HW Solutions # 7 - 8.01 MIT - Prof. Kowalski

Potential Energy Curves, Momentum and Center of Mass.

1) *7.61*

We use conservation of energy

$$K_i + U_i + W_{other} = E_i + W_{other} = E_f = K_f + U_f \qquad (1)$$

$$\frac{1}{2}mv_1^2 + mgy_1 + W_{other} = \frac{1}{2}mv_2^2 + mgy_2 \qquad (2)$$

I will measure the gravitation potential energy with respect to the ground.

a) First obtain the kinetic energy - denote it as K_h- as if he stepped off the platform: When he stepped of a distance h $\leq$ d, W_{other}=0 because of negligible air resistance the energy conservation implies:

$$0 + mgh = K_h + 0 \qquad (3)$$

For the case he slides down the pole W_{other} after moving down the distance d is just $f_{ave} d \cos 180^{\circ}$(force opposes the motion $\Rightarrow$ W is $<$ 0):

$$W_{other} = -f_{ave} d \qquad (4)$$

The next step is to calculate Kinetic energy when he slides down the pole and as it's said make it equal to K_h:

$$0 + mgd + W_{other} = K_d + 0 \qquad (5)$$

Combining equation (4) with equations (2) and (3) and $K_d = K_h$:

$$mgd - f_{ave} d = mgh \qquad (6)$$

Therefore:

$$\boxed{f_{ave} = mg(1 - \frac{h}{d})}$$

When $h = d, f = 0$; as expected: when there is no friction sliding down the pole has the same effect as stepping off the platform.

And when h=0 we expect $f_{ave} = mg$ which is consistent with what we derived.

b) Plugging the numbers give in the problem into the boxed equation:

$$f_{ave} = 75.80 \times (1 - \frac{1}{2.5}) = 441 \text{ N.} \tag{7}$$

c) Let's denote v at position y as $v(y)$ Again setting up the equation (1) -energy conservation- :

$$v_1 = 0 \quad y_1 = d \quad W_{other} = -f_{ave}(d - y) \tag{8}$$

$$v_2 = v(y) \quad y_2 = y \tag{9}$$

$$\therefore \quad 0 + mgd - mgf_{ave}(d - y) = \frac{1}{2}mv(y)^2 + mgy \tag{10}$$

$$\frac{1}{2}mv(y)^2 = (mg - f_{ave})(d - y) \tag{11}$$

Using the boxed equation for f_{ave} we get:

$$\frac{1}{2}mv(y)^2 = mg(\frac{h}{d})(d - y) = mgh(1 - \frac{y}{d}) \tag{12}$$

From which

$$\boxed{v(y) = \sqrt{2gh(1 - \frac{y}{d})}}$$

When $y = 0, \ v = \sqrt{2gh}$, which is the original condition. When $y = d, \ v = 0$: the firman is at the top of the pole.

2) *7.68*

`Please refer to figure 7.40 p.279.`

I use the index θ for the energy of the block at θ and index 0 for the initial point and $\boldsymbol{l}$ the length the spring is **stretched**.I will measure the gravitation potential energy with respect to the height of the dashed horizontal bar indicated in the figure 7.40. The energy conservation:

$$K_\theta + U_\theta = K_0 + U_0 + W_{other} \tag{13}$$

$$\frac{1}{2}mv_\theta^2 + \frac{1}{2}kl_\theta^2 + mgy_\theta = \frac{1}{2}mv_0^2 + \frac{1}{2}kl_0^2 + mgy_0 + W_{other} \tag{14}$$

I the block is moved slowly, the kinetic energy can be taken as constant:

$$K_0 = K_\theta = K \tag{15}$$

$$v_0 = v_\theta \quad l_0 = 0 \quad y_0 = 0 \tag{16}$$

$$l_\theta = a\theta \ \ (the\ length\ of\ the\ arc) \qquad y_\theta = a\sin\theta \tag{17}$$

W_{other} here is W_F and kinetic energy is same on both sides:

$$K + 0 + 0 + W_F = K + \frac{1}{2}k(a\theta)^2 + mga\sin\theta \tag{18}$$

$$\boxed{W_F = \frac{1}{2}k(a\theta)^2 + mga\sin\theta}$$

3) ***7.76***

Using energy conservation:

$$K_1 + U_1 + W_{other} = K_2 + U_2 \tag{19}$$

Which U includes both gravitational and elastic potential energy $(U = \frac{1}{2}kx^2 + mgy)$:

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 + mgy_1 + W_{other} = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2 + mgy_2 \tag{20}$$

Where x is compression of spring along incline, y is height, s is distance travelled along the ramp. Measure the gravitation potential energy with respect to the height of the horizontal bar passing from its initial position(when it's compressed).

$$v_1 = 0 \quad x_1 = x \quad y_1 = 0 \tag{21}$$

$$x_2 = 0 \quad y_2 = s \sin \alpha \tag{22}$$

($x_2 = 0$ because the box will leave the spring and there will be no energy associated with spring after box leaving it.)

So our energy conservation equation (20) becomes

$$0 + \frac{1}{2}kx^2 + 0 + W_{other} = K_2 + 0 + mg \sin \alpha \tag{23}$$

Applying Newton's law in the direction perpendicular to the ramp:

$$\sum F_\perp = ma_\perp = 0 \tag{24}$$

$$\sum F_\perp = N - mg \cos \alpha = 0 \Rightarrow N = mg \cos \alpha \tag{25}$$

$$f_k = N\mu_k = \mu_k mg \cos \alpha \tag{26}$$

f_k is against the relative motion of the box so

$$W_{other} = W_f = \overrightarrow{\mathbf{f}_k}.\overrightarrow{\mathbf{s}} = f_k s \cos 180^{\circ} = -f_k s \tag{27}$$

$$\boxed{W_f = -\mu_k mgs\cos\alpha}$$

combining this with (8) we get:

$$\boxed{K_2(s) = \frac{1}{2}kx^2 - mgs(\sin\alpha + \mu_k\cos\alpha)}$$

Minimizing the speed is equivalent to minimizing K_2. Hence and differente the above expression with respect to α and set $\frac{dK_2}{d\alpha} = 0$:

$$0 = -mgs(\cos\alpha - \mu_k\sin\alpha) \Rightarrow \tan\alpha = \frac{1}{\mu_k} \qquad (28)$$

$$\boxed{\alpha = \arctan(\frac{1}{\mu_k})}$$

Note: $\alpha = 90^{\mathrm{o}}$ maximizes increase in gravitational potential energy but as you see from our equation $W_f = 0$ at this angle and you wouldn't have *any* loss due to friction.As α decreases from 90^{o}, $\cos\alpha$ increases linearly but $\sin\alpha$ decreases only quadratically.That is the reason that minimum didn't happen at $\alpha = 90^{\mathrm{o}}$.

Please refer to the figure attached at the end for a graph with $\mu_k = 0.4$.

4) ***7.78***

a) To apply Newton's law $\overrightarrow{\mathbf{F}} = m\overrightarrow{\mathbf{a}}$ for finding the components of the force we should find the components of acceleration by twice differentiating:
The x & y positions:

$$x(t) = x_0 \cos \omega_0 \tag{29}$$

$$y(t) = y_0 \sin \omega_0 \tag{30}$$

This gives:

$$a_x = \frac{d^2x}{dt^2} = -x_0\omega_0^2 \cos \omega_0 t \tag{31}$$

$$a_y = \frac{d^2y}{dt^2} = -y_0\omega_0^2 \sin \omega_0 t \tag{32}$$

So we have:

$$F_x = ma_x = -mx_0\omega_0^2 \cos \omega_0 t = -\omega_0^2 x(t) \tag{33}$$

$$F_y = ma_y = -my_0\omega_0^2 \sin \omega_0 t = -\omega_0^2 y(t) \tag{34}$$

or

$$\overrightarrow{\mathbf{F}} = -\omega_0^2 \overrightarrow{\mathbf{r}} \tag{35}$$

In another words F is radial so it's curl free and the potential energy can be be calculated easily by:

$$U(x,y) = -\int_{\mathbf{0}}^{\overrightarrow{\mathbf{r}}} \overrightarrow{\mathbf{F}}\boldsymbol{.}\, d\overrightarrow{\mathbf{r}} \tag{36}$$

Choose $\overrightarrow{\mathbf{0}}$ as the lower point in integration because U(0,0)=0.
The integration becomes:

$$U(x,y) = -\int_0^x F_x dx - \int_0^y F_y dy \tag{37}$$

$$U(x,y) = (-1)(-m\omega_0^2)\{\int_0^x xdx + \int_0^y ydy\} \tag{38}$$

$$\boxed{U(x,y) = \frac{m\omega_0^2}{2}(x^2 + y^2)}$$

Note: This is the potential energy of a spring in two dimensions with $k = m\omega_0^2$.

b) The force is conservative so the energy is conserved:

$$E = K(x,y) + U(x,y) = const. \tag{39}$$

The next step is to find K(x,y):

$$v_x = \frac{dx}{dt} = -x_0\omega_0 \sin\omega_0 t \tag{40}$$

$$v_y = \frac{dy}{dt} = y_0\omega_0 \cos\omega_0 t \tag{41}$$

From the for of x(t) and y(t) we have:

$$\sin\omega_0 t = \frac{y}{y_0} \tag{42}$$

$$\cos\omega_0 t = \frac{x}{x_0} \tag{43}$$

Replacing into (40) and (41) we have:

$$v_x = -\frac{yx_0\omega_0}{y_0} \tag{44}$$

$$v_y = \frac{xy_0\omega_0}{x_0} \tag{45}$$

$$K(x,y) = \frac{1}{2}m(v_x^2 + v_y^2) \tag{46}$$

$$\boxed{\therefore \quad K(x,y) = \frac{1}{2}m\omega_0^2\{(\frac{xy_0}{x_0})^2 + (\frac{yx_0}{y_0})^2\}}$$

c) Now using (39) and the two boxed equations:

(**i**):

$$K(x_0, 0) = \frac{m\omega_0^2}{2} y_0^2 \tag{47}$$

$$U(x_0, 0) = \frac{m\omega_0^2}{2} x_0^2 \tag{48}$$

$$\therefore \quad E(x_0, 0) = K(x_0, 0) + U(x_0, 0) = \frac{m\omega_0^2}{2}(y_0^2 + x_0^2) \tag{49}$$

(**ii**):

$$K(0, y_0) = \frac{m\omega_0^2}{2} x_0^2 \tag{50}$$

$$U(0, y_0) = \frac{m\omega_0^2}{2} y_0^2 \tag{51}$$

$$\therefore \quad E(0, y_0) = K(0, y_0) + U(0, y_0) = \frac{m\omega_0^2}{2}(x_0^2 + y_0^2) \tag{52}$$

As it was expected the total energy is constant:

$$\boxed{E(x_0, 0) = E(0, y_0) = \frac{m\omega_0^2}{2}(x_0^2 + y_0^2)}$$

5) ***7.86***

`Please refer to figure 7.43 p.281.`

Force is related to the potential energy by:

$$F_x(x) = -\frac{dU}{dx} \tag{53}$$

Sign of $\frac{dU}{dx}$ can be read from the slope of U(x) at point x.

a) The slope of the curve U(x) is negative at A, so F_x is positive.

b) The slope of the curve U(x) is positive at B, so F_x is negative.

c) The force is conservative so the energy U+K=const. Therefore the kinetic energy is maximum when potential energy is a minimum and that appears to be around 0.75 m:

$$Max\{K\} \rightarrow Min\{U\} \Longrightarrow x \approx 0.75m \tag{54}$$

d) The curve at point C looks pretty close to flat, so the force is *zero.*

e) You may want to review problem 4 part c of HW #6. The object had zero kinetic energy at point A, and in order to reach a point with more potential energy U(A) -due to energy conservation the kinetic energy would need to be negative. Kinetic energy is never negative, so the object can never be at any point where the potential energy is larger than U(A). On the graph that seems to be at about 2.2 m.

f) The points of relative minimum in U(x) curve are stable equilibrium points. One of them is the one we found in part **c** the other is at x≈ 1.8 m.

e) The points of relative maximum in U(x) curve are unstable equilibrium points. the only potential maximum, and hence the only point of unstable equilibrium is at point C which is about 1.4 m.

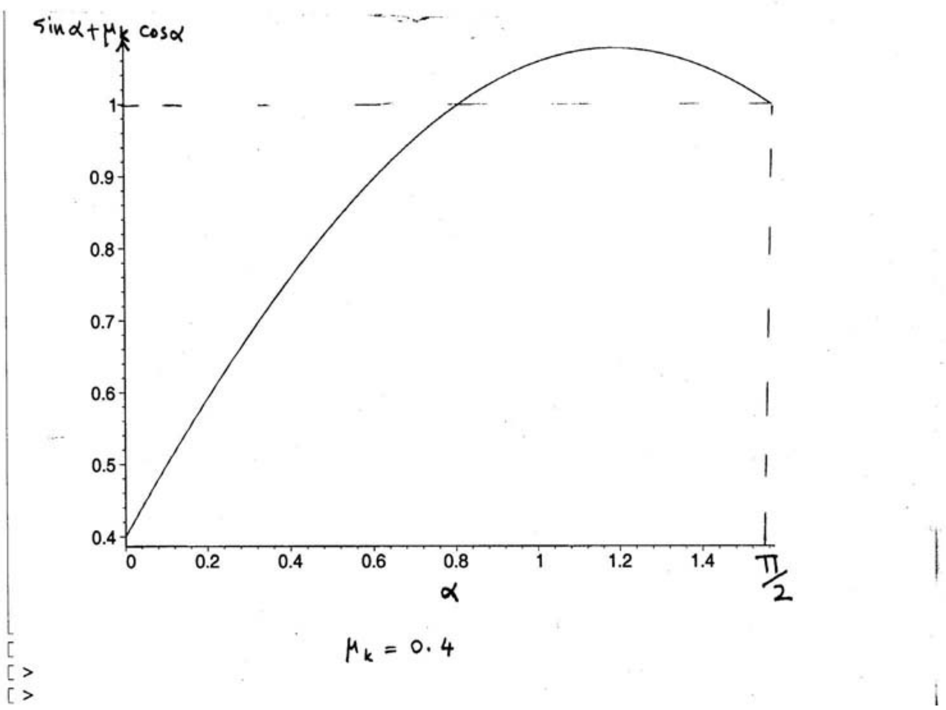

Figure 1: **7.76**