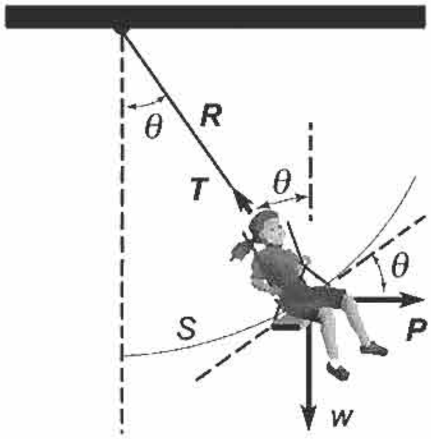
Example

A body of weight ω is supported on a swing of length R. A variable horizontal force $\vec{P}$, which starts from zero and gradually increases, pulls the swing over to a final angle θ_0. What is the work of $\vec{P}$?

Body moves slowly, $KE \equiv 0$.



A variable horizontal force *P* acts on a body while the displacement varies from zero to *s*.

$$P = T \sin \theta$$

$$\omega = T \cos \theta$$

$$\therefore P = \omega \tan \theta$$

$$W = \int \vec{P} \cdot \vec{dl} = \int P \cos\theta \, dl$$

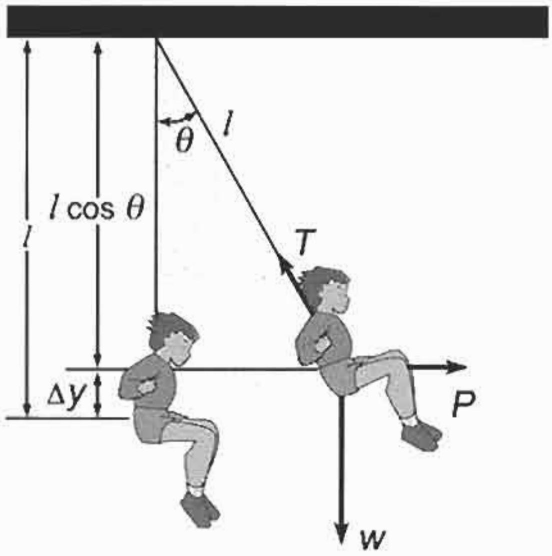
$$= \int_0^{\theta_0} \omega \tan\theta \cos\theta \; R \, d\theta$$

$$W = \omega R \int_0^{\theta_0} \sin\theta \, d\theta = \underbrace{\omega R (1 - \cos \theta_0)}_{\text{Increase in height of body.}}$$

$W = 0$, $\theta_0 = 0$

$W = \omega R$, $\theta_0 = \pi/2$

Consider all forces other than gravity:



$\Delta y = l(1 - \cos\theta)$.

$$W_{other} = W_P + W_T = \Delta K + \Delta U = \Delta E$$

$W_T = 0$ [$T \perp$ path of motion]

$\Delta K = 0$ [swing moved slowly!]

$$\therefore W_P = \Delta U = w\,\Delta y$$

$$\Delta y = l(1 - \cos\theta)$$

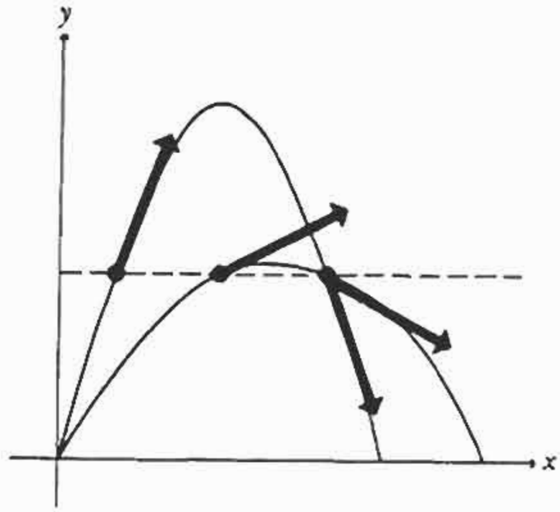
$$W_P = wl(1 - \cos\theta)$$

Example

Ballistic trajectories — only force is weight, neglect air resistance.

Two trajectories, same initial speed (i.e. same total energy) different launch angles.

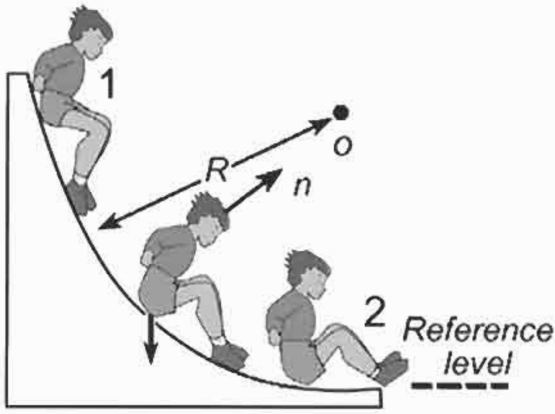
At points with the same elevation, PE is the same, $\therefore$ KE is the same and the speed is the same.



For the same initial speed, the speed is the same at all points at the same elevation.

Example:

A child slides down a curved playground slide that is a quadrant of a circle.
Start from rest. No friction.
What is speed at bottom?



A child sliding down a frictionless curved slide.

Cannot solve using constant acceleration, since $|a|$ changes during motion. Slope angle decreases.

Only other force than gravity, is the normal force $\vec{N}$.
It is $\perp$ to displacement, $\therefore W_{other} = 0$.

Let Point 1 $\Rightarrow$ starting point
2 $\Rightarrow$ bottom of slide.

$$K_2 + U_2 = K_1 + U_1$$

$$\frac{1}{2} m v_2^2 + 0 = 0 + mgR$$

$$v_2 = \sqrt{2gR}$$

If $R = 3.0\text{m}$, $v_2 = \sqrt{2 \times 9.80 \times 3.0} = 7.67 \text{ m/s}$.

What if friction present and $v_2 = 3.00 \text{ m/s}$. What is the work due to friction? Assume $m = 25 \text{ kg}$.

$$W_{other} = W_f$$

$$W_f = \left(\frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2\right) + (mgz_2 - mgz_1)$$

Total ME is decreased due to friction

$$= \frac{1}{2} \times 25 \times 3.0 - 0 + 0 - 25 \times 9.80 \times 3.0 = 112 - 735 = -623 \text{ J}$$

Conservation of Energy

- Discussed motion of a particle under the influence of gravity $\Longrightarrow$ conservation of mechanical energy.

- In many processes the sum of the kinetic and potential energies does not remain a constant. There are dissipative forces such as friction.

- The more general law of conservation of energy was established by including other forms of energy:

  - thermal energy
  - electrical
  - chemical
  - nuclear

- The changes in all forms of energy:

$$\Delta KE + \Delta U + \Delta(\text{all other forms}) \equiv 0$$

This is the Law of Conservation of Energy and is one of the most important principles of physics.

"The total energy is neither increased nor decreased in any process. Energy can be transformed from one form to another, and transferred from one body to another, but the total amount remains a constant."

- The validity of this law rests on experimental observations.

e.g. Newton's Laws fail in the submicroscopic world of atoms and nuclei. The Law of Conservation of Energy holds there and in every experiment so far tested. It is one of the great unifying principles of science.

Why is it important in the study of mechanics? It gives us another tool for solving complicated problems. The detailed solution of Newton's Laws of motion of an object are often difficult or impossible.

Using the conservation laws
- Energy
- Momentum
- Angular Momentum

allows us to obtain many interesting characteristics of a system and its motion easily!

We can treat in a more straightforward way cases involving forces which are not constant with position.

i.e. Spring force was easy, others are more complicated.

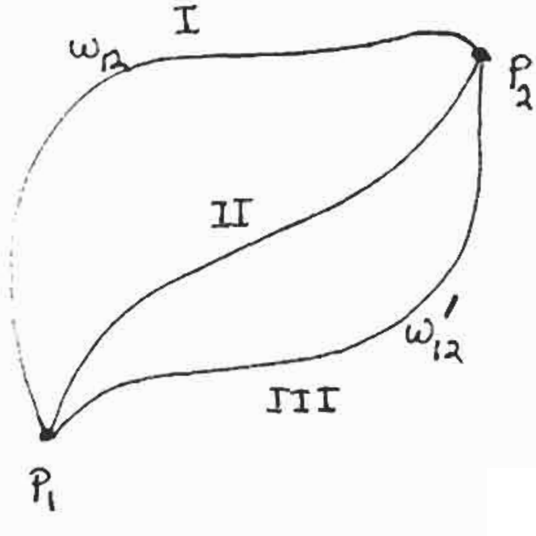
One notes that the information obtained is less than that contained in a complete time-dependent solution. The latter would tell us everything about the motion of the system. Time evolution of the system is normally only available by solving the dynamics.

Conservative Forces

For the gravitational force we obtained Conservation of Energy by summing the kinetic and potential energies:

$$E = K + \mathcal{U}(z) \qquad [\text{Constant - Conserved}]$$

What are the essential requirements for other such conservative forces, i.e. that a potential energy is to exist?



Consider a particle which moves from P_1 to P_2 with a force $\vec{F}$ acting on it. Assume $\vec{F}$ is a function only of position but does not depend explicitly on time. Since particle moves, force is a function of time but we assume that at any given location the force is the same no matter when the particle arrived there.

The work done by the force $\vec{F}$ on the particle

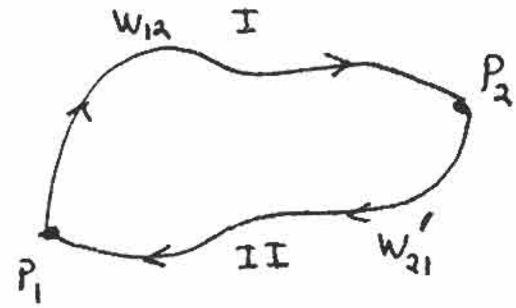
$$W_{12} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$ in moving from $1 \to 2$ along path - I

$$W_{12}' = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$ work done in moving from $1 \to 2$ along primed path - II.

<u>Definition</u>: $\vec{F}$ is a <u>conservative force</u> if the work depends only on the position of the points P_1 and P_2 but not on the shape of the path between P_1 and P_2.

i.e. $W_{12} = W_{12}'$ for any two paths.

<u>Closed Path - Round Trip</u>



Particle moves from $P_1 \to P_2$ and then back to P_1. If the force is conservative the total work is exactly zero for a round trip along a closed path.

$$W_{12} + W_{21}' = W_{12} - W_{12}' = 0$$

$$\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} + \int_{P_2}^{P_1} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} - \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} \equiv 0.$$

$$\oint \vec{F} \cdot d\vec{r} = 0$$

Line integral around a closed loop.

- Gravity is a conservative force.
- Spring is a conservative force.
- Friction is not a conservative force since it always opposes motion.

$$\oint \vec{F} \cdot d\vec{r} \neq 0 \qquad \text{[Friction]}$$

Work done against a non-conservative force is not recoverable ; it goes into heat or is dissipated in some other way. Work also depends on path length and not simply on the location of end points. [Macroscopic Viewpoint]

Potential Energy of Conservative Forces

PE can only be defined for conservative forces. Take a reference point P_0 to which is assigned a potential energy which has some value $\mathcal{U}(P_0)$ — any arbitrary number. Often convenient to take $\mathcal{U}(P_0) \equiv 0$.

Then any other point P has the potential energy which is given by:

$$\boxed{\mathcal{U}(P) = -\int_{P_0}^{P} \vec{F}\cdot d\vec{r} + \mathcal{U}(P_0)}$$

Evaluated along any path between P_0 and P since for conservative forces it is path independent.

Does this function for PE have the correct properties? Calculate the change in PE between the points P_1 and P_2.

$$\mathcal{U}(P_2) - \mathcal{U}(P_1) = -\int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} + \mathcal{U}(P_0) - \left[-\int_{P_0}^{P_1} \vec{F}\cdot d\vec{r} + \mathcal{U}(P_0)\right]$$

$$= -\int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} + \int_{P_0}^{P_1} \vec{F}\cdot d\vec{r}$$

$$= -\int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} - \int_{P_1}^{P_0} \vec{F}\cdot d\vec{r}$$

$$\mathcal{U}(P_2) - \mathcal{U}(P_1) = -\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$

The change in PE equals the <u>negative of the work done by the force</u> between the two points.

$\mathcal{U}(P_0)$: Drops out in final result. Choice of P_0 does not matter. In all applications only differences in potential are significant and we often use $\mathcal{U}(P_0) \equiv 0$.

<u>Mechanical Energy</u>

We had previously that the change in kinetic energy of the particle equals the work;

$$K_2 - K_1 = W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$

$\therefore$ For <u>any</u> conservative force we must have

$$K_2 - K_1 = \mathcal{U}(P_1) - \mathcal{U}(P_2)$$

$$K_2 + \mathcal{U}(P_2) = K_1 + \mathcal{U}(P_1)$$

$$E = K + \mathcal{U} = [\text{constant}]$$

$E \equiv$ Total Mechanical Energy

[Law of Conservation of Mechanical Energy]

PE: Gravity (Near Earth)

$$\vec{F} = -mg\,\hat{z}$$

$$\mathcal{U}(P) = -\int_{P_0}^{P} \vec{F}\cdot d\vec{r}\,' + \mathcal{U}(P_0)$$

Assume $\mathcal{U}(P_0) = 0$ for $P_0 \Rightarrow x=0,\ y=0,\ z=0$

$$\mathcal{U}(P) = -\cancel{\int_0^x F_x\,dx'} - \cancel{\int_0^y F_y\,dy'} - \int_0^z F_z\,dz'$$

$$= -\int_0^z F_z\,dz' = -\int_0^z (-mg)\,dz'$$

$$\mathcal{U}(z) = mgz$$

Total mechanical energy is

$$E = K + \mathcal{U} = \frac{1}{2}mv^2 + mgz.$$

The potential energy U = mgx of an 80-kg mass is shown versus height x near the surface of the earth.