HW Solutions # 11 - 8.01 MIT - Prof. Kowalski

Universal Gravity.

1) *12.23* Escaping From Asteroid

Please study example 12.5 "from the earth to the moon".

a) The escape velocity derived in the example (from energy conservation) is :

$$v_{esc} = \sqrt{\frac{2Gm_A}{R_A}} \tag{1}$$

Where:
$m_A \equiv$ Asteroid's mass=3.6×10^{12} kg
$R_A \equiv$ Asteroid's radius=700 m
$G = 6.673 \times 10^{-11}$ Nm^2/kg^2
Plugging in these numbers into equation (1):

$$\boxed{v_{esc} = 0.83\ m/s}$$

You can certainly walk that fast on earth. However, you could not *walk* on the asteroid faster and faster to achieve this speed because you would go into orbit at a lower velocity ($v_{esc}/\sqrt{2}$) at which point the normal force of the ground would be zero so there would be no more friction to accelerate you!

To get a feeling of how small this gravity is let's do some estimation of the time it takes to reach this velocity on the asteroid. The gravitational force $F_g = Gm_Am/R_A^2$ on the surface of planet for a mass, $m \sim 100$ kg is ~ 0.05 N. Let's take $\mu = 1$. The friction would be ~ 0.05 N so the acceleration $a \sim 0.05/100 = 0.0005$ m/s^2 and the time t it takes to reach $v_{esc} = 0.83$ m/s is:

$$t \sim \frac{0.83}{0.0005} = 1660 \text{ s} \sim 30 \text{ min}$$

b) The question is about the comparison with vertical leap on earth.
Using:

$$v_y^2 - v_{0y}^2 = -2g_{earth}d$$

we have:

$$d = \frac{v_{0y}^2}{2gd} \approx 0.03m$$

$$\boxed{d \approx 3cm}$$

Even octogenarians can jump $\sim 10 \times$ this length.

2) *12.24* `Satellite's altitude and mass`

m_S := Satellite's mass.
m_E := Earth's mass.
R := The distance between the *center* of earth and satellite.
F_g :=the gravitational force between the two masses.
U := the gravitational energy between the two masses.

a) F_g and U are given. By writing them in terms of m_S and m_E:

$$F_g = \frac{G m_E m_S}{R^2} \tag{2}$$

$$U = -\frac{G m_E m_S}{R} \tag{3}$$

you can eliminate R:

$$R = -\frac{U}{F_g} \tag{4}$$

With the numbers given in the problem (F=19.0 kN; U=-1.39×10^{11} J) you'll get:

$$\boxed{R = 7.31 \times 10^6\ m}$$

To find the altitude above the earth denoted by H you should subtract it from Earth radius:

$$H = R - R_E = 7.31 \times 10^6 - 6.38 \times 10^6 = 9.3 \times 10^5\ m$$

$$\boxed{H = 9.3 \times 10^5\ m}$$

b) You can use the value of R and use either of (2) or (3) to find m_S:

$$m_S = -\frac{RU}{G m_E}$$

Where $m_E = 5.97 \times 10^{24}$ kg:

$$\boxed{m_S = 2.55 \times 10^3\ kg}$$

3) *12.46* `Gravitation from 3 masses`

Let's use three indices appropriate for 3 masses namely:
$R_{ight} \equiv$ the mass at (0.5m, 0): $r_{_{PR}} = 0.5$m; $m_{_R} = 1.0$ kg.
$U_p \equiv$ the mass at (0,0.5m): $r_{_{PU}} = 0.5$ m; $m_{_U} = 1.0$ kg.
$D_{iagonal} \equiv$ the mass at (0.5m,0.5 m): $r_{_{PD}} = 0.5\sqrt{2}$ m; $m_{_D} = 2.0$ kg.

a) Because of symmetry we expect to get the total F acting on P along the diagonal. We it more more generally though.

The three forces acting on mass P at origin are ($F_{_{PR}} \equiv$ F acting on P from R):

$$\overrightarrow{\mathbf{F}}_{_{PR}} = +\frac{Gm_{_P}m_{_R}}{r_{_R}^2}\hat{\mathbf{x}}$$

$$\overrightarrow{\mathbf{F}}_{_{PU}} = +\frac{Gm_{_P}m_{_U}}{r_{_U}^2}\hat{\mathbf{y}}$$

$$\overrightarrow{\mathbf{F}}_{_{PD}} = +\frac{Gm_{_P}m_{_D}}{r_{_D}^2}(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}})$$

Where $\sqrt{2}$ comes from the projection of $\overrightarrow{\mathbf{F}}_{_D}$ (which is oriented at 45° with respect to x axis) on x and y axis.
and

$$\overrightarrow{\mathbf{F}}_{_P} = \overrightarrow{\mathbf{F}}_{_{PR}} + \overrightarrow{\mathbf{F}}_{_{PU}} + \overrightarrow{\mathbf{F}}_{_{PD}}$$

Using the values given in the problem you'll get the magnitude:

$$F_P = 9.67 \times 10^{-12} N$$

which orients along the diagonal with unit vector $\hat{\mathbf{n}} \equiv \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}})$.

$$\boxed{\overrightarrow{\mathbf{F}}_{_P} = 9.67 \times 10^{-12} N\hat{\mathbf{n}}}$$

b) Use energy conservation

$$K_2 + U_2 = K_1 + U_1 \tag{5}$$

where 1 denotes the situation that mass P is 300m far away from origin. U_1 is practically "zero" at this large distance. (this distance is 2 orders of magnitude larger so within 1% approximation you can ignore U_1:

$$K_1 = 0 \quad U_1 \approx 0$$

$$U_2 = -Gm_P(\frac{m_R}{r_{PR}} + \frac{m_U}{r_{PU}} + \frac{m_D}{r_{PD}})$$

$$U_2 + \frac{1}{2}m_P v_2^2 \approx 0$$

$$v_2 = \sqrt{-\frac{2U_2}{m_P}}$$

The factor of m_P will be cancelled:

$$\boxed{v_2 = \sqrt{2G(\frac{m_R}{r_{PR}} + \frac{m_U}{r_{PU}} + \frac{m_D}{r_{PD}})}} \tag{6}$$

Plugging the given numbers into (6):

$$v_2 = 30.2 \pm 0.3 \ \mu\text{m/s}$$

$$\boxed{v_2 \approx 30 \ \mu\text{m/s}}$$

NOTE: Gravity is a very weak force (e.g. compared with electrostatic forces), even if it had the speed v_2 for the entire journey, it would take $\sim$ 1 year to make this trip. In In fact it will take $\sim$ 1 year if no other forces (e.g. from sunlight forces) come into play.

3) ***12.68*** `Gravitational Potential`

a) From the definition given the units of ϕ is the unit of energy divided by mass.We use the convention "[ ]" to denote the units:

$$[\phi] = \frac{[U]}{[m]} = \frac{[m][v^2]}{[m]} = [v^2] = \mathrm{m}^2/\mathrm{s}^2$$

b) The gravitational potential energy of two masses m and m_E separated by a distance r (assuming *Zero* energy at *infinity* separation) is:

$$U = -\frac{Gm_E m}{r}$$

From the "*definition*"

$$\phi = \frac{U}{m} \tag{7}$$

we get:

$$\boxed{\phi(r) = -\frac{Gm_E}{r}}$$

c) The problem asks for the quantity

$$\Delta\phi = \phi(R_E + H) - \phi(R_E).$$

Where H denotes the altitude of the space station (400 km in this problem).

Using:

$G = 6.673 \times 10^{-11}\ \ \mathrm{Nm}^2/\mathrm{kg}^2.$
$m_E = 5.97 \times 10^{24}\ \ \mathrm{kg}.$
$R_E = 6.38 \times 10^{6}\ \mathrm{m}.$
$H = 400 \times 10^{3}\ \ \mathrm{m}.$

You'll get:

$$\boxed{\Delta\phi = 3.68 \times 10^6\ m^2/s^2}$$

d) If you assume that the initial and final velocities are "Zero" which is equivalent to a very gradual process. Using

$$K_i + U_i + W_{lift} = K_f + U_f$$

Here $K_i = K_f = 0$ and W_{lift} is the work that must be done against the gravity:

$$W_{lift} = U_f - U_i = m\Delta\phi$$

Using m=15,000 kg and the result of part (c) you'll get:

$$\boxed{\Delta U = 5.53 \times 10^{10}\ J}$$

e) To dock, you have to get up to the speed of the orbiting space station. So the K_f should be its final Kinetic energy as an orbiting payload. For a *circular orbit* we have: $K_f = -\frac{U_f}{2}^{*}$. Now going through the same procedure as part **d** but with $K_f = -\frac{U_f}{2}$:

$$W'_{orbit} = U_f - U_i + K_f = U_f - U_i - \frac{U_f}{2} = +\frac{U_f}{2} - U_i$$

$$\frac{W'_{orbit}}{W_{lift}} = \frac{U_f - U_i + K_f}{U_f - U_i} = \frac{+\frac{U_f}{2} - U_i}{U_f - U_i} = \frac{\frac{1}{2} - \frac{U_i}{U_f}}{1 - \frac{U_i}{U_f}}$$

$$\frac{U_i}{U_f} = \frac{r_f}{r_i} = \frac{R_E + H}{R_E} = 1 + \frac{H}{R_E} = 1.06$$

$$\therefore \boxed{\frac{W'_{orbit}}{W_{lift}} = \frac{0.5 - 1.06}{1 - 1.06} = 9.33}$$

So getting there is only $\sim$ 11% of the work- most is getting up to orbit speed.

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$$F_r = ma_r \Rightarrow -\frac{Gm_E m}{r^2} = -\frac{mv^2}{r} \Rightarrow K = \frac{1}{2}mv^2 = \frac{Gm_E m}{2r} = -\frac{U}{2}$$

3) *12.70* `Effect of Air Drag on Satellite's Motion`

a) In moving to a lower orbit by whatever means, gravity does positive work, and so the speed does increase.

b) From Calculus you know that for a general function $f(r)$ we have:

$$f(r - \Delta r) = f(r) - \frac{df(r)}{dr}\Delta r \Rightarrow \boxed{\Delta f = -\frac{df(r)}{dr}\Delta r} \tag{8}$$

Where Δr is much smaller than r ($\frac{\Delta r}{r} \ll 1$).

From the expression for $v(r)$ in *circular* motion:

$$v(r) = \sqrt{\frac{Gm_E}{r}}$$

$$\frac{d(r^\alpha)}{dr} = -\alpha r^{\alpha-1} \tag{9}$$

Combined with (8) you'll get:

$$\boxed{\Delta v = +(\Delta r/2)\sqrt{\frac{Gm_E}{r^3}} > 0}$$

Combining $K = 1/2mv^2$ with (8) and (9) you'll get:

$$\boxed{\Delta K = +\frac{Gm_E m}{2r^2}\Delta r > 0}$$

Combining $U == -\frac{Gm_E m}{r}$ with (8) and (9) you'll get:

$$\Delta U = -\frac{Gm_E m}{r^2}\Delta r$$

$$\boxed{\Delta U = -\frac{Gm_E m}{r^2}\Delta r = -2\Delta K}$$

Total energy is $E = K + U$ so:

$$\Delta E = \Delta K + \Delta U = \Delta K - 2\Delta K = -\Delta K$$

$$\boxed{W = \Delta E = -\Delta K < 0}$$

c) Since Δr=50 km is much smaller than R_E itself so you can safely use equation (8) for functions v, K and U. For the rest you should just plug in the numbers and use:

$r = R_E + H = 6.38 \times 10^6 + 300 \times 10^3 = 6.68 \times 10^6$ m.
$\Delta r = 50 \times 10^3$ m.
$m = 3000$ kg.

$$v = \sqrt{\frac{Gm_E}{r}} = 7.72 \times 10^3 \text{ m/s}$$

$$\Delta v = +(\Delta r/2)\sqrt{\frac{Gm_E}{r^3}} = +28.9 \text{ m/s}$$

$$E = K + U = -\frac{Gm_E m}{2r} = -8.95 \times 10^{10} \text{ J}$$

$$\Delta K = +\frac{Gm_E m}{2r^2}\Delta r = 6.70 \times 10^8 \text{ J}$$

$$\Delta U = -2\Delta K = -1.34 \times 10^9 \text{ J}$$

$$W = -\Delta K = -6.70 \times 10^8 \text{ J}$$

As the term "burns up" suggests, the energy is converted to heat or is dissipated in the collisions of the debris with the ground.