

Example: CM of Three Particles

$x_{cm} = \frac{\sum m_i x_i}{M} = \frac{2md + m(d+b) + 4m(d+b)}{7m}$

$= d + \frac{5}{7} b$

$y_{cm} = \frac{\sum m_i y_i}{M} = \frac{2m(0) + m(0) + 4mh}{7m}$

$= 4/7\, h$

$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j}$

$= \left(d + \frac{5b}{7}\right)\hat{i} + \frac{4}{7} h\,\hat{j}$

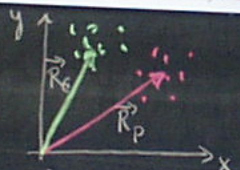
CM: Groups of Particles

$M\vec{R}_{cm} = \sum m_k \vec{r}_k$

$= \sum m_i \vec{r}_i + \sum m_j \vec{r}_j$

Let $\vec{R}_G = \frac{1}{M_G} \sum m_i \vec{r}_i \qquad M_G = \sum m_i$

Let $\vec{R}_P = \frac{1}{M_P} \sum m_j \vec{r}_j \qquad M_P = \sum m_j$



$M\vec{R}_{cm} = M_G \vec{R}_G + M_P \vec{R}_P$

$M = M_G + M_P$

CM of Solid Objects

Δm_i : Coord (x, y, z)

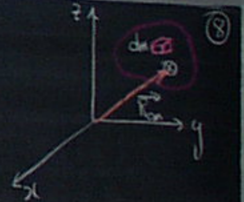
$X_{cm} = \sum \Delta m_i x_i / M$

Let $i \to \infty$

$x_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum \Delta m_i x_i}{M} = \frac{1}{M}\int x\, dm$

$y_{cm} = \frac{1}{M}\int y\, dm \qquad z_{cm} = \frac{1}{M}\int z\, dm.$

$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$



Lecture 17, Blackboard #5

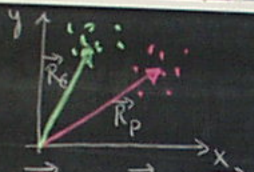
CM: Groups of Particles

$$M\vec{R}_{cm} = \sum m_k \vec{r}_k$$

$$= \sum m_i \vec{r}_i + \sum m_j \vec{r}_j$$

Let $\vec{R}_G = \frac{1}{M_G}\sum m_i \vec{r}_i$ $\qquad M_G = \sum m_i$

Let $\vec{R}_P = \frac{1}{M_P}\sum m_j \vec{r}_j$ $\qquad M_P = \sum m_j$



$$M\vec{R}_{cm} = M_G\vec{R}_G + M_P\vec{R}_P$$

$$M = M_G + M_P$$

CM of Solid Objects

(8)

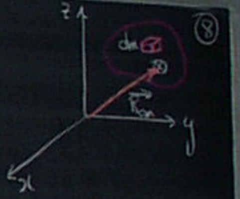
Δm_i : Coord (x, y, z)

$$X_{cm} = \sum \Delta m_i x_i / M$$

Let $i \to \infty$

$$x_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum \Delta m_i x_i}{M} = \frac{1}{M}\int x\,dm$$

$$y_{cm} = \frac{1}{M}\int y\,dm \qquad z_{cm} = \frac{1}{M}\int z\,dm.$$



$$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

(9)

CM of homogeneous symmetric objects lies on an axis of symmetry.

point sym. $\longrightarrow$ cm on point

plane sym $\longrightarrow$ cm on plane

line sym $\longrightarrow$ cm on line

Convenient to express mass in terms of local density.

$\rho(x,y,z)$ = mass/volume

$$dm = \rho(x,y,z)\,dv$$

$$M = \int \rho(x,y,z)\,dv$$

$$\boxed{\begin{aligned} x_{cm} &= \frac{1}{M}\int x\rho\,dv \\ y_{cm} &= \frac{1}{M}\int y\rho\,dv \\ z_{cm} &= \frac{1}{M}\int z\rho\,dv \end{aligned}}$$

$dv = dx\,dy\,dz$
element of volume
3D integral

If $\rho(x,y,z)$ = constant

$$x_{cm} = \frac{\rho}{M}\int x\,dv$$

CM $\Rightarrow$ 1'st moments of mass distribution of object.

Lecture 17, Blackboard #6