Massachusetts Institute of Technology - Physics Department
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Solutions for Assignment # 7

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Problem 7.1 (Ohanian, page 271, problem 55)

Throughout this problem we will use the rocket equation:

$$v_f - v_i = u \ln\left(\frac{M_i}{M_f}\right)$$

Consider two rockets: (1) a multiple-stage rocket with initial mass M_0 and with a final mass of the last stage M; (2) a single-stage rocket with initial mass M_0 and final mass M. Suppose all stages of (1) and the single stage of (2) have the same exhaust speed u. We know that the terminal speed of the single-stage rocket is

$$v_2 = u \ln\left(\frac{M_0}{M}\right)$$

Rocket (1) has less fuel than rocket (2) because each jettisoned stage has some mass; we should expect that rocket (1) will have a smaller terminal speed than rocket (2). To see this, consider a two stage rocket. Suppose the mass of the fuel for the first stage is M_{F1}, the mass of the first jettisoned stage is M_1, and the mass of the fuel for the second stage is M_{F2}. Then the final mass will be $M = M_0 - M_{F1} - M_1 - M_{F2}$. After burning $MF1$, the speed of the rocket is

$$v' = u \ln\left(\frac{M_0}{M_0 - M_{F1}}\right)$$

Then the first stage is jettisoned, so the mass is reduced to $M_0 - M_{F1} - M_1$. If we assume the first stage moves away at zero velocity relative to the remainder of the rocket then the remainder of the rocket will not speed up due to the jettisoned stage. Then M_{F2} is burned, and the speed of the rocket is

$$v'' = u \ln\left(\frac{M_0 - M_{F1} - M_1}{M_0 - M_{F1} - M_1 - M_2}\right) = u \ln\left(\frac{M_0 - M_{F1} - M_1}{M}\right)$$

Thus the terminal speed for rocket (1) is

$$
\begin{aligned}
v_1 = v' + v'' &= u \ln\left(\frac{M_0}{M_0 - M_{F1}}\right) + u \ln\left(\frac{M_0 - M_{F1} - M_1}{M}\right) \\
&= u \ln\left(\frac{M_0}{M_0 - M_{F1}} \cdot \frac{M_0 - M_{F1} - M_1}{M}\right) \\
&= u \ln\left(\frac{M_0}{M}\right) - u \ln\left(\frac{M_0 - M_{F1}}{M_0 - M_{F1} - M_1}\right) \\
&= v_2 - u \ln\left(\frac{M_0 - M_{F1}}{M_0 - M_{F1} - M_1}\right)
\end{aligned}
$$

Since $M_1 > 0$ the second piece in the above expression is negative, so

$$v_1 < v_2$$

For each stage that is jettisoned there will be similar terms which further reduce v_1.

Problem 7.2

Consider velocities positive if along the initial direction of the spacecraft. Let the final velocity of the spacecraft be v' and the final velocity of the planet be V'. Conservation of momentum gives

$$mv + MV = mv' + MV' \qquad \Longrightarrow \qquad V' - V = \frac{m}{M} \cdot (v - v') \tag{1}$$

Mechanical energy is conserved. If we consider the initial and final moments of the system to occur when there is sufficient separation between the planets, then the gravitational potential energy can be ignored. Therefore kinetic energy is conserved.

$$
\begin{aligned}
&\frac{1}{2}mv^2 + \frac{1}{2}MV^2 = \frac{1}{2}mv'^2 + \frac{1}{2}MV'^2 \qquad \Longrightarrow \\
&\frac{m}{M} \cdot (v - v') \cdot (v + v') = (V' - V) \cdot (V' + V)
\end{aligned}
$$

Using Equation (**??**) we can rewrite the above equation as

$$
\begin{aligned}
\frac{m}{M} \cdot (v - v') \cdot (v + v') = \frac{m}{M} \cdot (v - v') \cdot (V' + V) \qquad \Longrightarrow \\
V' = v + v' - V
\end{aligned}
$$

Substituting this expression back into Equation (**??**) we obtain

$$v + v' - 2V = \frac{m}{M} \cdot (v - v') \qquad \Longrightarrow \qquad v' = \frac{\frac{m}{M} - 1}{1 + \frac{m}{M}} \cdot v + \frac{2}{1 + \frac{m}{M}} \cdot V$$

(b) The mass of a spacecraft is very small compared to the mass of a planet, i.e. $\frac{m}{M} \ll 1$. Therefore,

$$v' \approx -v + 2V$$

If $v = 10$ km/s $= 10^4$ m/s and $V = -13$ km/s $= 13 \times 10^3$ m/s then

$$v' \approx -36 \text{ km/s} = -36 \times 10^3 \text{ m/s}$$

(c) The change in energy, where again we assume the initial and final moments are such that the spacecraft is sufficiently far from Jupiter that gravitational energy can be ignored, is given by the change in kinetic energy. If $m = 2000$ kg then

$$\Delta E = \frac{1}{2}mv'^2 - \frac{1}{2}mv^2 \approx 1.2 \times 10^{12} \text{ J}$$

Problem 7.3 (Ohanian, page 320, problem 23)

Examples 7 and 8 on page 306 illustrate a "thin rod".

Let $d = 0.20$ m be the distance from her shoulders to the vertical axis of rotation through the center of her body, $l = 0.60$ m be the length of each arm, and $M = 2.8$ kg be the mass of each arm. First consider the configuration for which her arms are down vertically at her sides. The moment of inertia about a vertical axis through the center of her body is given by

$$I_1 = \sum m \cdot d^2 = 2Md^2 = 0.224 \text{ kg m}^2$$

where we could use the simple expression above because all the mass is the same distance, d, from the axis of rotation. Now consider the configuration for which her arms are stretched out horizontally. The moment of inertia about a vertical axis through the center of her body is given by the parallel axis theorem as

$$I_2 = 2\left(\frac{1}{12}Ml^2 + M\left(d + \frac{l}{2}\right)^2\right) = 1.568 \text{ kg m}^2$$

where the factor of 2 accounts for both arms, $\frac{1}{12}Ml^2$ is the moment of inertia about the center of mass (given in Table 12.1 on page 309), and $d + \frac{l}{2}$ is the distance from the center of mass to the axis of rotation. The difference in her moment of inertia is

$$\Delta I = I_2 - I_1 = 1.344 \text{ kg m}^2$$

Problem 7.4 (Ohanian, page 320, problem 26)

For a sphere of uniform density, the center of mass is actually the geometric center; thus the moment of inertia about a diameter is actually the moment of inertia about an axis through the center of mass. Therefore in the language of the parallel axis theorem

$$I_{CM} = \frac{2}{5} MR^2$$

Now we note that any axis tangent to the surface is parallel to some diameter, thus we can use the parallel axis theorem to find the moment of inertia, I, about such an axis

$$I = I_{CM} + MR^2 = \frac{2}{5} MR^2 + MR^2 = \frac{7}{5} MR^2$$

Problem 7.5 (Ohanian, page 322, problem 41)

Let $M = 1.5 \times 10^{30}$ kg, $R = 20$ km $= 20 \times 10^3$ m, $\omega_0 = 2.1$ rev/s $= 2.1 \cdot 2\pi$ radian/s, $\alpha_0 = -1.0 \times 10^{-15}$ rev/s^2 $= -1.0 \times 10^{-15} \cdot 2\pi$ radian/s. (Remember it is necessary to convert all angular quantities from "rev" to "radian.") The rotational kinetic energy is

$$K = \frac{1}{2} I \omega^2$$

where the moment of inertia is given in Table 12.1 on page 309 as

$$I = \frac{2}{5} MR^2 = 2.4 \times 10^{38} \text{ kg m}^2$$

Therefore the time rate of change of K is

$$\frac{dK}{dt} = \frac{I}{2} \frac{d\omega^2}{dt} = I\omega\alpha$$

Initially $\omega = \omega_0$ and $\alpha = \alpha_0$, so the initial rate of change of K is

$$r_0 = I\omega_0\alpha_0 \approx -1.99 \times 10^{25} \text{ J/s}$$

If this rate is to remain constant then

$$\frac{dK}{dt} = \frac{I}{2} \frac{d\omega^2}{dt} = r_0 \qquad \Longrightarrow \qquad \omega^2 = \frac{2r_0}{I} \cdot t + \omega_0^2$$

The time T for the rotation to stop is given by

$$0 = \frac{2r_0}{I} T + \omega_0^2 \qquad \Longrightarrow \qquad T = -\frac{\omega_0^2 I}{2r_0} = 1.05 \times 10^{15} s \approx 3.3 \times 10^7 \text{ years}$$

Problem 7.6 (Ohanian, page 322, problem 45)

Choose an x-y-z coordinate system with origin at the center of the square, z axis perpendicular to the square, and x and y axes parallel to sides of the square. In the language of the perpendicular axis theorem, we want to calculate I_z, but it is easier to calculate I_x and I_y instead. Let M be the mass of the square and l be the length of each side. Also let h be the small depth for the square. Then the volume of the square is

$$V = hl^2$$

and the density of each rod is

$$\rho = \frac{M}{V} = \frac{M}{hl^2}$$

which is a constant that can be "pulled out" of the integration. Then the moment of inertia about the x axis is

$$\begin{aligned}
I_x &= \int \rho r^2 \; dV \\
&= \int_0^l dx \int_{-\frac{l}{2}}^{\frac{l}{2}} \rho \; hdxdy \\
&= \rho h \cdot l \cdot \frac{1}{3}\frac{2l^3}{8} \\
&= \frac{M}{hl^2} \cdot h \cdot \frac{l^4}{12} \\
&= \frac{1}{12} Ml^2
\end{aligned}$$

By symmetry, the moment of inertia about the y axis is the same

$$I_y = \frac{1}{12} Ml^2$$

Using the perpendicular axis theorem gives

$$I_z = I_x + I_y = \frac{1}{12}Ml^2 + \frac{1}{12}Ml^2 = \frac{1}{6}Ml^2$$

Problem 7.7 (Ohanian, page 324, problem 59)

First we need the equations of motion for projectile flight. For the angle 45° we find $\cos 45° = \frac{1}{\sqrt{2}}$ and $\sin 45° = \frac{1}{\sqrt{2}}$ The equations of motion are given by

$$x = \frac{1}{\sqrt{2}} v_0 t$$

$$y = -\frac{1}{2}gt^2 + \frac{1}{\sqrt{2}}v_0 t$$

where the origin is chosen to coincide with the launch point.

At the instant of launch the velocity and position are

$$\vec{r} = 0$$

$$\vec{v} = \frac{1}{\sqrt{2}}v_0\hat{x} + \frac{1}{\sqrt{2}}v_0\hat{y}$$

The angular momentum is given by

$$\vec{L} = m\vec{r} \times \vec{v} = 0 \qquad \Longrightarrow \qquad |L| = 0$$

because $\vec{r} = 0$.

At the instant it reaches maximum height, the y value is given by mechanical energy conservation

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{1}{\sqrt{2}}v_0\right)^2 + mgy \qquad \Longrightarrow \qquad y = \frac{1}{4}\frac{v_0^2}{g}$$

and the x value is irrelevant. At this instant the velocity is horizontal and given by

$$\vec{v} = \frac{1}{\sqrt{2}}v_0\hat{x}$$

The magnitude of the angular momentum is given by the product of the speed and the component of position perpendicular to $\vec{v}$, thus

$$|L| = m|y| \cdot |\vec{v}| = \frac{1}{4}\frac{v_0^2}{g} \cdot \frac{m}{\sqrt{2}}v_0 = \frac{m}{4\sqrt{2}}\frac{v_0^3}{g}$$

The direction is perpendicular to the plane of motion.

At the instant it reaches the ground, the x value is given by the range of motion and the y value is zero.

$$\vec{r} = \frac{v_0^2}{g}\hat{x}$$

Due to the symmetry of the parabola of motion, the velocity is given by

$$\vec{v} = \frac{1}{\sqrt{2}} v_0 \hat{x} - \frac{1}{\sqrt{2}} v_0 \hat{y}$$

The magnitude of the angular momentum is given by the product of the distance and the component of velocity perpendicular to $\vec{r}$,

$$|L| = m|\vec{r}| \cdot |v_y| = \frac{v_0^2}{g} \cdot \frac{m}{\sqrt{2}} v_0 = \frac{m}{\sqrt{2}} \frac{v_0^3}{g}$$

The direction is perpendicular to the plane of motion.

For each of the three instants, the magnitude of angular momentum was different; hence the angular momentum was indeed not conserved. (This indicates that at least one component of angular momentum was not conserved. The torque is due to gravity and is perpendicular to the plane of motion; hence the components of angular momentum in the other directions are conserved.)

Problem 7.8 (Ohanian, page 348, problem 11)

Please see Figure 13.34 on page 348.

The only forces acting on the mass m are gravity and the tension due to the string T. If we let a be the acceleration *down* then

$$ma = mg - T \tag{2}$$

The only forces acting on the mass M are gravity, the force of the support, and the tension due to the string. All three forces balance to give no translational motion, but only tension creates a torque about the axis of the disk. (The other two forces act through the center of the disk; hence there torque vanishes.) If the string is massless the tension throughout the string from mass m to mass M is constant. If we let α be the angular acceleration *clockwise* then

$$I\alpha = RT \tag{3}$$

where R is the radius of the disk. If the disk has a uniform density then I is given by Table 12.1 on page 309 as

$$I = \frac{1}{2} M R^2 \tag{4}$$

Finally if we assume the string does not slip against the disk then

$$a = \alpha R \tag{5}$$

Substituting Equations (**??**) and (**??**) into Equation (**??**) gives

$$\frac{1}{2}MR^2 \cdot \frac{a}{R} = RT \qquad \Longrightarrow \qquad T = \frac{aM}{2}$$

Substituting this result into Equation (**??**) gives

$$ma = mg - \cdot\frac{aM}{2} \qquad \Longrightarrow \qquad a = \frac{g}{1 + \frac{M}{2m}}$$

We can check that for $M \ll m$ that $a \to g$.

Problem 7.9

Choose the y axis parallel to the rod and the x axis along the direction of the hit, which is said to be perpendicular to the rod. Let $M = 3$ kg, $l = 50$ cm, and $d = 15$ cm.

(a) The impulse of the hit has magnitude $I = 4$ kg $\cdot$ m/s and direction $\hat{x}$, so let $\vec{I} = I\hat{x}$; this is given by

$$\vec{I} = \int \vec{F}\ dt = \int \frac{d\vec{P}}{dt}\ dt = \Delta\vec{P}$$

where $\vec{P}$ is the total momentum of the system. The object is initially at rest, so the total momentum before the hit is zero. The total momentum after the hit, which is related to the center of mass velocity, is then

$$M\vec{v}_{CM} = \vec{P}_{after} = \Delta\vec{P} = \vec{I}$$

The point C is the center of the rod, which is the center of mass assuming the rod has uniform mass; therefore the translational speed of C is

$$|\vec{v}_{CM}| = \frac{|\vec{I}|}{M} = \frac{I}{M} = \frac{4}{3}\ \text{m/s}$$

(b) If we assume that the hit occurs over a short enough time that the rod does not move appreciably, i.e. $\vec{r}$ is constant, then

$$\int \vec{r} \times \vec{F}\ dt = \vec{r} \times \int \vec{F}\ dt = \vec{r} \times \vec{I}$$

The change in angular momentum is then given by

$$\Delta L = \int \frac{dL}{dt}\ dt = \int \vec{\tau}\ dt = \int \vec{r} \times \vec{F}\ dt = \vec{r} \times \vec{I}$$

The object is initially at rest, so the total angular momentum before the hit is zero, and the total angular momentum after the hit is

$$\vec{L} = \vec{r} \times \vec{I}$$

If we choose the origin of the coordinate system to coincide with C then the angular velocity about C is given by

$$I_{CM}\omega = |\vec{L}| = |\vec{r} \times \vec{I}| = dI$$

where $I_{CM} = \frac{1}{12}Ml^2$ is the moment of inertia through the center of the rod and perpendicular to the length of the rod. (Please see Table 12.1 on page 309.) Therefore

$$\omega = \frac{dI}{I_{CM}} = \frac{12 \cdot dI}{Ml^2} = 9.6 \text{ radian/s}$$

(c) After the hit there are no external forces, hence linear and angular momentum are conserved. The center of mass continues to move with the speed above; thus the distance it travels in 8 s is

$$D = v_{CM} \cdot 8 = \frac{4}{3} \cdot 8 \approx 11 \text{ m}$$

The rod also continues to rotate with the velocity above; thus the total angular rotation is

$$\Theta = \omega \cdot 8 = 76.8 \text{ radian}$$

This is clearly larger than 2π, so the angle between the direction of the rod before and after it is hit is given by

$$\theta = 76.8 - 12 \cdot 2\pi = 1.4 \text{ radian}$$

where $0 < 1.4 < 2\pi$ is the correct angle.

(d) The total kinetic energy of a rotating system is given by

$$K = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 \approx 5.5 \text{ J}$$

Problem 7.10

Please see the sketch on the assignment.

Consider the direction to the right to be positive, and consider clockwise rotations to be positive. The force equation for horizontal motion is

$$T\cos\alpha - F = ma \qquad \Longrightarrow \qquad F = T\cos\alpha - ma \tag{6}$$

where T is the tension due to the pull, F is the friction, and a is the acceleration in the positive horizontal direction. The torque equation about the center of the yo-yo is

$$FR_2 - TR_1 = I\alpha \tag{7}$$

where α is the angular acceleration. If the pull is "gentle" enough, then the friction force will be able to create rolling without sliding, i.e. the relationship between a and α is

$$a = R_2\alpha$$

We can use this relationship to eliminate α from Equation (**??**) giving

$$FR_2 - TR_1 = \frac{I}{R_2}a \tag{8}$$

Now we use Equation (**??**) to eliminate F from the above equation giving

$$(T\cos\alpha - ma)R_2 - TR_1 = \frac{I}{R_2}a \qquad \Longrightarrow \qquad a = \frac{R_2(\cos\alpha - \frac{R_1}{R_2})}{\frac{I}{R_2} + R_2 M}\cdot T$$

Therefore, if $\cos\alpha > \frac{R_1}{R_2}$ then $a > 0$ and the yo-yo will roll in the direction of the pull; if $\cos\alpha < \frac{R_1}{R_2}$ then $a < 0$ and the yo-yo will roll in the opposite direction of the pull. For "gentle" enough pulls, the yo-yo will not roll at the critical angle $\cos\alpha^\star = \frac{R_1}{R_2}$. (Once you pull hard enough the yo-yo will either slide or lift off the floor.)

Problem 7.11

We need to be careful because:

(1) Angular momentum is NOT conserved and

(2) Kinetic energy of rotation is NOT conserved

(a) The frictional force will dissipate some of the kinetic energy.

(b) An external torque is needed; you will sense this in your hands as you push the wheels together.

(c) The applied torque will change the angular momentum.

(d) When the two disks are no longer slipping against one another their circumferential speeds must be the same. Thus

$$\omega_1 R_1 = \omega_2 R_2 \quad \text{and} \quad \frac{\omega_1}{\omega_2} = \frac{R_2}{R_1}$$

F_2 and F_1 are the frictional forces on disks 2 and 1, respectively. $|F_1| = |F_2| = F$. For a uniform disk the moment of inertia is $\frac{1}{2}MR^2$. Since the torque equals the moment of inertia times the angular acceleration:

$$FR_1 = \frac{1}{2}M_1R_1^2\alpha_1 \qquad -FR_2 = \frac{1}{2}M_2R_2^2\alpha_2$$

α_1 and α_2 are the angular accelerations of the disks. Disk #1 is spun down; disk #2 is spun up. Notice the vectors F and the radii R are perpendicular to each other.

Hence $|\frac{\alpha_1}{\alpha_2}| = \frac{M_2R_2}{M_1R_1}$. Since $\alpha = \frac{d\omega}{dt}$

$$d\omega_1 = -\frac{M_2R_2}{M_1R_1}d\omega_2$$

$$\omega_1 - \omega = -\frac{M_2R_2}{M_1R_1}\omega_2$$

Using the relation in Part (d) we have two equations and two unknowns. Solving, we get

$$|\omega_1| = \frac{\omega}{1+\frac{M_2}{M_1}} \qquad |\omega_2| = \frac{\omega\frac{R_1}{R_2}}{1+\frac{M_2}{M_1}}$$

Since the disks are in all respects identical except for their radii, $\frac{M_2}{M_1} = \left(\frac{R_2}{R_1}\right)^2$ and thus

$$|\omega_1| = \frac{\omega}{1+\left(\frac{R_2}{R_1}\right)^2} \qquad |\omega_2| = \frac{\omega\frac{R_1}{R_2}}{1+\left(\frac{R_2}{R_1}\right)^2}$$

Notice that for $R_2 = R_1$ we obtain $\omega_2 = \omega_1$ as one would expect (why?); they both equal to $\frac{\omega}{2}$ (not so obvious!).