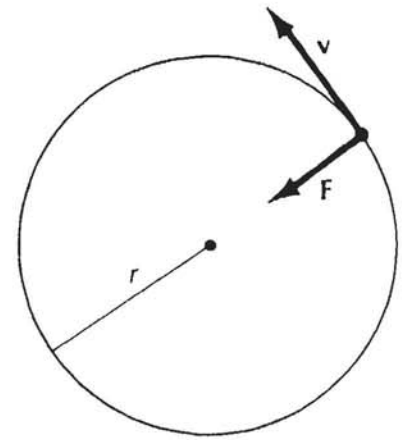


## Dynamics of Uniform Circular Motion

13-16

For a particle moving with speed  $v$  in a circle of radius  $R$ , the centripetal acceleration is

$$a_c = \frac{v^2}{R}$$



Centripetal force for a particle in uniform circular motion.

If the particle has a mass  $m$ , the required force is given by

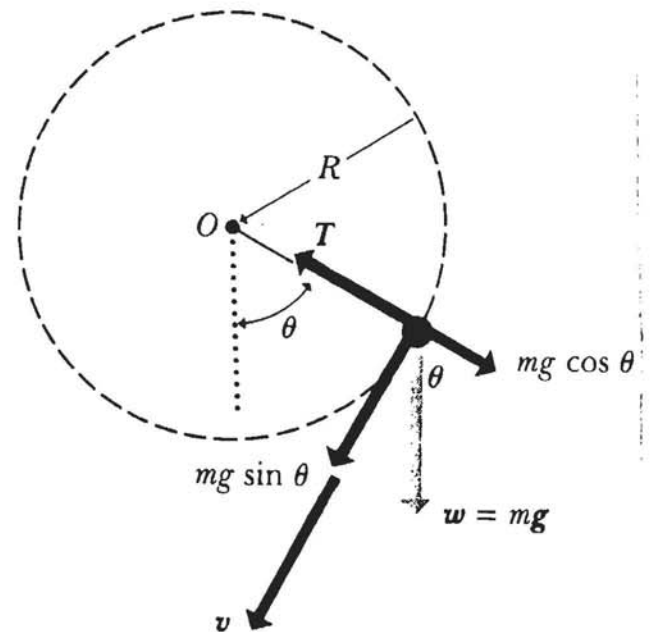
$$F_c = ma_c = \frac{mv^2}{R}$$

Whenever a particle moves in a circle with a speed  $v$ , this force must be provided by some external agent.

## Motion in a Vertical Circle.

13-17

- Ball whirled in a vertical circle about point O.
- Motion circular but is not uniform.
- Speed increases on the way down, decreases on the way up.
- $v$  changes continuously around the path.
- $\therefore$  We must have  $a_{\perp}$  and  $a_{\parallel}$
- Forces on ball are gravity and tension.



Forces on a body whirling in a vertical circle with center at O.

$$F_{\parallel} = mg \sin \theta \quad (1)$$

$$F_{\perp} = T - mg \cos \theta \quad (2)$$

The tangential acceleration:

$$a_{\parallel} = \frac{F_{\parallel}}{m} = g \sin \theta \quad (3)$$

$$a_{\perp} = \frac{F_{\perp}}{m} = \frac{T - mg \cos \theta}{m} = \frac{v^2}{R} \quad (4)$$

Solve (4) for  $T = m \left( \frac{v^2}{R} + g \cos \theta \right)$

Lowest Point:  $\theta = 0$

$F_{\parallel} = 0$ ,  $a_{\parallel} = 0$ , acceleration is purely radial.

$$T = m \left( \frac{v^2}{R} + g \right)$$

Highest Point:  $\theta = 180^\circ$

- acceleration purely radial

$$T = m \left( \frac{v^2}{R} - g \right)$$

If speed equals a critical value  $v_c$ , tension vanishes,  $T = 0$ .

$$0 = m \left( \frac{v_c^2}{R} - g \right)$$

$$v_c = \sqrt{Rg}$$

The speed at any point can be determined from energy considerations given a value at some initial point.

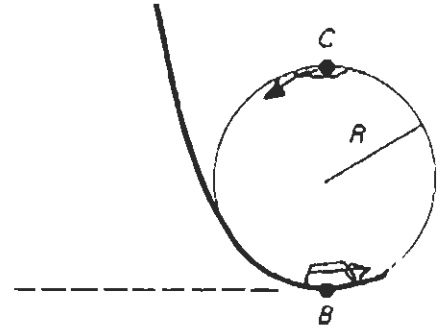
## Example : Loop-the-loop

13-19

$$m = 75 \text{ kg}$$

$$R = 10 \text{ m}$$

$$v_0 = 15 \text{ m/s}$$



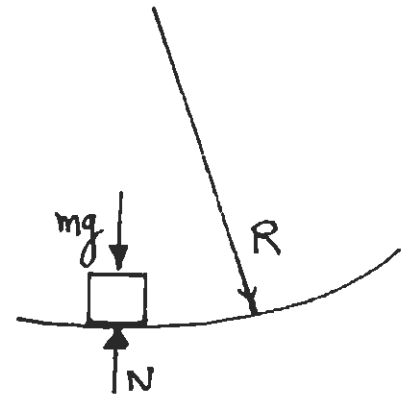
Man in a car executes motion in a vertical loop at a constant speed  $v_0$ .

Calculate the apparent force the man feels pressing against him at the bottom and top of the loop. i.e. Calculate the apparent weight.

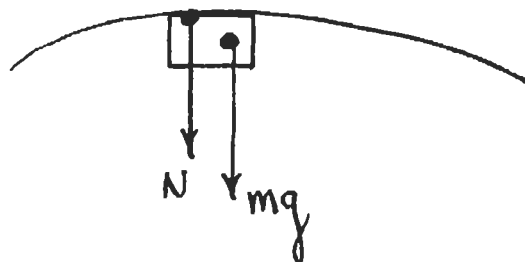
Bottom

$$N - mg = \frac{mv_0^2}{R} \quad (\text{circle})$$

↑ Normal force pushing on man (weight)



$$\begin{aligned} N &= m \left( g + \frac{v_0^2}{R} \right) = mg \left[ 1 + \frac{1}{g} \frac{v_0^2}{R} \right] \\ &= mg \left[ 1 + \frac{1}{9.81} \frac{15^2}{10} \right] = mg [1 + 2.29] \\ &= 3.29(mg) \end{aligned}$$

Top

$$N + mg = \frac{mv_0^2}{R}$$

$$\begin{aligned} N &= mg \left[ \frac{v_0^2}{Rg} - 1 \right] \\ &= mg [2.29 - 1] \\ &= 1.29(mg) \end{aligned}$$

What happens if  $v_0$  is decreased?

In particular what if

$$\begin{aligned} \frac{v_0^2}{Rg} = 1 &\quad \Rightarrow \quad v_0 = \sqrt{gR} = \sqrt{9.81 \times 10} \\ &= 9.90 \text{ m/s} \end{aligned}$$

$N \equiv 0$  ; No normal force.

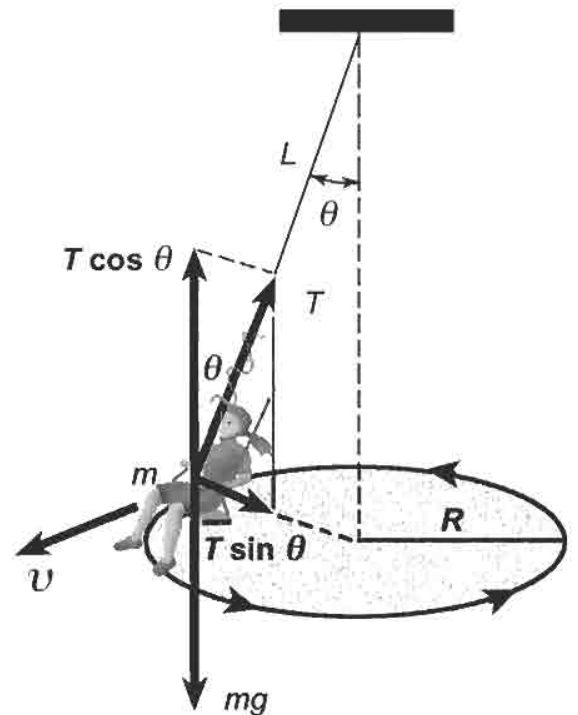
What if  $v_0 < \sqrt{gR}$  ??

## Conical Pendulum

Object at end of rope of length  $l$ . Total mass is  $m$ . Object swings in a horizontal circle of radius  $R$  at constant speed  $v$ .

Assume period of revolution is  $\tau$ .

Find tension in rope  $T$  and angle  $\theta$ .



A child's indoor swing.

$$\sum \vec{F} = m\vec{a}$$

$$T \sin \theta = m \frac{v^2}{R} \quad (1)$$

$$T \cos \theta - mg = 0 \quad (2)$$

$$\tan \theta = \frac{v^2}{Rg} \quad (3)$$

$$R = L \sin \theta$$

$$v = \frac{2\pi L \sin \theta}{\tau}$$

Then Eq. (3) becomes

$$\cos \theta = \frac{g \tau^2}{4\pi^2 L}$$

$$\text{or } \tau = 2\pi \sqrt{\frac{L}{g \cos \theta}}$$

## Work and Energy

14-1

- Newton's laws of motion relating acceleration to forces acting enable us to predict future values of the position and velocity of a particle.
- In this lecture we will see how to relate force to particle motion in a second way. The scalar product of force and displacement defines work. The product of mass and the square of a particle's velocity gives twice the kinetic energy.
- Combining work and K.E. we derive the work-energy principle. This principle plays a role which is analogous to that of Newton's Second Law.

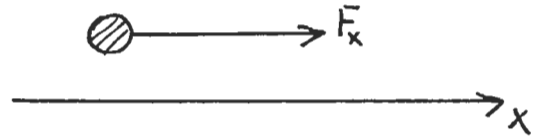
Work-Energy Principle  $\rightarrow$  Conservation of Energy  
 $\Delta t$  (Force) = Linear Impulse  $\rightarrow$  Conservation of Momentum  
Force  $\otimes$  Position  $\rightarrow$  Torque  $\rightarrow$  Conservation of Angular Momentum

## Conservation laws $\leftrightarrow$ Symmetry

- Conservation laws are closely connected to a symmetry associated with our mathematical description of the physics:

Momentum  $\longleftrightarrow$  Translation in Space  
Energy  $\longleftrightarrow$  Translation in Time  
Angular Momentum  $\longleftrightarrow$  Invariance to Rotations  
Electric Charge  $\longleftrightarrow$  QM shift in Phase

Work - 1 Dimension



Force  $F_x$  acting on a particle moving along  $x$  does an amount of work :

$$W = F_x \Delta x$$

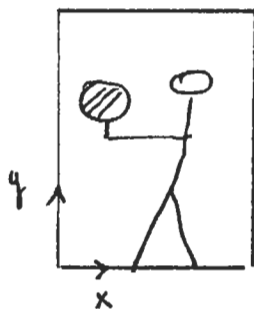
Work done by the force  $F_x$ .  
Work is a scalar quantity.

$$[W] = N \cdot m = \underline{\text{Joules}}$$

- $W > 0$  • Force and displacement are in the same direction
- $< 0$  • Force and displacement are opposed
- $< 0$  • Represents work done by the particle.

- when several forces act, add individual contributions.

Work is dependent on the frame of reference.



$v = \text{const.}$

$W = 0$   $\Delta x = 0$   
Elevator Frame



$v = \text{const}$

$W \neq 0$   $\Delta x \neq 0$   
laboratory Frame



## Variable Force/Work.

14-3

$$F_x = F_x(x)$$

- Force is a function of position, spring, gravity, etc.

What is  $w(a \rightarrow b)$  in moving a particle from  $x=a$  to  $x=b$ .

Divide the total displacement into a large number of small intervals  $\Delta x$ .

For each interval

$$\Delta W_i = F_x(x_i) \Delta x \Rightarrow \text{Area of a rectangle of height } F_x \text{ and width } \Delta x.$$

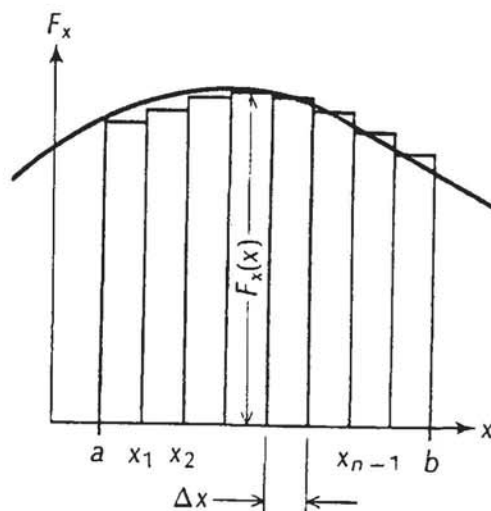
Total work from  $x=a$  to  $x=b$  is the sum of all such intervals.

$$W = \sum_{i=0}^{i=n-1} \Delta W_i = \sum_{i=0}^{i=n-1} F_x(x_i) \Delta x$$

In the limiting case  $\Delta x \rightarrow 0$   
 $n \rightarrow \infty$

$$W = \lim_{\Delta x \rightarrow 0} \sum_i F_x(x_i) \Delta x$$

$$W = \int_a^b \underbrace{F_x(x)}_{\text{integrand}} dx$$



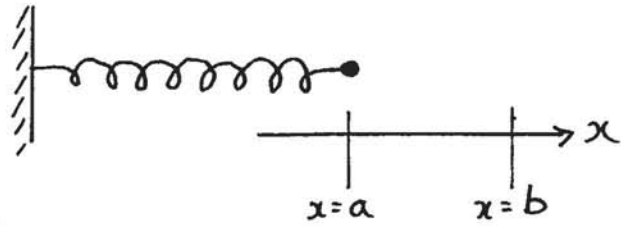
The total displacement from  $a$  to  $b$  has been divided into small intervals of  $\Delta x$ .

[Definite Integral]

Work  $\equiv$  Area bounded by the curve  $F_x(x)$  and the lines  $x=a$  and  $x=b$  and the  $x$ -axis

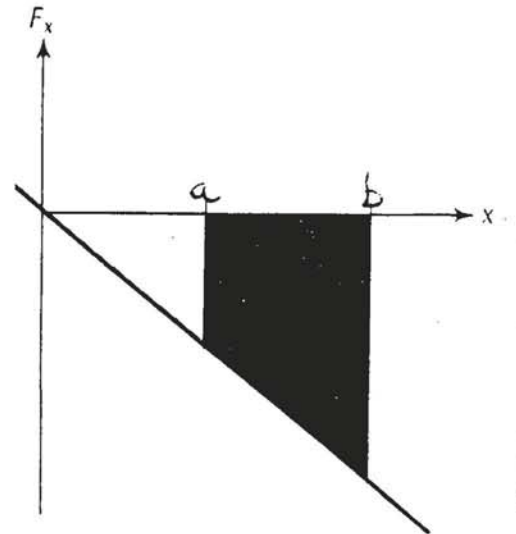
### Example: Spring Force

$$F(x) = -kx$$



How much work is needed to move a spring (fixed at one end) from  $x=a$  to  $x=b$ ?

$$\begin{aligned} W &= \int_a^b F(x) dx \\ &= \int_a^b (-kx) dx \\ &= \left. -\frac{kx^2}{2} \right|_a^b \\ &= -\frac{k}{2} (b^2 - a^2) \end{aligned}$$



The straight line is a plot of the function  $F_x(x) = -kx$ . The quadrilateral area to the right of  $a$ , and the left of  $b$  represents the work.

$W < 0$ , work is done on the spring !!

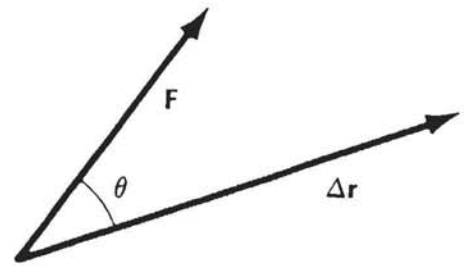
## Work - 3 Dimensions

14-5

In general:

$$W = \vec{F} \cdot \Delta \vec{r}$$

↑ displacement.  
↑ constant force



A constant force  $F$  acts during a displacement  $\Delta r$ .

$$= F (\Delta r) \cos \theta$$

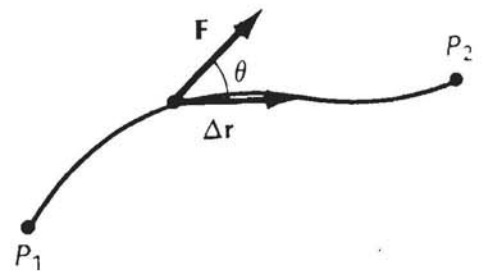
$$W = 0 \text{ if } \vec{F} \perp \Delta \vec{r}$$

$$W = F_x (\Delta x) + F_y (\Delta y) + F_z (\Delta z)$$

[Using components]

IF  $F = F(x)$  is not constant,

$$\Delta W = F(\Delta r) \cos \theta$$



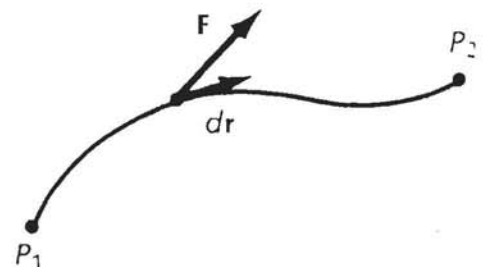
A force  $F$  acts during a small displacement  $\Delta r$ .

Limiting case  $\Delta r \rightarrow 0$

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} \quad [\text{Line-Integral}]$$

$$= \int_{P_1}^{P_2} (F_x dx + F_y dy + F_z dz)$$

$$W = \int_{P_1}^{P_2} F_x dx + \int_{P_1}^{P_2} F_y dy + \int_{P_1}^{P_2} F_z dz$$



The infinitesimal displacement  $dr$  is tangent to the path.

## Example: Gravitational Force

14-6

$$\left. \begin{array}{l} F_x = 0 \\ F_y = 0 \\ F_z = -mg \end{array} \right\} \begin{array}{l} \text{Force of Gravity} \\ \text{close to earth's} \\ \text{surface.} \end{array}$$

$$W = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$$

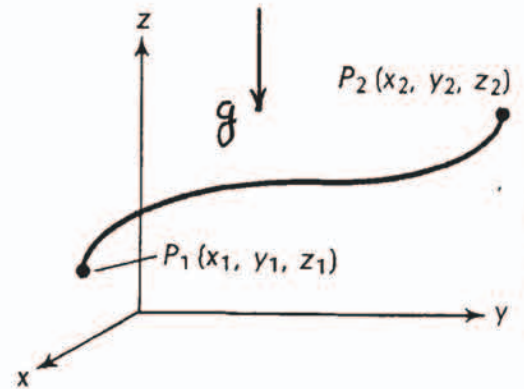
$$W = \int_{z_1}^{z_2} -mg dz = -mg(z_2 - z_1)$$

$$W = -mg(\Delta z)$$

↑ change in height.

Work done by gravity depends on the vertical separation between  $P_1$  and  $P_2$ .

NB: complication or details of the motion do not count — only  $(\Delta z)$  !!



Path of a particle. The z axis is directed vertically upward.

## Example

14-7

A block is pushed along a horizontal plane at constant velocity. Coefficient of sliding friction is  $\mu_k$ . How much work is done in a distance  $s$  by the force  $F$ ?

$$F \cos \theta - f = 0 \quad (\text{constant } v)$$
$$-F \sin \theta + N - mg = 0$$

$$f = \mu_k N = \mu_k (F \sin \theta + mg)$$

$$\therefore F \cos \theta - \mu_k (F \sin \theta + mg) = 0$$

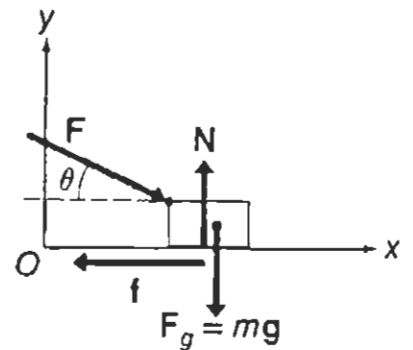
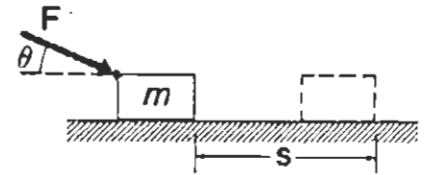
$$\therefore F = \frac{\mu_k mg}{\cos \theta - \mu_k \sin \theta}$$

$$W_F = \vec{F} \cdot d\vec{s} \quad [\text{Constant Force}]$$

$$= F s \cos \theta = \frac{\mu_k mg s}{1 - \mu_k \tan \theta}$$

Q: What happens when  $\mu_k \tan \theta = 1$  ?

$$\text{i.e. } \tan \theta = \frac{1}{\mu_k} \quad W \rightarrow \infty \quad ???$$



$$f = F \cos \theta$$

$$= \frac{\mu_k mg \cos \theta}{\cos \theta - \mu_k \sin \theta}$$

$$W_f = \vec{f} \cdot d\vec{s}$$

$$= \frac{-\mu_k mgs \cos \theta}{\cos \theta - \mu_k \sin \theta} \quad [\vec{f} \text{ and } d\vec{s} \text{ are opposite}]$$

$$W_N = 0 \quad \text{Since } \vec{N} \text{ and } d\vec{s} \text{ are } \perp.$$

$$W_{mg} = 0 \quad \checkmark \quad m\vec{g} \text{ and } d\vec{s} \text{ are } \perp$$

Total work on object:

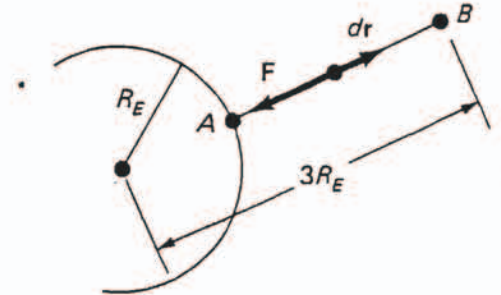
$$W = W_F + W_f + W_N \stackrel{+W_{mg}}{\equiv} 0.$$

True since object moves at constant velocity,  
net force on object  $\vec{F}_R \equiv 0$ .

## Example: Work Done on an Astronaut

14-8

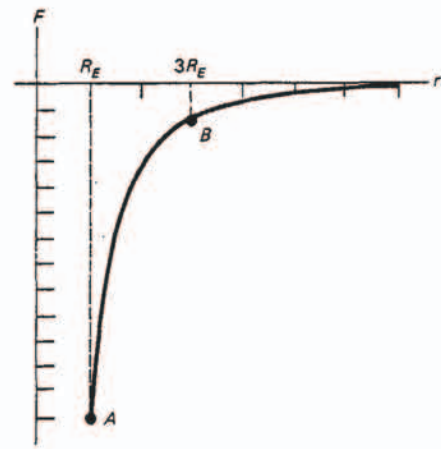
What is the work done by the force of gravity on an 80-kg astronaut in a displacement from point A at the earth's surface to point B whose altitude is  $2R_E$  (earth radii).



$$F = -\frac{GM_E m}{r^2} \hat{r}$$

$$W = -\int_{R_E}^{3R_E} \frac{GM_E m}{r^2} dr$$

$$\int \frac{1}{r^2} dr = -\frac{1}{r}$$



Figure

The force acting on an object that is moving away from the earth yields the negative shaded area.

$$W = +\frac{GM_E m}{r} \Big|_{R_E}^{3R_E} = GM_E m \left( \frac{1}{3R_E} - \frac{1}{R_E} \right)$$

$$= -\frac{2}{3} \frac{GM_E m}{R_E} = -\frac{2}{3} mg R_E$$

Negative work: Force directed towards earth, displacement is away from earth.

$$W = -3.34 \times 10^9 \text{ J.}$$