Solutions for Assignment # 4

by Dru Renner

Throughout these solutions, the following symbol will be used frequently.
n **will represent an arbitrary integer:** $0, \pm 1, \pm 2, \ldots$

Problem 4.1

We are in Regime I when

$$v_{\text{term}} \ll v_{\text{crit}}$$

thus when

$$\frac{mg}{C_1 r} \ll \frac{C_1}{C_2 r}$$

$$m = \frac{4}{3}\pi r^3 \rho_{\text{oil}}$$

thus

$$r^3 \ll \frac{3C_1^2}{4\pi C_2 \rho_{\text{oil}} g} \approx 4 \times 10^{-12}$$

$$r \ll 1.6 \times 10^{-4} \text{ m} \approx 160 \text{ microns}$$

Problem 4.2

(a) There are only two forces: gravity, $mg\hat{y}$, and drag force, $-C_1 r\vec{v}$. (The downward vertical direction is labeled by y.) Newton's Second Law gives

$$m\vec{a} = m\frac{d^2\vec{R}}{dt^2} = \vec{F} = mg\hat{y} - C_1 r\vec{v} = mg\hat{y} - C_1 r\frac{d\vec{R}}{dt}$$

where $\vec{R}$ is the vector position as a function of time. The differential equations for motion in the x and y direction follow by taking the x and y components of the above vector equation. The differential equation for x motion is

$$m\ddot{x} = -C_1 r\dot{x}$$

The differential equation for y motion is

$$m\ddot{y} = mg - C_1 r\dot{y}$$

(b) The differential equation for the x motion can be written as

$$\frac{dv_x}{dt} = -\frac{C_1 r}{m} v_x$$

where $v_x = \frac{dx}{dt}$. Define τ by

$$\tau = \frac{m}{rC_1}$$

Then the equation becomes

$$\frac{dv_x}{dt} = -\frac{1}{\tau} v_x$$

This equation can be solved by following this sequence of steps.

$$\frac{dv_x}{v_x} = -\frac{dt}{\tau}$$

$$\ln v_x - \ln u = \int_0^t \frac{dv_x}{v_x} = \int_0^t -\frac{dt}{\tau} = -\frac{t}{\tau}$$

$$\ln v_x = -\frac{t}{\tau} + \ln u$$

$$v_x = e^{\ln v_x} = e^{-\frac{t}{\tau} + \ln u} = ue^{-\frac{t}{\tau}}$$

Therefore v_x is given by

$$v_x = ue^{-\frac{t}{\tau}}$$

(c) The differential equation for the y motion can be written as

$$\frac{dv_y}{dt} = g - \frac{C_1 r}{m} v_y$$

where $v_y = \frac{dy}{dt}$. Again using τ the equation becomes

$$\frac{dv_y}{dt} = g - \frac{1}{\tau} v_y$$

This equation can be solved by following this sequence of steps. First define the intermediate quantity f by

$$g - \frac{1}{\tau} v_y = -\frac{1}{\tau} f \qquad \Longrightarrow \qquad f = -g\tau + v_y$$

Then

$$\frac{df}{dt} = \frac{d(-g\tau + v_y)}{dt} = \frac{dv_y}{dt} = g - \frac{1}{\tau} v_y = -\frac{1}{\tau} f$$

which leads to the equation

$$\frac{df}{dt} = -\frac{1}{\tau} f$$

To solve this equation follow these steps.

$$\frac{df}{f} = -\frac{dt}{\tau}$$

$$\ln f - \ln f_0 = \int_0^t \frac{df}{f} = \int_0^t -\frac{dt}{\tau} = -\frac{t}{\tau}$$

$$\ln f = -\frac{t}{\tau} + \ln f_0$$

$$f = e^{\ln f} = e^{-\frac{t}{\tau} + \ln f_0} = f_0 e^{-\frac{t}{\tau}}$$

Then

$$-g\tau + v_y = f = f_0 e^{-\frac{t}{\tau}} = (-g\tau + v_{y0}) e^{-\frac{t}{\tau}} = -g\tau e^{-\frac{t}{\tau}}$$

Therefore v_y is given by

$$v_y = g\tau(1 - e^{-\frac{t}{\tau}})$$

(d) The equation for v_y is

$$v_y = g\tau(1 - e^{-\frac{t}{\tau}})$$

We see that

$$t \to \infty \qquad \Longrightarrow \qquad v_y \to g\tau$$

Thus the terminal speed is $g\tau = \frac{mg}{rC_1}$. For $v_y = 0.99 \cdot g\tau$ then

$$\begin{aligned}
0.99 \cdot g\tau &= g\tau(1 - e^{-\frac{t}{\tau}}) \\
0.99 &= (1 - e^{-\frac{t}{\tau}}) \\
e^{-\frac{t}{\tau}} &= 0.01 \\
t &= -\ln(0.01) \cdot \tau \\
t &\approx 4.6 \cdot \tau
\end{aligned}$$

Thus for $t \approx 4.6 \cdot \tau$ the speed is already 99% of the terminal speed.

For Karo Corn Syrup, C_1 is approximately 1.6×10^2 kg/m/s at room temperature; it depends strongly on its temperature, and it may well have been 10% different when we did the demonstration in lectures. (Please see the 8.01 Home Page, Lecture #12 of Wed Oct 6.)

The mass of the 1/4 inch diameter ($r = 1/8$ inch $\approx 3.18 \times 10^{-3}$ m) steel ball is about 1.0 gram. (The density of steel is about 7.8×10^3 kg/m^3. Please see the 8.01 Home Page.)

Thus for the lecture demonstration τ is about 2 msec.

$$\tau = \frac{1.0 \times 10^{-3} \text{ kg}}{3.18 \times 10^{-3} \cdot 1.6 \times 10^2} \approx 2.0 \times 10^{-3} \text{ s} = 2.0 \text{ msec}$$

If you wait $4.6 \cdot \tau$ (which is about 9 msec) you have already reached about 99% of the terminal velocity for the 1/4 inch ball bearing which had speeds as listed below in the table. You can find the results (two measurements) of the 10 AM lecture also on PIVoT.

Class	Time to Drop 4.0 cm	Average Speed
10:00	1.68 ± 0.2 s	2.4 ± 0.3 cm/s
10:00	1.40 ± 0.2 s	2.9 ± 0.4 cm/s
11:00	1.75 ± 0.2 s	2.3 ± 0.3 cm/s

The 10:00 class made two measurements with an average result of 2.7 ± 0.3 cm/s, and the 11:00 class made a measurement with a result of 2.3 ± 0.3 cm/s. The terminal velocity is mg/rC_1, which gives you about 2.0 cm/s (not such a bad agreement given the large uncertainty in C_1).

(e) For $t \to \infty$ the horizontal speed $v_x \to 0$ and the vertical speed $v_y \to \frac{mg}{C_1 r}$.

Problem 4.3 (Ohanian, page 405, problem 1)

(a) The period of the motion is given as $\tau = 1.2$ s, thus the frequency is

$$\nu = \frac{1}{\tau} = \frac{1}{1.2\text{ s}} \approx 0.83\text{ Hz}$$

The angular frequency is

$$\omega = 2\pi\nu = \frac{2\pi}{\tau} = \frac{2\pi}{1.2\text{ s}} \approx 5.2\text{ rad/s}$$

(b) For simple harmonic motion, the amplitude is half the total range of motion; thus the amplitude, which we take to be positive, is

$$A = \frac{1}{2}|0.20\text{ m} - (-0.20\text{ m})| = 0.20\text{ m}$$

(c) We are told that the motion is simple harmonic, so we know x as a function of t is given by

$$x = A\cos(\omega t + \delta)$$

where δ is the phase constant. We also know that $x = 0$ when $t = 0$; this gives the equation

$$0 = A\cos(\omega 0 + \delta) = A\cos(\delta) \qquad \Longrightarrow \qquad \delta = \frac{\pi}{2} + n2\pi \quad \text{or} \quad \delta = -\frac{\pi}{2} + n2\pi$$

The two possible choices (ignoring multiples of 2π) are $\delta = \frac{\pi}{2}$ and $\delta = -\frac{\pi}{2}$. To decide which choice is correct, we need to examine the velocity.

$$v = -\omega A\sin(\omega t + \delta)$$

We are told that the velocity is positive at $t = 0$; therefore

$$v = -\omega A\sin\delta > 0 \qquad \Longrightarrow \qquad \delta = -\frac{\pi}{2}$$

(d) Using the quantities from above we have

$$x = 0.2\cos\left(\omega t - \frac{\pi}{2}\right)$$

We need to consider what times t correspond to $x = 0.20$ m.

$$x = 0.2\cos\left(\omega t - \frac{\pi}{2}\right) = 0.2 \qquad \Longrightarrow$$

$$\begin{aligned}
\cos\left(\omega t - \frac{\pi}{2}\right) &= 1 \\
\omega t - \frac{\pi}{2} &= 2n\pi \\
t &= \frac{1}{\omega}\left(\frac{\pi}{2} + 2n\pi\right) \\
t &= \frac{\tau}{2\pi}\left(\frac{\pi}{2} + 2n\pi\right) \\
t &= \frac{\tau}{4} + n\tau
\end{aligned}$$

The first time the particle reaches $x = 0.20$ m is just

$$t = \frac{\tau}{4} = \frac{1.2 \text{ s}}{4} = 0.30 \text{ s}$$

Now we need to consider what times t correspond to $x = -0.10$ m.

$$x = 0.2\cos\left(\omega t - \frac{\pi}{2}\right) = -0.1 \qquad \Longrightarrow$$

$$\begin{aligned}
\cos\left(\omega t - \frac{\pi}{2}\right) &= -\frac{1}{2} \\
\omega t - \frac{\pi}{2} &= \pm\frac{2\pi}{3} + 2n\pi \\
t &= \frac{1}{\omega}\left(\pm\frac{2\pi}{3} + \frac{\pi}{2} + 2n\pi\right) \\
t &= \frac{\tau}{2\pi}\left(\pm\frac{2\pi}{3} + \frac{\pi}{2} + 2n\pi\right) \\
t &= \tau\left(\frac{1}{4} \pm \frac{1}{3}\right) + n\tau
\end{aligned}$$

The first time the particle reaches $x = -0.10$ m is just

$$t = \frac{7\tau}{12} = \frac{7 \times 1.2 \text{ s}}{12} = 0.70 \text{ s}$$

(e) The speed (which for simple harmonic motion is independent of direction) is given by

$$\left|\frac{dx}{dt}\right| = |A\omega \sin(\omega t + \delta)|$$

The point $x = 0$ corresponds to $t = 0$, so

$$\left|\frac{dx}{dt}\right| = A\omega \left|\sin\left(-\frac{\pi}{2}\right)\right| = A\omega = \frac{2\pi A}{\tau} = \frac{2\pi \times 0.20 \text{ m}}{1.2 \text{ s}} \approx 1.0 \text{ m/s}$$

The point $x = -0.1$ m corresponds to $t = \frac{7\tau}{12}$, so

$$\left|\frac{dx}{dt}\right| = A\omega \left|\sin\left(\frac{2\pi}{3}\right)\right| = \frac{\sqrt{3}A\omega}{2} = \frac{\sqrt{3}\pi A}{\tau} = \frac{\sqrt{3} \times \pi \times 0.20 \text{ m}}{1.2 \text{ s}} \approx 0.91 \text{ m/s}$$

Problem 4.4 (Ohanian, page 405, problem 4)

We are told that the position of the particle as a function of time is given by

$$x = 3.0 \text{ m} \times \cos\left(2.0 \text{ rad/s } t + \frac{\pi}{3}\right)$$

(a) This motion is simple harmonic. We can compare the above to the usual form

$$x = A\cos(\omega t + \delta)$$

From this we identify that

$$A = 3.0 \text{ m}$$

$$\omega = 2.0 \text{ rad/s}$$

$$\nu = \frac{\omega}{2\pi} = \frac{2.0 \text{ rad/s}}{2\pi} \approx 0.32 \text{ Hz}$$

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{2.0 \text{ rad/s}} \approx 3.1 \text{ s}$$

Additionally, note that

$$\delta = \frac{\pi}{3}$$

(b) The equilibrium point corresponds to $x = 0$. We can use the equation of motion to find all times for which $x = 0$.

$$x = A\cos(\omega t + \delta) = 0 \qquad \Longrightarrow$$

$$\begin{aligned}
\cos(\omega t + \delta) &= 0 \\
\omega t + \delta &= \frac{\pi}{2} + n\pi \\
t &= \frac{1}{\omega}\left(\frac{\pi}{2} + n\pi - \delta\right) \\
t &= \frac{\tau}{2\pi}\left(\frac{\pi}{6} + n\pi\right) \\
t &= \frac{\tau}{12} + \frac{n\tau}{2}
\end{aligned}$$

The earliest positive time is

$$t = \frac{\tau}{12} = \frac{2\pi}{12\omega} = \frac{\pi}{6 \cdot 2.0 \text{ rad/s}} \approx 0.26 \text{ s}$$

The turning point corresponds to $x = \pm A$. We can use the equation of motion to find all times for which $x = \pm A$.

$$x = A\cos(\omega t + \delta) = \pm A \qquad \Longrightarrow$$

$$\begin{aligned}
\cos(\omega t + \delta) &= \pm 1 \\
\omega t + \delta &= n\pi \\
t &= \frac{1}{\omega}(n\pi - \delta) \\
t &= \frac{\tau}{2\pi}\left(-\frac{\pi}{3} + n\pi\right) \\
t &= -\frac{\tau}{6} + \frac{n\tau}{2}
\end{aligned}$$

The earliest positive time is

$$t = -\frac{\tau}{6} + \frac{\tau}{2} = \frac{\tau}{3} = \frac{2\pi}{3\omega} = \frac{2\pi}{3 \times 2.0 \text{ rad/s}} \approx 1.0 \text{ s}$$

(c) All the times for which $x = 0$ and $x = \pm A$ were already given above.

$$x = 0 \quad \text{at} \quad t = \frac{\tau}{12} + \frac{n\tau}{2}$$

and

$$x = \pm A \quad \text{at} \quad t = -\frac{\tau}{6} + \frac{n\tau}{2}$$

The part $\frac{n\tau}{2}$ indicates that the particle will move through a turning point or equilibrium exactly at intervals of $\frac{\tau}{2}$. But we must be careful. There are two turning points: half of the times above correspond to $x = +A$, and the other half correspond to $x = -A$. The time $t = -\frac{\tau}{6}$ corresponds to $x = +A$. So the possible times for $x = +A$ are

$$t = -\frac{\tau}{6} + n\tau$$

The possible times for $x = -A$ are

$$t = -\frac{\tau}{6} + \frac{\tau}{2} + n\tau = \frac{\tau}{3} + n\tau$$

Problem 4.5

(a) The motion is simple harmonic, so the position as a function of time is

$$x = A\cos(\omega t + \delta)$$

and the velocity as a function of time is

$$v = -\omega A \sin(\omega t + \delta)$$

The angular frequency is given by

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.4} \approx 15.7 \text{ rad/s}$$

If we choose the direction for x to point along the same direction as the initial displacement then the initial ($t = 0$) displacement from origin is $x = 0.1$ m, so

$$0.1 = A\cos\delta \tag{1}$$

The initial ($t = 0$) velocity is $v = -3$ m/s (it is directed *toward* the origin from the *positive* side), so

$$-3 = -\omega A \sin\delta \tag{2}$$

Using equation (1) and equation (2) we can find δ.

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{3}{0.1\ \omega} = \frac{3\tau}{0.1 \cdot 2\pi}$$

Thus

$$\delta = \tan^{-1}\left(\frac{3\cdot\tau}{0.1\cdot 2\pi}\right) = \tan^{-1}\left(\frac{3\cdot 0.4}{0.1\cdot 2\pi}\right) \approx 1.088$$

(Remember that any value $\delta = 1.088 + 2n\pi$ will do.) Then using equation (1) gives

$$A = \frac{0.1}{\cos\delta} \approx 0.216 \text{ m}$$

Therefore the motion is given by

$$x = 0.216 \cdot \cos(15.7\ t + 1.088)$$

(b) Equilibrium corresponds to $x = 0$.

$$x = A\cos(\omega t + 1.088) = 0 \qquad \Longrightarrow$$

$$\begin{aligned}
\cos(\omega t + 1.088) &= 0 \\
\omega t + 1.088 &= \frac{\pi}{2} + n\pi \\
t &= \frac{1}{\omega}\left(\frac{\pi}{2} + n\pi - 1.088\right) \\
t &= \frac{\tau}{2\pi}(0.48 + n\pi) \\
t &= 7.7 \times 10^{-2}\cdot\tau + \frac{n\tau}{2}
\end{aligned}$$

The first time the mass will pass through equilibrium is

$$t = 7.7 \times 10^{-2} \cdot \tau = 3.1 \times 10^{-2} \text{ s}$$

The total energy is conserved.

$$E = K + U$$

with

$$K = \frac{1}{2}mv^2 \quad \text{and} \quad U = \frac{1}{2}kx^2 = \frac{m\omega^2 x^2}{2} = \frac{2\pi^2 m}{\tau^2}x^2$$

At $t = 0$ the energy is given by

$$E = \frac{1}{2}mv_0^2 + \frac{2\pi^2 m}{\tau^2}x_0^2 = \frac{1}{2}m \cdot (3)^2 + \frac{2\pi^2 m}{\tau^2} \cdot (0.1)^2 \approx 17.2 \text{ J}$$

By energy conservation, this will be the total energy at any other time.

At equilibrium, $x = 0$, and the potential energy is

$$U = \frac{1}{2}k(0)^2 = 0$$

Thus the total energy, which is now just the kinetic energy, is

$$E = K = \frac{1}{2}mv^2 \qquad \Longrightarrow \qquad |v| = \sqrt{\frac{2E}{m}}$$

Thus the kinetic energy is

$$K = E \approx 17.2 \text{ J}$$

and the speed is

$$|v| = \sqrt{\frac{2E}{m}} \approx 3.4 \text{ m/s}$$

The acceleration at $x = 0$ is given by

$$a = \frac{d^2x}{dt^2} = -\omega^2 x \qquad \Longrightarrow \qquad a = 0$$

(c) The turning points corresponds to $x = \pm A$. These points also correspond to $v = 0$.

$$v = -\omega A \sin(\omega t + 1.088) = 0 \qquad \Longrightarrow$$

$$\begin{aligned}
\sin(\omega t + 1.088) &= 0 \\
\omega t + 1.088 &= n\pi \\
t &= \frac{1}{\omega}(n\pi - 1.088) \\
t &= \frac{\tau}{2\pi}(n\pi - 1.088) \\
t &= -0.18 \cdot \tau + \frac{n\tau}{2}
\end{aligned}$$

The first time the mass will pass through a turning point is

$$t = -0.18 \cdot \tau + \frac{\tau}{2} \approx 0.13 \text{ s}$$

By substituting $t = 0.13$ s into the equation of motion we find that the corresponding position is $x = -0.216$ m. (We knew it had to be $x = \pm 0.216$ m by the above.)

At a turning point, $v = 0$, and the kinetic energy is

$$K = \frac{1}{2}m(0)^2 = 0$$

Thus the total energy, which is now just the potential energy, is

$$U = E \approx 17.2 \text{ J}$$

The acceleration at $x = -0.216$ m is given by

$$a = \frac{d^2x}{dt^2} = -\omega^2 A \qquad \Longrightarrow \qquad a = 53.2 \text{ m/s}^2$$

Problem 4.6 (Ohanian, page 406, problem 11)

Springs and gravity are discussed in example 2 on page 386. The key point in that example is that gravity only lowers the equilibrium position of the spring. The motion is still simple harmonic with the same frequency. The equilibrium length is moved down by an amount

$$\Delta x = \frac{mg}{k} \approx 0.27 \text{ m}$$

Thus, holding the spring at the unstretched position is actually holding it at a displacement of Δx up from equilibrium. At this point, determining the motion will be very similar to problem 4.5 part (a). Label the vertical direction with x; define the increasing direction of x to point down; and choose $x = 0$ to coincide with the *new* point of equilibrium. The initial velocity is zero, so $\delta = 0$. The initial displacement is $-\Delta x$. Thus the motion is given by

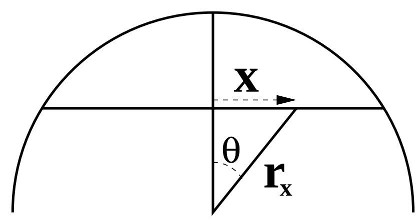
$$x = -\frac{mg}{k}\cos\left(\sqrt{\frac{k}{m}}t\right) = -0.27 \text{ m} \times \cos(6 \text{ rad/s } t)$$

Other choices of labeling the motion lead to different, but equivalent, forms for the motion.

Problem 4.7 (Ohanian, page 410, problem 43)

(a) By examining the picture, the relationship between x, the distance from the center of the tunnel, and r_x, the distance to the center of the planet, is

$$x = r_x \sin\theta$$



By equation (45) on page 232, the acceleration due to gravity is given by

$$g = \left(\frac{GM}{R^3}\right) r_x$$

The component of the acceleration of gravity along the track is

$$g_x = -g\sin(\theta) = -\left(\frac{GM}{R^3}\right) r_x \sin(\theta) = -\left(\frac{GM}{R^3}\right) x$$

(b) Newton's Second Law gives the following equation of motion

$$a = g_x = -\left(\frac{GM}{R^3}\right) x$$

which is of the form

$$\ddot{x} = -\omega^2 x$$

which has the solution

$$x = A\cos(\omega t + \delta)$$

with

$$\omega = \sqrt{\frac{GM}{R^3}}$$

The period is then given by

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}}$$

(c) Since the train starts at rest from San Fransico, the amplitude of the motion is half the tunnel length. Thus the time to travel from San Fransisco to Washington, D.C. would be

$$\Delta T = \frac{\tau}{2} = \pi\sqrt{\left(\frac{R^3}{GM}\right)} \approx 2550 \text{ s}$$

The tunnel length, l, is worked out in example 5 on page 18. Thus the amplitude of motion is

$$A = \frac{l}{2} = \frac{3.8 \times 10^3 \text{ km}}{2} = 1.9 \times 10^3 \text{ km}$$

The motion is then given by

$$x = A\cos(\omega t) \qquad \Longrightarrow \qquad v = -\omega A\sin(\omega t)$$

It takes the train the time $t = \frac{\tau}{4}$ to reach the midpoint of the tunnel, thus the maximum speed is given by

$$|v| = \omega A\sin\left(\frac{2\pi}{\tau}\cdot\frac{\tau}{4}\right) = \omega A\sin\left(\frac{\pi}{2}\right) = \omega A = \frac{2\pi A}{\tau} = \frac{2\pi \times 1.9 \times 10^3 \text{ km}}{5100 \text{ s}} \approx 2.3 \text{ km/s}$$

Problem 4.8

Let $l = 2$ m and $m = 3$ kg and $g = 10$ m/s^2. We are going to measure the vertical height from the bottom of the swing, and let θ be the angle of the string relative to the vertical. If you work out the geometry, the height of the bob, relative to the bottom, as a function of angle, θ, is given by

$$h = l(1 - \cos\theta)$$

(Please see Figure 15.16 on page 391 for a clear depiction of the geometry.) Therefore the potential energy as a function of θ is

$$U = mgh = mgl(1 - \cos\theta)$$

The total energy is conserved for such a pendulum. (The tension always acts perpendicular to the motion, thus it does no work; hence energy is conserved. This is also shown on page 392.) The bob is released from rest at an angle $\theta = 30°$ relative to the vertical. Thus the total energy initially, which is solely potential, is

$$E = mgl(1 - \cos 30°) \approx 8.038 \text{ J}$$

(a) At the bottom of the swing the total energy, which is solely kinetic, is

$$E = \frac{1}{2}mv^2 \qquad \Longrightarrow \qquad |v| = \sqrt{\frac{2E}{m}} = \sqrt{2gl(1 - \cos 30°)} \approx 2.3 \text{ m/s}$$

(b) The consider when the string makes an angle $\theta = 10°$ relative to the vertical. The energy is given by

$$E = mgl(1 - \cos 10°) + K \qquad \Longrightarrow \qquad K = mgl(\cos 10° - \cos 30°) \approx 7.13 \text{ J}$$

(c) The answer to part (a) was independent of m; thus the speed at the bottom wouldn't change if the mass doubled. The answer to part (b) was linear in m; thus the kinetic energy at $10°$ would double if the mass doubled.

Problem 4.9 (Ohanian, page 178, problem 8)

The work done by a force on a particle as it travels from position P_1 to position P_2 is given by

$$W = \int_{P_1}^{P_2} F dx$$

For the current problem, we can evaluate this integral by summing areas of four regions. The regions are defined as: (1) $0 \leq x \leq 2$, (2) $2 \leq x \leq 4$, (3) $4 \leq x \leq 6$, and (4) $6 \leq x \leq 8$. In region (1) the force as a function of position is

$$F_x = \left(\frac{2 \text{ N} - 0 \text{ N}}{2 \text{ m} - 0 \text{ m}}\right) x = (1 \text{ N/ m})x$$

In region (2) the force as a function of position is

$$F_x = 2 \text{ N}$$

In region (3) and (4) the force as a function of position is

$$F_x = \left(\frac{-2 \text{ N} - 2 \text{ N}}{8 \text{ m} - 4 \text{ m}} \right) x + 2 \text{ N} = (-1 \text{ N/ m})x + 2 \text{ N}$$

Thus the work done is

$$W = \int_0^2 F dx + \int_2^4 F dx + \int_4^6 F dx + \int_6^8 F dx$$

You could evaluate these integrals by using the usual rules of integration and the functions for F_x written above. But it is probably easier to simply add the areas. (Remember that area beneath the line $y = 0$ is counted as negative.) So the work is

$$W = \underbrace{\frac{1}{2} \times 2 \text{ N} \times 2 \text{ m}}_{0 \le x \le 2} + \underbrace{2 \text{ N} \times 2 \text{ m}}_{2 \le x \le 4} + \underbrace{\frac{1}{2} \times 2 \text{ N} \times 2 \text{ m}}_{4 \le x \le 6} - \underbrace{\frac{1}{2} \times 2 \text{ N} \times 2 \text{ m}}_{6 \le x \le 8} = 6 \text{ J}$$

Problem 4.10 (Ohanian, page 178, problem 10)

This man pushes the box at constant speed, so the force of his push must be just enough to balance all forces on the box. The various forces are given by

$$N = mg \cos 30^\circ$$

$$\mathcal{F} = \mu mg \cos 30^\circ$$

$$F = mg(\mu \cos 30^\circ + \sin 30^\circ) \approx 520 \text{ N}$$

Where F is the force which this man exerts on the box, $m = 60$ kg, $\mu = 0.45$. (The other two forces are the normal force, N, and the frictional force, $\mathcal{F}$.) But to push the box *up* the height 2.5 m he must push the box *along* the ramp a length

$$l = \frac{2.5 \text{ m}}{\sin 30^\circ} = 5 \text{ m}$$

He applies a constant force along the whole length, so

$$W = Fl \approx 2.6 \times 10^3 \text{ J}$$

Problem 4.11 (Ohanian, page 182, problem 52)

If we ignore friction, there are only two forces on the coaster: gravity, which always points down, and the normal force that the track exerts on the car. The normal force cancels the component of acceleration of the car perpendicular to the track. If the speed of the car is too low the normal force will vanish and the car will leave the track. Now, we consider a case for which the normal force just vanishes.

The top of the loop is a circular arc; the force of gravity is pointed straight down. The gravitational force must provide the centripetal acceleration. Thus

$$a_{cp} = \frac{v^2}{r} = g \qquad \Longrightarrow \qquad v = \sqrt{rg}$$

For the radius R, the critical speed is

$$v_{cr} = \sqrt{Rg} \approx 9.9 \text{ m/s}$$

If the speed is less than the critical speed, v_{cr}, then the coaster car will fall off the track. If the speed is larger than v_{cr} then the car will stay on the track and there will be a normal force pointing down at the highest point.

The minimum speed at the bottom is given by energy conservation. If we measure height relative to the bottom of the loop, then the energy at the top of the loop is

$$E = \frac{mv_{top}^2}{2} + mgh$$

where $h = 40$ m is the height of the highest point of the loop and v_{top} is the speed at the top of the loop. The energy at the bottom of the loop is

$$E = \frac{mv_b^2}{2}$$

where v_b is the speed at the bottom of the loop. Thus energy conservation gives

$$v_b \geq \sqrt{v_{top}^2 + 2gh} \approx 29.7 \text{ m/s}$$

Problem 4.12

Let $m = 3.0$ kg, $\mu_s = 0.30$, $\mu_k = 0.20$, and $k = 80$ N/m.

(a) There are two vertical forces: gravity (mg) and the normal force (N) exerted by the table. These forces must balance.

$$N = mg$$

There are two horizontal forces: friction ($\mathcal{F}$) exerted by the table and the pull (F_s) of the spring. By the nature of friction these forces will be in opposite directions. If the block is to remain stationary, then these forces must balance.

$$F_s = \mathcal{F}$$

If the forces balance, then there will be no acceleration. If there is no motion at all, the frictional force is static, and thus

$$|\mathcal{F}| \le \mu_s mg \qquad \Longrightarrow \qquad |F_s| \le \mu_s mg$$

The pull of the spring is related to the extension of the spring, x, by

$$|F_s| = k|x| \qquad \Longrightarrow \qquad |x| \le \frac{\mu_s mg}{k}$$

Thus the maximum extension for which the block will remain stationary is

$$x_{max} = \frac{\mu_s mg}{k} \approx 0.11 \text{ m}$$

(b) The 'very gentle push' has the effect of making the friction kinetic, not static. At this maximum extension, the kinetic friction will not hold the block stationary. (The static friction just barely held the block stationary, and the kinetic friction is less than the static.) Thus the block will begin to move towards the equilibrium point. The spring force will speed the block up, but friction will eventually stop the block. So it is clear that there will be some maximum speed.

The object will accelerate at first as the magnitude of the spring force, F_s, will be larger than that of the frictional force. This acceleration will continue till the two forces have become equal in magnitude. After that, the frictional force will be larger in magnitude than the spring force, and thus the speed will decrease. Therefore, the maximum speed is reached when these two forces are equal (the acceleration is then zero). Thus

$$\mu_k mg = kx \qquad \Longrightarrow \qquad x \approx 0.074 \text{ m}$$

Problem 4.13

This pendulum will behave like any other pendulum until the moment it strikes the pin. In particular, the bob is released 'without speed,' thus the first phase of motion (until the string strikes the pin) will be identical to that discussed in problem 4.8. There it was shown that the potential energy is

$$U = mgl(1 - \cos\theta)$$

where l, m, and θ have identical meanings. We can use energy conservation to find the speed of the bob just before impact, which corresponds to the bottom of the swing. The total energy at θ_0, which is solely potential, is

$$E = U = mgl(1 - \cos\theta_0)$$

Now the string strikes the pin. We assume the string is massless, so it carries no energy; therefore the bob retains all the energy it had just before the impact. The system will now behave like a pendulum fixed to the pin. The length of free string is

$$l' = l - L$$

The bob will swing until energy conservation prevents it from swinging further. At this maximum angle, the energy is solely potential. The situation is analogous to the one we started with. The energy is given by

$$E = U = mgl'(1 - \cos\alpha) = mg(l - L)(1 - \cos\alpha)$$

Conservation of energy implies that both energies are the same, thus

$$E = mg(l - L)(1 - \cos\alpha) \qquad \Longrightarrow \qquad \cos\alpha = 1 - \frac{E}{mg(l - L)}$$

Using the energy given above gives the result for α

$$\cos\alpha = \frac{l\cos\theta_0 - L}{l - L}$$

(We must be careful here. If $|\theta_0|$ is too large then the bob will complete a full 'loop' about the pin; and our above result will be incorrect. This problem will occur when the above equation indicates that $\cos\alpha < -1$.)

(b) If the bob had been released with an initial tangential velocity, the only part of the above discussion that would change is the first calculation for energy which would now include the kinetic energy as well. Thus the result for α

$$\cos\alpha = 1 - \frac{E}{mg(l - L)}$$

would not change. But the value for E would become

$$E = mgl(1 - \cos\theta_0) + \frac{mv_0^2}{2} \qquad \Longrightarrow \qquad \cos\alpha = \frac{l\cos\theta_0 - L}{l - L} - \frac{v_0^2}{2g(l - L)}$$

(We must be careful here. If $|v_0|$ is too large or if $|\theta_0|$ is too large then the bob will complete a full 'loop' about the pin; and our above result will be incorrect. This problem will occur when the above equation indicates that $\cos\alpha < -1$.)

The dependence of α on v_0 is quadratic, i.e. the sign of v_0 does not seem to matter. Physically there are three different cases for the motion:

(1) The bob moves immediately toward the pin, strikes the pin, and then moves to the maximum angle α;

(2) The bob initially moves away from the pin, the value of v_0 was large enough that the bob can complete a full 'loop,' the bob strikes the pin from the other side, and then moves to the maximum angle α;

(3) The bob initially moves away from the pin, the value of v_0 was *not* large enough that the bob can complete a full 'loop,' at some point the bob swings back toward the pin as for case (1), strikes the pin, and then moves to the maximum angle α.

(Note that there is a case between (2) and (3) in which the bob moves partly around the 'loop' but drops during the process. If we ignore any energy that is loss during the bouncing then even this case will eventually swing to the maximum angle α.)

Problem 4.14 (Ohanian, page 182, problem 52)

(a) Initially, the suitcase is 'stationary,' and the conveyor belt is 'moving.' Thus there is relative motion between the two, and the friction is kinetic.

(b) Friction always opposes the *relative* motion, but depending on the frame of reference, friction may or may not be aligned with the motion in that frame. To see this most clearly, we will use two frames of reference to answer this question.

First, imagine the situation from the perspective of the conveyor belt. The suitcase is moving left, and the conveyor belt is stationary. The suitcase slams onto the conveyor belt, and skids until it stops. The direction of friction, which caused the skidding, and the apparent direction of motion are opposite. Clearly friction did negative work: initially there was kinetic energy, and eventually there was none.

Now, imagine the situation from the perspective of a person standing next to the conveyor belt. The suitcase is initially stationary, and the conveyor belt is moving right. The suitcase lands on the conveyor belt, and accelerates right until it's speed matches the speed of the conveyor belt. The direction of friction, which caused the acceleration, and the apparent direction of motion were the same. Clearly friction did positive work: initially there was no kinetic energy, and eventually there was a non-zero kinetic energy.

The conclusion is that work is a quantity that depends on the frame of reference.

(c) Presumably the conveyor belt is moving at a constant speed. If the suitcase has reached the speed of the conveyor belt then it too has a constant speed, thus the net horizontal force must be zero. If the conveyor belt and suitcase have identical velocities, then there is no relative motion and no force attempting to cause relative motion; therefore there is no friction.