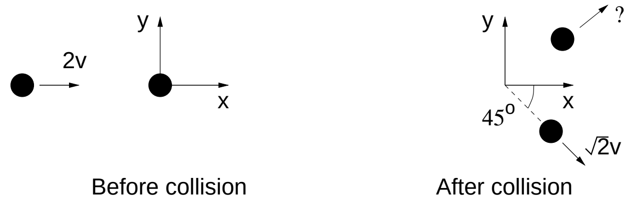
Problem 1: 2D collision (15 pts)

A particle of mass m collides with a second particle of mass m. Before the collision, the first particle is moving in the x-direction with a speed $2v$ and the second particle is at rest. After the collision, the second particle is moving in the direction $45°$ below the x-axis and with a speed $\sqrt{2}v$.



(a) Find the velocity of the first particle after the collision. (*ie* find the x- and y-components of the velocity.)

(b) Find the total kinetic energy of the two particles before and after the collision.

(c) Is the collision elastic or inelastic?

Solution:

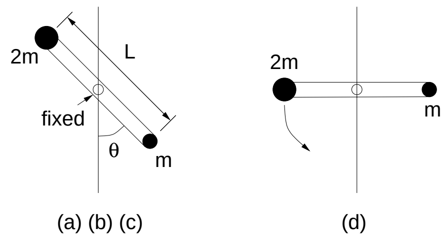
(a) The velocity of the second particle after the collision is $(v_{2x}, v_{2y}) = (v, -v)$. From momentum conservation in x-direction $2vm = v_{1x}m + v_{2x}m$, we find $\boxed{v_{1x} = v}$. From momentum conservation in y-direction $0 = v_{1y}m + v_{2y}m$, we find $\boxed{v_{1y} = v}$.

(b) Before: $K = \frac{1}{2}m(2v)^2 = \boxed{2mv^2}$.
After: $K = \frac{1}{2}m(v_{1x}^2 + v_{1y}^2) + \frac{1}{2}m(v_{2x}^2 + v_{2y}^2) = \boxed{2mv^2}$

(c) The collision is elastic.

Problem 2: Dynamics (15 pts)

Two balls of mass m and $2m$ are connected by a rod of length L. The mass of the rod is small and can be treated as zero. The size of the balls can also be neglected. We also assume the center of the rod is fixed, but the rod can rotate about its center in the vertical plane without friction.



(a) Find the center of mass of the two balls. That is find the distance between the center of mass and the ball of mass $2m$.

(b) Find the moment of inertia of the two balls about the rotation axis.

(c) Find the gravity induced angular acceleration of the rod when the angle between the rod and the vertical line is θ as shown.

(d) If the rod starts its swing from the horizontal position, what is the angular velocity of the rod when it reaches the vertical position?

Solution:

(a) Let d be the distance between the center of mass and the ball of mass $2m$. We have $(2m)d = m(L - d)$. We find $\boxed{d = L/3}$.

(b) Moment of inertia about the axis (the center of the rod): $I = 2m(L/2)^2 + m(L/2)^2 = \boxed{\frac{3}{4}mL^2}$.

(c) Torque about the axis (the center of the rod): $T = 2mg\frac{L}{2}\sin\theta - mg\frac{L}{2}\sin\theta$. Angular acceleration $= T/I = \boxed{\frac{2g}{3L}\sin\theta}$.

(d) Change in the potential energy $\Delta U = 2mg(L/2) - mg(L/2)$. From the energy conservation $\Delta U = K = I\omega^2/2$, we find the angular velocity $\omega = \sqrt{\frac{2\Delta U}{I}} = \boxed{\sqrt{\frac{4g}{3L}}}$.

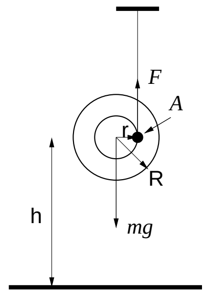
Problem 3: Dynamics (15 pts)

A yo-yo can be treated as a solid disk of mass m, radius R and thickness d. A string is wrapped around a small axis of radius r in the center of the yo-yo. The mass of the string can be ignored. Before we release the yo-yo, the string is stretched in the vertical direction. Initially, the yo-yo is at rest and is a distance h above the ground.
After the yo-yo is released:

(a) Find the angular acceleration of the yo-yo.

(b) Find the acceleration of the center of the yo-yo.

(c) Find the tension in the string.



Solution:

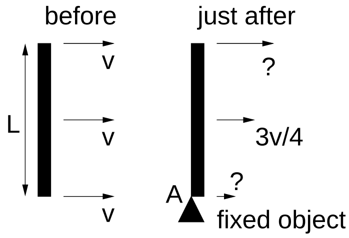
(a) Choosing A as the reference point, the torque due to the gravity mg is $T = mgr$. (The string tension F does not contribute to the torque about point A.) The moment of inertia about point A is $I = \frac{1}{2}mR^2 + mr^2$. The angular acceleration$= T/I = \boxed{\frac{gr}{\frac{1}{2}R^2+r^2}}$.

(b) The acceleration of the center of the yo-yo $a = \boxed{\frac{gr^2}{\frac{1}{2}R^2+r^2}}$

(c) Let F be the tension in the string. We have $ma = mg - F$. Thus $F = mg - ma = \boxed{mg\frac{\frac{1}{2}R^2}{\frac{1}{2}R^2+r^2}}$.

Problem 4: Collision (15 pts)

Initially, a rod of mass m and length L moves without rotation on a frictionless surface in a direction perpendicular to the rod. The speed of the rod is v. At time $t = 0$, one end of the rod collides with (or brushes over) a fixed object. Just after the collision, the rod is still parallel to the rod before the collision and the center of the rod still moves in the same direction as before. But the speed of the center of the rod is reduced to $\frac{3}{4}v$



(a) Find the angular velocity ω of the rod just after the collision. (Hint: the total angular momentum about the collision point A is conserved during the collision.)

(b) Find the total kinetic energy of the rod after the collision.

(c) Find the speeds of the two ends of the rod just after the collision.

Solution:

(a) Angular momentum about point A:
Before collision: $mv\frac{L}{2}$
After collision: $m\frac{3v}{4}\frac{L}{2} + I_c\omega$.
(After collision, the angular momentum contains both contributions from the orbital motion of the center of mass and the spinning motion about the center of mass.)
$I_c = \frac{1}{12}mL^2$ is the moment of inertia about the center of mass of the rod.
From the conservation of the angular momentum $mv\frac{L}{2} = m\frac{3v}{4}\frac{L}{2} + I_c\omega$, we find the angular velocity $\omega = \frac{mv\frac{L}{2} - m\frac{3v}{4}\frac{L}{2}}{I_c} = \boxed{\frac{3v}{2L}}$.

(b) Before collision: $K = \boxed{\frac{1}{2}mv^2}$.

After collision: $K = \frac{1}{2}m(\frac{3v}{4})^2 + \frac{1}{2}I_c\omega^2 = \boxed{\frac{3}{8}mv^2}$.

(c) Speed of the top end: $\frac{3}{4}v + \frac{L}{2}\omega = \boxed{\frac{3}{2}v}$.

Speed of the bottom end: $\frac{3}{4}v - \frac{L}{2}\omega = \boxed{0}$.