Massachusetts Institute of Technology - Physics Department
Physics-8.01 Fall 1999

Solutions for Assignment # 8

by Dru Renner dru@mit.edu

November 14, 1999 6:20 pm

Problem 8.1

The subsequent motion will be elliptical with the CENTER of the planet at one focus. (Kepler's First Law) Angular momentum of the satellite is conserved if we measure it relative to the center of the planet. (This is what is called "orbital angular momentum.") Angular momentum is NOT conserved relative to ANY OTHER point! Initially the magnitude of angular momentum relative to the center of the planet is

$$L = mRv_0 \sin(20^\circ)$$

where m is the mass of the satellite. At the point of maximum distance, the apogee, the magnitude of angular momentum is

$$L = m5Rv \sin(90^\circ)$$

where v is the corresponding speed. (At the apogee and perigee, the angle between the velocity vector and the radius vector from either foci will always be 90°.) Conservation of angular momentum and hence conservation of the magnitude of angular momentum gives

$$mRv_0 \sin(20^\circ) = m5Rv \sin(90^\circ) \qquad \Longrightarrow \qquad v = \frac{\sin(20^\circ)}{5} \cdot v_0$$

The gravitational force is conservative; therefore mechanical energy is conserved. Initially the mechanical energy is

$$E = -\frac{GMm}{R} + \frac{1}{2}mv_0^2$$

At the point of maximum distance the mechanical energy is

$$E = -\frac{GMm}{5R} + \frac{1}{2}mv^2 = -\frac{GMm}{5R} + \frac{1}{2}m\left(\frac{\sin(20^\circ)}{5} \cdot v_0\right)^2$$

Conservation of mechanical energy gives

$$-\frac{GMm}{R}+\frac{1}{2}mv_0^2=-\frac{GMm}{5R}+\frac{1}{2}m\left(\frac{\sin(20°)}{5}\cdot v_0\right)^2$$

Solving the above for v_0 gives

$$v_0=\sqrt{\frac{8GM}{5R\left(1-\frac{\sin^2(20°)}{25}\right)}}\approx 1.27\cdot\sqrt{\frac{GM}{R}}$$

Notice that this satellite will slam into the planet before it has made one rotation about the planet (why?).

Problem 8.2

(a) The distance apart is $f2\pi R\approx 1710$ m.

(b) $n_sT_s=T(n_a-f)$ (Please see handout.); thus $2T_s=T(2-0.04)$. The period T_s is smaller than T. The sandwich should be thrown backward.

(c) Mary's (and Peter's) orbital period $T=2\pi(R^3/MG)^{0.5}=5580$ s ≈ 93 min. Mary must wait $T(n_a-f)=(2-0.04)\times 93$ min ≈ 3.04 hrs before she will make the catch.

(d) $n_sT_s=T(n_a-f)$, thus $n_s(4\pi^2a^3/MG)^{0.5}=(4\pi^2R^3/MG)^{0.5}(n_a-f)$ which leads to $a=R[(n_a-f)/n_s]^{2/3}$; thus $a\approx 6800(0.987)\approx 6710$ km.

(e)

(Need Picture)

(f) The velocity of the spacecraft, v_a, is $2\pi R/T$. $v_a\approx 7.66$ km/s. The velocity, v_s, of the sandwich at the point X can be obtained from the relation

$$\text{ENERGY}=-GMm/2a=\frac{1}{2}mv_s^2-GMm/R$$
$$R=7000\text{ km}$$

Since

$$v_a=\sqrt{GM/R}$$
$$v_s=v\sqrt{2-R/a}$$
$$v_s\approx 7.61\text{ km/s}$$

The sandwich must be thrown relative to Peter's motion with a velocity $v_s-v_a\approx 7.61-7.66\approx -52.1$ m/s ≈ -116.5 mph. Too high for any human being! The "-" sign indicates that the sandwich must be thrown backwards. To reduce the speed with which the sandwich has to be thrown you could increase the number of orbits for both Mary and the sandwich before the catch is made. Using Dave Pooley's program (as shown in lectures) we find that for $n_s=n_a=3$ the speed is about 77.1 mph (Roger Clemens could just do it!); for $n_s=n_a=4$ the speed will be about 57.7 mph, and for $n_s=n_a=5$ the speed is only about 46.0 mph (that is certainly doable!).

Problem 8.3

(a) The speed of the earth in a circular orbit, with a radius $R = 1.5 \times 10^{11}$ m, about the sun, with a mass $M = 1.99 \times 10^{30}$ kg, is given by Equation (12) on page 216 as

$$v_0 = \sqrt{\frac{GM}{R}} \approx 2.98 \times 10^4 \text{ m/s}$$

(b) The impulse is given by the change in momentum. The final momentum is zero; therefore the impulse is given by

$$I_0 = \Delta p = m v_0$$

(c) As discussed in Problem 1, mechanical energy and angular momentum are conserved after the rocket is finished firing. Consider the initial and final points, which are respectively the perihelion and aphelion of the elliptical orbit. Conservation of angular momentum gives

$$m v_1 R = m v_2 r \qquad \Longrightarrow \qquad v_2 = \frac{R}{r} \cdot v_1$$

where R is the distance from the sun to the perihelion and v_1 is the velocity there and r is the distance from the sun to the aphelion and v_2 is the velocity there. Conservation of mechanical energy gives

$$-\frac{GMm}{R} + \frac{1}{2} m v_1^2 = -\frac{GMm}{r} + \frac{1}{2} m v_2^2 = -\frac{GMm}{r} + \frac{1}{2} m \left(\frac{R}{r} \cdot v_1\right)^2$$

Solving the above for v_1 gives

$$v_1 = \sqrt{\frac{2GM\left(1 - \frac{R}{r}\right)}{R\left(1 - \left(\frac{R}{r}\right)^2\right)}} = v_0 \cdot \sqrt{\frac{2}{1 + \frac{R}{r}}}$$

The impulse is given by the change in momentum.

$$I_1 = \Delta p = m v_1 - m v_0 = I_0 \left(\sqrt{\frac{2}{1 + \frac{R}{r}}} - 1\right)$$

(d) Using the result above following from angular momentum conservation, the final velocity is

$$v_2 = v_1 \cdot \frac{R}{r} = v_0 \cdot \sqrt{\frac{2}{1 + \frac{R}{r}}} \cdot \frac{R}{r}$$

(e) The impulse is given by the change in momentum. The final momentum is zero; therefore the impulse is given by

$$I_2 = \Delta p = mv_2 = I_0 \cdot \sqrt{\frac{2}{1+\frac{R}{r}}} \cdot \frac{R}{r}$$

(f) The sum of each impulse is

$$I = I_1 + I_2 = \Delta p = mv_1 - mv_0 = I_0 \left(\sqrt{\frac{2}{1+\frac{R}{r}}} - 1 \right) + I_0 \cdot \sqrt{\frac{2}{1+\frac{R}{r}}} \cdot \frac{R}{r}$$

$$= I_0 \left(\sqrt{\frac{2}{1+\frac{R}{r}}} \cdot \left(1 + \frac{R}{r}\right) - 1 \right) = I_0 \left(\sqrt{2\left(1+\frac{R}{r}\right)} - 1 \right)$$

Therefore the difference is

$$I_0 - I = I_0 \cdot \left(2 - \sqrt{2\left(1+\frac{R}{r}\right)} \right) \geq 0$$

since $r \geq R$.

(g) For $r = 20 \cdot R$ the above equation gives

$$I_0 - I \approx 0.55 \cdot I_0$$

The quantity I_0 is defined above as

$$I_0 = mv_0 \approx 2.98 \times 10^4 \cdot m$$

Therefore, in terms of the mass of the spacecraft, m, the difference in the impulses is

$$I_0 - I \approx 1.6 \times 10^4 \cdot m$$

Problem 8.4

(a)

$$\frac{Gm_1m_2}{(r_1+r_2)^2} = \frac{m_1v_1^2}{r_1} = \frac{m_2v_2^2}{r_2}$$

$$v_1 = \frac{2\pi r_1}{T} \qquad v_2 = \frac{2\pi r_2}{T}$$

$$\frac{Gm_2}{(r_1+r_2)^2} = \frac{4\pi^2 r_1}{T} \qquad \frac{Gm_1}{(r_1+r_2)^2} = \frac{4\pi^2 r_2}{T}$$

Adding and solving for T yields:

$$T^2 = \frac{4\pi^2(r_1+r_2)^3}{G(m_1+m_2)}$$

(b)

$$r_2 = \frac{v_2 T}{2\pi} \qquad v_2 = 148\times 10^3 \text{ m/s} \qquad T = 5.6 \text{ days} \times 86,400 \text{ sec/day}$$

$$\Longrightarrow \qquad r_2 = 1.14\times 10^{10} \text{ m}$$

(c)

$$m_1 r_1 = m_2 r_2$$

If $x = \frac{r_1}{r_2}$, then $m_1 = \frac{m_2}{x}$. The period is given by

$$T^2 = \frac{4\pi^2(r_1+r_2)^3}{G(m_1+m_2)} = \frac{4\pi^2 r_2^3\left(\frac{r_1}{r_2}+1\right)^3}{Gm_2\left(\frac{m_1}{m_2}+1\right)} = \frac{4\pi^2 r_2^3(x+1)^3}{Gm_2\left(\frac{1}{x}+1\right)}$$

This means, upon substituting for T and r_2

$$15.9 = x^3 + 2x^2 + x$$

Either by plotting the right hand side as a function of x or by noting that x is nearly equal to 2 and using trial and error, you will find $x \approx 1.90$; thus $r_1 = xr_2 \approx 2.17\times 10^{10}$ m.

(d)

$$m_1 = m_2/x \approx 15.8 \text{ M}_\odot$$

Problem 8.5 (Ohanian, page 352, problem 45)

We will calculate the required static friction for rolling at a given angle of inclination θ. The force equation along the incline of the ramp is

$$ma = mg\sin\theta - F_{\text{fric}}$$

where F_{fric} is the static friction force. The torque equation, about the center of the hoop, is

$$I\alpha = RF_{\text{fric}}$$

where $I = mR^2$ is the moment of inertia of a hoop. (Please see Table 12.1 on page 309.) The rolling constraint is

$$a = R\alpha$$

Combining the torque equation and the rolling constraint gives

$$a = \frac{F_{\text{fric}}}{m}$$

Substituting this relation into the force equation gives

$$F_{\text{fric}} = \frac{1}{2} mg \sin\theta$$

Static friction must satisfy the constraint $F_{\text{fric}} \leq \mu_s N$. This gives

$$\frac{1}{2} mg \sin\theta \leq \mu_s mg \cos\theta \qquad \Longrightarrow \qquad \tan\theta \leq 2\mu_s$$

Problem 8.6 (Ohanian, page 409, problem 34)

The frequency of a physical pendulum is given by Equation (72) on page 395 as

$$\omega = \sqrt{\frac{Mgl}{I}}$$

hence the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgl}}$$

where the quantities above are explained in Section 15.5 on page 393. In particular I is the moment of inertia of the physical pendulum about the point of oscillation. (Please see Figure 15.38 on page 409 for an illustration of the physical pendulum.) Consider a cylinder placed with its length perpendicular to the axis of rotation. Let d be the distance between the axis of rotation and the center of mass of the cylinder. The parallel axis theorem gives the moment of inertia of this cylinder as

$$I = Md^2 + \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

where M is the mass, R the radius, L the length, and $I_{CM} = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$ is given in Table 12.1 on page 309. The mass is given by

$$M = \rho L \pi R^2$$

where ρ is the density of brass, which we are not given. For the thinner cylinder: $d_1 = \frac{L_1}{2} = 0.45m$, $R_1 = 0.005$ m and $L_1 = 0.90$ m, and the moment of inertia is

$$I_1 = \rho L_1 \pi R_1^2 \left(d_1^2 + \frac{1}{4} R_1^2 + \frac{1}{12} L_1^2 \right) = 1.91 \times 10^{-5} \rho$$

where ρ must be in metric units. For the thicker cylinder: $d_2 = 1.0$ m, $R_2 = 0.03$ m and $L_2 = 0.20$ m, and the moment of inertia is

$$I_2 = \rho L_2 \pi R_2^2 \left(d_2^2 + \frac{1}{4} R_2^2 + \frac{1}{12} L_2^2 \right) = 5.67 \times 10^{-4} \rho$$

where ρ must be in metric units. The total moment of inertia is

$$I = I_1 + I_2 = 5.87 \times 10^{-4} \rho$$

The total mass is the sum of each mass and is given by

$$M = \rho (L_1 \pi R_1^2 + L_2 \pi R_2^2) = 6.36 \times 10^{-4} \rho$$

where ρ must be in metric units.

The quantity l in the formula for the period is the distance from the point of oscillation to the center of mass of the physical pendulum. This is given by

$$Ml = M_{\text{thin}} \left(\frac{L_1}{2} \right) + M_{\text{thick}} \left(L_1 + \frac{L_2}{2} \right)$$

where ρ must be in metric units. Therefore the quantity l is

$$l = \frac{2.39 \times 10^{-3} \rho}{2.54 \times 10^{-3} \rho} \approx 0.941 \text{ m}$$

and the period T is

$$T = 2\pi \sqrt{\frac{5.87 \times 10^{-4} \rho}{6.36 \times 10^{-4} \rho \cdot g \cdot 0.941}} \approx 1.988 \text{ s}$$

Problem 8.7

We assume that the mass m_2 is sufficiently larger than the mass m_1 so that m_2 accelerates down. This determines which direction the kinetic friction acts for each block. The force equation for m_2 in the direction down the slope is

$$m_2 a = \sum F = m_2 g \sin\theta_2 - \underbrace{\mu m_2 g \cos\theta_2}_{\text{fric}} - T_2 \qquad \Longrightarrow$$

$$T_2 = m_2 g \sin\theta_2 - \mu m_2 g \cos\theta_2 - m_2 a \tag{1}$$

where a is the acceleration of m_2 directed *down* the slope and T_2 is the tension for that portion of the string. The force equation for m_1 in the direction up the slope is

$$m_1 a = \sum F = T_1 - m_1 g \sin\theta_1 - \underbrace{\mu m_1 g \cos\theta_1}_{\text{fric}} \qquad \Longrightarrow$$

$$T_1 = m_1 a + m_1 g \sin\theta_1 + \mu m_1 g \cos\theta_1 \tag{2}$$

where a is also the acceleration of m_1 directed *up* the slope (assuming the string does not stretch) and T_1 is the tension for that portion of the string. The torque equation for the pulley is

$$I\alpha = \sum \tau$$

$$\frac{1}{2}MR^2\alpha = RT_2 - RT_1 \qquad \Longrightarrow$$

$$\alpha = \frac{2(T_2 - T_1)}{MR} \tag{3}$$

where α is the angular acceleration clockwise in the picture and $I = \frac{1}{2}MR^2$ is the moment of inertia about the axle. (Please see Table 12.1 on page 309.) The coefficient of RT_2 is positive because T_2 acts to increase α; the coefficient of RT_1 is negative because T_1 acts to decrease α. The constraint of no slipping between the pulley and the string requires that

$$a = R\alpha \tag{4}$$

Substituting Equation (**??**) into Equation (**??**) gives

$$a = \frac{2(T_2 - T_1)}{M} \tag{5}$$

Substituting Equations (**??**) and (**??**) into Equation (**??**) and solving for a gives

$$a = \frac{m_2 g(\sin\theta_2 - \mu\cos\theta_2) - m_1 g(\sin\theta_1 + \mu\cos\theta_1)}{m_1 + m_2 + \frac{1}{2}M} \tag{6}$$

Substituting Equation (**??**) into Equation (**??**) gives

$$T_2 = m_2 g\sin\theta_2 - \mu m_2 g\cos\theta_2 - m_2 \cdot \frac{m_2 g(\sin\theta_2 - \mu\cos\theta_2) - m_1 g(\sin\theta_1 + \mu\cos\theta_1)}{m_1 + m_2 + \frac{1}{2}M}$$

Substituting Equation (**??**) into Equation (**??**) gives

$$T_1 = m_1 \cdot \frac{m_2 g(\sin\theta_2 - \mu\cos\theta_2) - m_1 g(\sin\theta_1 + \mu\cos\theta_1)}{m_1 + m_2 + \frac{1}{2}M} + m_1 g\sin\theta_1 + \mu m_1 g\cos\theta_1$$

Substituting Equation (**??**) into Equation (**??**) gives

$$\alpha = \frac{a}{R} = \frac{m_2 g(\sin\theta_2 - \mu\cos\theta_2) - m_1 g(\sin\theta_1 + \mu\cos\theta_1)}{R\left(m_1 + m_2 + \frac{1}{2}M\right)}$$

Problem 8.8 (Ohanian, page 462, problem 29)

The trumpeters on the train act as a moving emitter with speed $V_E = 60$ km/h = 16.67 m/s and with frequency $\nu = 329.7$ Hz. The frequency heard by a listener on the ground (rest frame of the air) is given by Equation (13) on page 449 as

$$\nu' = \nu\left(\frac{1}{1 \mp \frac{V_E}{v_S}}\right)$$

where the $-$ sign corresponds to an approaching emitter and the $+$ sign corresponds to a receding emitter and $v_S = 331$ m/s is assumed to be the speed of sound in air. While the train is approaching, the frequency heard is

$$\nu' \approx 347.2 \text{ Hz}$$

which is closest to the musical note E. (See Table 17.2 on page 441.) While the train is receding, the frequency heard is

$$\nu' \approx 313.9 \text{ Hz}$$

which is closest to the musical note D#. (See Table 17.2 on page 441.)

Problem 8.9

Initially the plane is moving too fast and the tires too slow to satisfy the rolling constraint. Friction acts on the wheels. This creates a linear acceleration that slows the plane down and an angular acceleration that spins the tires. At some moment the linear speed of the plane and angular speed of the tires are appropriate for rolling. Static friction will then maintain the rolling.

Each tire supports a weight Mg where the total mass is given by $M_{TOT} = nM$ where n is the number of tires. Now suppose the runway exerts a friction force F_{fric} on each tire. The force equation for the plane is

$$nMa = nF_{\text{fric}} \qquad \Longrightarrow \qquad a = \frac{F_{\text{fric}}}{M}$$

The friction force while sliding is constant; hence the velocity is given by

$$v = v_0 - at = v_0 - \frac{F_{\text{fric}}}{M} \cdot t$$

The torque equation (about the center of a tire) is

$$I\alpha = RF_{\text{fric}} \qquad \Longrightarrow \qquad \alpha = \frac{RF_{\text{fric}}}{I}$$

Again, the friction force is constant; hence the angular velocity is given by

$$\omega = \frac{RF_{\text{fric}}}{I} \cdot t$$

Rolling occurs when

$$v = R\omega \qquad \Longrightarrow \qquad v_0 - \frac{F_{\text{fric}}}{M} \cdot t = R \cdot \frac{RF_{\text{fric}}}{I} \cdot t$$

The solution to the above equation is

$$t = \frac{v_0}{F_{\text{fric}} \left(\frac{1}{M} + \frac{R^2}{I} \right)}$$

The velocity at this time is

$$v = v_0 \cdot \frac{\frac{MR^2}{I}}{1 + \frac{MR^2}{I}} = \frac{v_0}{1 + \frac{I}{MR^2}}$$

Problem 8.10 (Ohanian, page 353, problem 50)

The precession frequency is given by Equation (41) on page 344 as

$$\omega_P = \frac{rMg}{L}$$

where the quantities above are explained in Section 13.6 on page 343. The angular momentum is given by

$$L = I\omega$$

where I is the moment of inertia given by Table 12.1 on page 309 as

$$I = \frac{1}{2}MR^2$$

Combining the above gives

$$\omega_P = \frac{2rg}{R^2\omega}$$

For this child's toy top the values are $r = 6.0$ cm $= 6.0 \times 10^{-2}$ m, $M = 0.15$ kg, $R =$ 5.0 cm $= 5.0 \times 10^{-2}$ m, and $\omega = 200$ rev/s $= 200 \cdot 2\pi$ radian/s. These values give

$$\omega_P \approx 0.37 \text{ radian/s}$$

Problem 8.11 (Ohanian, page 353, problem 51)

(a) The ship will rotate clockwise in the water when viewed from above. The wave exerts an upward force on the right axle and a downward force on the left axle; both forces create a torque that points to the back of the boat. The angular momentum will attempt to align with the torque; thus causing the ship to rotate.

(b) The ship will roll left, or capsize. The wave exerts a backward force on the left axle and a forward force on the right axle; both forces create a torque that point upward. The angular momentum will attempt to align with the torque; thus causing the ship to capsize.

Problem 8.12

(a) Let $I =$ the moment of inertia of the rod through an axis perpendicular to the plane of the page. An angular impulse is delivered to the rod of magnitude $Fd\Delta t$. This gives the rod an angular momentum $I\omega$, therefore $\omega = Fd\Delta t/I$. The rod tumbles with this angular velocity.

(b) L is the angular momentum of the spinning rod. The situation is now exactly the same as a gyroscope with its angular momentum horizontal acted on by gravity. (F replaces mg). The torque is Fd which causes a precessional angular velocity Fd/L. (See Ohanian page 344). The precession is in the plane perpendicular to $\vec{F}$. In a time Δt, the axis moves through an angle $Fd\Delta t/L$ instead of tumbling as in part a). The larger L is the smaller the angle will be.