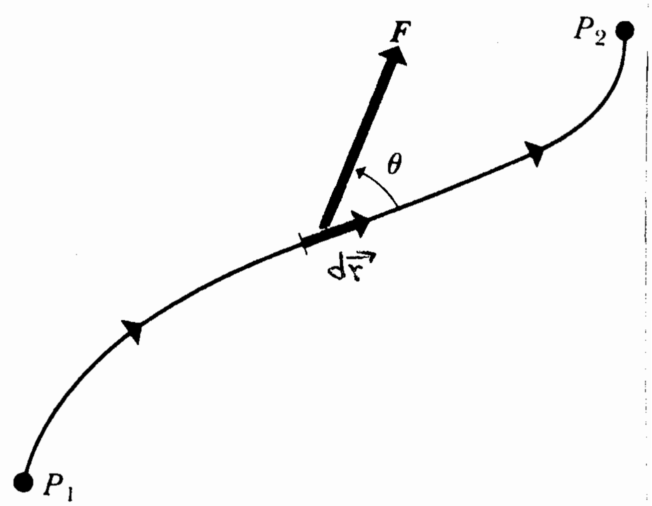
Kinetic Energy

Want to develop a relationship between the work done and the change in speed of a particle.

Particle moves from P_1 to P_2 under the action of a net force $\vec{F}$ (position).

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$



A particle moves along a curved path from point P1 to P2, acted on by a force *F* that varies in magnitude and direction

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

$$W = \int_{P_1}^{P_2} (F_x dx + F_y dy + F_z dz)$$

$$F_x = m a_x = m \frac{dv_x}{dt}$$

$$\int_{P_1}^{P_2} F_x dx = \int_{P_1}^{P_2} m \frac{dv_x}{dt} dx$$

v_x is a function of time.

Look at v_x as a function of position:

$$\frac{dv_x}{dt} = \frac{dv_x}{dx} \cdot \left(\frac{dx}{dt}\right) = \frac{dv_x}{dx} \cdot v_x = v_x \frac{dv_x}{dx}$$

$$\therefore \int_{P_1}^{P_2} F_x\,dx = \int_{P_1}^{P_2} m \frac{dv_x}{dt}\,dx = \int_{P_1}^{P_2} m\, v_x \frac{dv_x}{dx}\,dx$$

$$= \int_{P_1}^{P_2} m\, v_x\, dv_x = \frac{1}{2} m\, v_x^2 \Big|_{v_{x1}}^{v_{x2}}$$

$$= \frac{1}{2} m \left(v_{x2}^2 - v_{x1}^2 \right)$$

v_{x1} = velocity in x-direction at P_1
v_{x2} = velocity in x-direction at P_2.

Do the same for terms in y and z.

$$W = \frac{1}{2} m \left[v_{x2}^2 + v_{y2}^2 + v_{z2}^2 - \left(v_{x1}^2 + v_{y1}^2 + v_{z1}^2 \right) \right]$$

$$= \frac{1}{2} m \left(v_2^2 - v_1^2 \right)$$

$$\boxed{W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2}$$

Define: $\boxed{K = \frac{1}{2} m v^2}$ $\equiv$ Kinetic Energy of Particle

KE: Potential for a particle to do work by virtue of its velocity.

The work done on the particle by the <u>net force</u> equals the change in kinetic energy of the particle.

$$W = K_2 - K_1$$

or

$$\boxed{W = \Delta K} \implies \text{Work-Energy Theorem.}$$

For a particle $\vec{p} = m\vec{v}$ (Linear Momentum)

$$\therefore \quad K = \frac{1}{2m} p^2$$

Example:

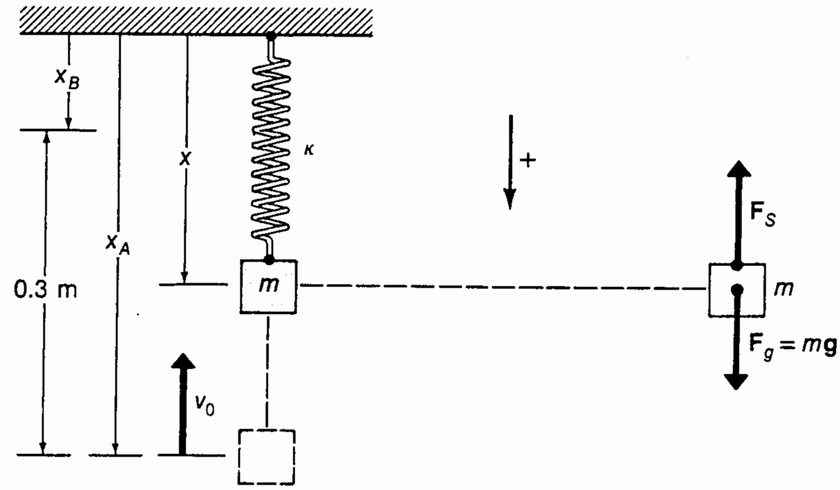
$m = 0.50$ kg.

$k = 50\,N/m$

$x_A = 0.50m$

$x_B = 0.20m$

$v_0 = 2.0m/s.$



Mass pulled down to x_A, released by imparting an upward velocity v_0. What is velocity of block at x_B ?

Net force on block:

$$F = F_g - F_s = mg - kx.$$

Work done, integrate $F\,dx$:

$$W(x_A \to x_B) = \int_{x_A}^{x_B} F\,dx = \int_{x_A}^{x_B} (mg - kx)\,dx = \left(mgx - \frac{1}{2}kx^2\right)\Big|_{x_A}^{x_B}$$

$$W = mg(x_B - x_A) - \frac{1}{2}k\,(x_B^2 - x_A^2)$$

$$= 0.5 \times 9.8\,(0.2 - 0.5) - \frac{1}{2} \times 50\left(0.2^2 - 0.5^2\right)$$

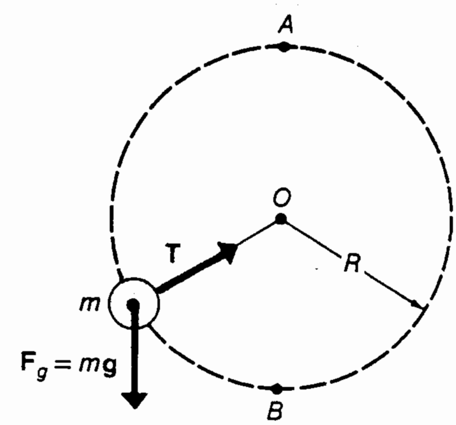
$$= -1.47 + 5.25 = 3.78\,J$$

Also $\quad W = \frac{1}{2}m v_B^2 - \frac{1}{2}m v_A^2$

$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = \left[(-2.0)^2 + \frac{2 \times 3.78}{0.5}\right] \Rightarrow v_B = \pm 4.37\ m/s$$

Example

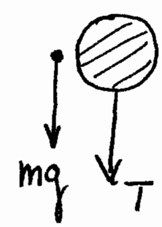
A rock of mass m is tied to the end of a string and whirled in a circular path in the vertical plane.



What is the minimum speed v_0 that the rock has at the bottom (B) if the rock is to pass the top (A) with the string remaining taut?

- Energy considerations alone not sufficient.
- Need to apply Newton's 2ND Law.

At the top the minimum speed occurs as tension vanishes, $T \rightarrow 0$.



Net downward force is

$$F_g = mg = m\frac{v_T^2}{R}$$

[centripetal acceleration in circular path]

$$v_T^2 = gR$$

$$W(A \rightarrow B) = mgh = K_B - K_A$$

$$mg(2R) = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_T^2$$

$$v_0^2 = v_T^2 + 4gR = gR + 4gR = 5gR$$

$$v_0 = \sqrt{5gR}$$

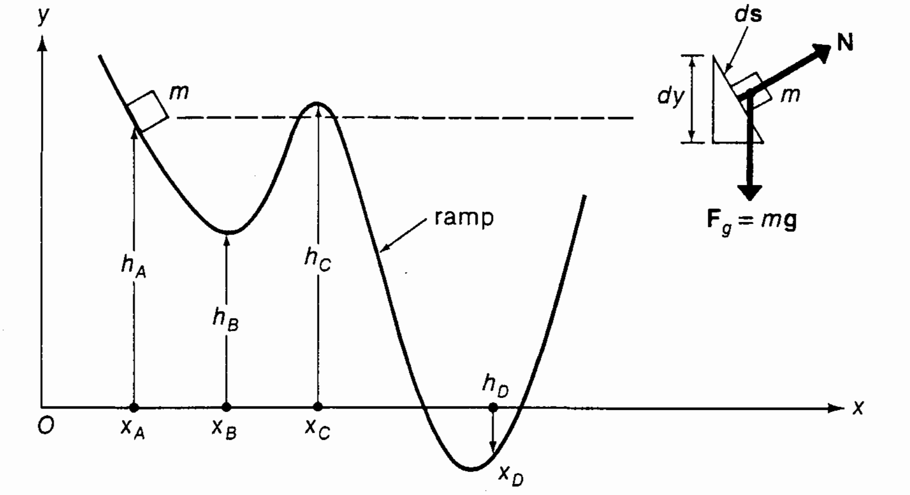
Example

$h_A = 7\,m$

$h_B = 4\,m$

$h_C = 7.2\,m$

$h_D = -1\,m$



$v_A = 3\,m/s$ downward and tangent to ramp.

What is speed of particle at $x = x_B, x_C, x_D$?

$\vec{N} \cdot \vec{ds} \equiv 0$, so work is done only by gravitational force.

$$W(A \rightarrow B) = \int_A^B \vec{F} \cdot \vec{ds} = \int_A^B (-mg)\,y$$

$$W = mg(h_A - h_B)$$

Also $W = K_2 - K_1 = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$

$$\therefore v_B^2 = v_A^2 + \frac{2W}{m} = v_A^2 + 2g(h_A - h_B)$$

$$= 3^2 + 2 \times 9.8\,(7-4)$$

$$= 67.8\ (m/s)^2$$

$$v_B = 8.23\ m/s.$$

How far up the ramp will particle go after passing $x = x_D$?

h_m will be reached at x_m where $v_m = 0$.

$0 = v_A^2 + 2g(h_A - h_m) \Rightarrow h_m = h_A + v_A^2/2g \qquad h_m = 7.46\,m$

Example

Car: $m = 1000\,kg$

$v_1 = 20\,m/s$

$v_2 = 30\,m/s$

$$W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$= \frac{1}{2} m \left(v_2^2 - v_1^2\right)$$

$$= \frac{1}{2} \times 1000 \times (900 - 400)$$

$$= 2.5 \times 10^5\ J$$

Gravitational Potential Energy

KE: Represents the capacity of a particle to do work by virtue of its velocity.

PE: Represents the capacity of a particle to do work by virtue of its position in space.

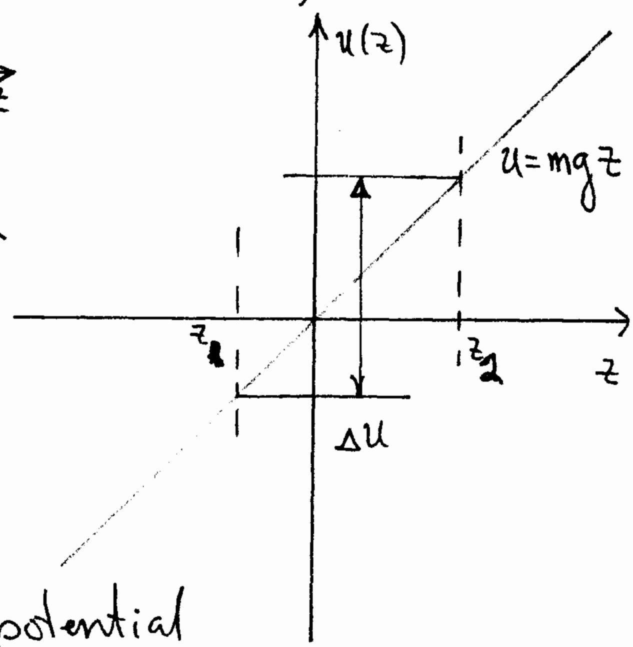
Consider a constant force of gravity

$$F_z = -mg,$$

acting on a particle which undergoes a displacement from $(x_1 y_1 z_1)$ to $(x_2 y_2 z_2)$. The force does an amount of work;

$$W_{gr} = -g(z_2 - z_1) = \int_{z_1}^{z_2} \vec{F} \cdot d\vec{z}$$

$$= -U(z_2) + U(z_1) = -\Delta U$$



where,

$$U(z) = mgz,$$

is called the gravitational potential energy.

The change in potential energy between the points z_1 and z_2 is the negative of the work done by gravity on the particle.

Gravitational Potential Energy:

$\Rightarrow$ Capacity of a particle to do work by virtue of its height above the surface of an attracting mass (earth).

<u>If</u> the only force acting is gravity, then using the work-energy theorem,

$$W_{gr} = K_2 - K_1$$

and

$$W_{gr} = -\mathcal{U}(z_2) + \mathcal{U}(z_1)$$

$$\therefore \quad K_1 + \mathcal{U}(z_1) = K_2 + \mathcal{U}(z_2)$$

$$\therefore \quad K + \mathcal{U}(z) \equiv \text{constant of the motion.}$$

$\boxed{E = K + \mathcal{U}(z)}$: Mechanical Energy.

$\hookrightarrow$ Represents the total capacity of a particle to do work by virtue of both its <u>velocity</u> and its position.

<u>If</u> only force acting is gravity;

$$\boxed{E = K + U(z) = \text{constant}}$$

Law of Conservation of Mechanical Energy

$$\boxed{E = \frac{1}{2}mv^2 + mgz = \text{constant}}$$

z - increases $\longrightarrow$ v - decreases

z - decreases $\longrightarrow$ v - increases

Consider two different positions:

$$\frac{1}{2}mv_1^2 + mgz_1 = \frac{1}{2}mv_2^2 + mgz_2$$

$$v_1^2 + 2gz_1 = v_2^2 + 2gz_2.$$

$$v_2^2 - v_1^2 = 2g(z_1 - z_2)$$

$$= -2g(z_2 - z_1)$$

Recall constant acceleration kinematics results.

Gravity + other forces

Suppose other forces act besides gravity: friction, etc.
Let W_{other} represent work by all forces other than gravity.
Total work by all forces equals change in KE.

$$W = W_{grav} + W_{other} = K_2 - K_1 = \Delta K$$

$$W_{other} - (mgz_2 - mgz_1) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$W_{other} = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) + (mgz_2 - mgz_1)$$

$$= \Delta K + \Delta U$$

ΔU ← change in PE

ΔK ← change in KE.

Also

$$W_{other} = \left(\frac{1}{2}mv_2^2 + mgz_2\right) - \left(\frac{1}{2}mv_1^2 + mgz_1\right)$$

$$= (K_2 + U_2) - (K_1 + U_1)$$

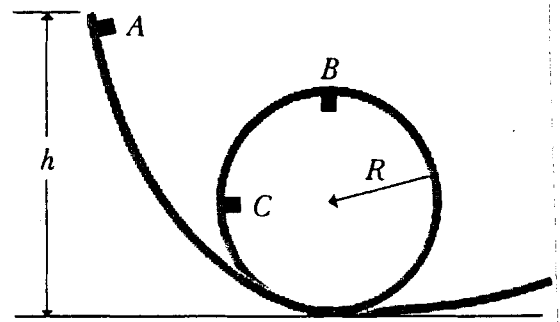
$$= E_2 - E_1 = \Delta E$$

The work done by all other forces acting on the body, with the exception of the gravitational force, equals the change in the total mechanical energy of the body.

$W_{other} > 0 \Rightarrow$ Mechanical Energy increases
$W_{other} < 0 \Rightarrow$ " " decreases

Example: Loop-the-Loop

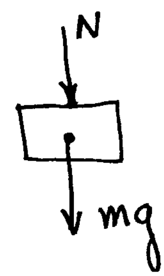
What is the minimum height h (in terms of R) such that an object moves around the loop without falling off the top (B)?



Top of Hoop:

velocity: v_T

$$F_c = mg + N = \frac{mv_T^2}{R}$$



Minimum $v_T \Rightarrow N \equiv 0$. [contact force vanishes]

$$\therefore \; v_T = \sqrt{gR}$$

Represents minimum velocity to make the loop.

Conservation of Energy

$$mgH + 0 = mg(2R) + \frac{1}{2}mv_T^2$$

$$v_T^2 = 2g\left[H - 2R\right]$$

$$v_T = \sqrt{2g(H-2R)}$$

[Energy Considerations]

$$\therefore \; gR = 2g(H-2R) \Longrightarrow H = \frac{5}{2}R$$