22.615, MHD Theory of Fusion Systems
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Lecture 2: Derivation of Ideal MHD Equation

Review of the Derivation of the Moment Equation

1. Starting Point: Boltzmann Equation for electrons, ions and Maxwell Equations

2. Moments of Boltzmann Equation: conservation of mass, momentum and energy.

$$\int\left[\frac{dF_\alpha}{dt}-\left(\frac{\partial F_\alpha}{\partial t}\right)_c\right]\left\{\begin{matrix}1\\ m_\alpha \underline{v}\\ m_\alpha v^2/2\end{matrix}\right\}d\underline{v}\qquad\begin{matrix}\text{mass}\\ \text{momentum}\\ \text{energy}\end{matrix}$$

3. Accounting: $\underline{v}=\underline{u}_\alpha(e,t)+\underline{\tilde{v}}$, $\underline{u}_\alpha=$ fluid velocity, $\underline{\tilde{v}}=$ random velocity

$n_\alpha=\int F_\alpha d\underline{v}$ density

$\underline{u}_\alpha=\frac{1}{n_\alpha}\int \underline{v}F_\alpha d\underline{v}=\langle\underline{v}\rangle$ fluid velocity

$\overleftrightarrow{P}_\alpha=n_\alpha m_\alpha\langle\underline{\tilde{v}}\,\underline{\tilde{v}}\rangle$ pressure tensor

$p_\alpha=\frac{1}{3}m_\alpha n_\alpha\langle\tilde{v}^2\rangle$ scalar pressure

$h_\alpha=\frac{n_\alpha m_\alpha}{2}\langle\tilde{v}^2\underline{\tilde{v}}\rangle$ heat flux

$\underline{R}_\alpha=\int m_\alpha\underline{\tilde{v}}C_{\alpha\beta}d\underline{\tilde{v}}$ friction due to collisions

$Q_\alpha=\int\frac{m_\alpha\tilde{v}^2}{2}C_{\alpha\beta}d\tilde{v}$ heat generated due to collisions

General 2 Fluid Equations

$$\nabla\times\underline{E}=-\frac{\partial B}{\partial t}\qquad\nabla\cdot B=0$$

$$\nabla\times\underline{B}=\mu\cdot\underline{J}+\frac{1}{c^2}\frac{\partial\underline{E}}{\partial t}\qquad\nabla\cdot\underline{E}=\frac{\sigma}{\epsilon_0}$$

$$\frac{\partial n_\alpha}{\partial t}+\nabla\cdot(n_\alpha\underline{u}_\alpha)=0$$

$$m_\alpha n_\alpha \frac{d\underline{u}_\alpha}{dt} = q_\alpha n_\alpha \left(\underline{E} + \underline{u}_\alpha \times \underline{B}\right) - \nabla \cdot \underline{\underline{P}}_\alpha + \underline{R}_\alpha$$

$$\frac{3}{2} n_\alpha \frac{dT_\alpha}{dt} + \underline{\underline{P}}_\alpha : \nabla \underline{u}_\alpha = Q_\alpha - \nabla \cdot \underline{h}_\alpha$$

e, i

$$\sigma = e\left(n_i - n_e\right)$$

$$\underline{J} = e\left(n_i \underline{u}_i - n_e \underline{u}_e\right)$$

Physical Assumptions Leading to Ideal MHD

1. Moment equations as they now stand are exact, but not closed.

2. Certain assumptions lead to closure - 1 fluid MHD model

Asymptotic Assumptions

1. MHD is concerned with low frequency - long wavelength macroscopic behavior

2. The first simplification of the 2 fluid equations eliminates short wavelength, fast time scale phenomena: well satisfied assumptions experimentally

3. Asymptotic assumptions change basic mathematical structure of the time evolution.

   speed of light $\to \infty$

   electron inertia $\to 0$

First Asymptotic Assumption $c \to \infty$

1. Maxwell equations $\to$ low frequency Maxwell equations

2. Formally let $\in_0 \to 0$

   $$\nabla \times \underline{B} = u_0 \underline{J} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \approx \mu_0 \underline{J}$$ neglect displacement current

   $$n_i - n_e = \frac{\in_0}{e} \nabla \cdot \underline{E} \approx 0$$ quasineutrality

3. Equations are now Gallilean invariant

4. Conditions for validity:

   $$\omega \ll \omega_{pe} \qquad \lambda_d \equiv \frac{v_{Te}}{\omega_{pe}} \ll a$$ no plasma oscillations

$\frac{\omega}{k} \sim v_{Ti} \ll v_{Te} \ll c$ no high frequency waves

5. Note: $n_e = n_i \equiv n$ does not imply $\underline{E}$ or $\nabla \cdot \underline{E} = 0$. Only that

   $\epsilon_0 \nabla \cdot E / en \ll 1$

Second Asymptotic Assumption $m_e \to 0$

1. The electron response time is essentially instantaneous because $m_e \ll m_i$

2. We then neglect electron inertia in the momentum equation

   $0 \approx -en_e \left(\underline{E} + \underline{u}_e \times \underline{B}\right) - \nabla \overleftrightarrow{P}_e + \underline{R}_e$

3. Conditions for validity

   $\omega \ll \omega_{pe} \quad \lambda_d \ll a$ no electron plasma oscillations $\parallel$ to B

   $\omega \ll \omega_{ce} \quad r_{ce} \ll a$ no electrons cyclotron oscillations

4. Both $c \to \alpha,\ m_e \infty 0$ assumptions are well satisfied for MHD behavior

Subtle Effect

1. Neglect of electron inertia along B can be tricky

2. For long wavelengths, electrons can still require a finite response time even though m_e is small. This is region of the drift wave

3. We shall see that MHD consistently treats $\parallel$ motion poorly, but for MHD behavior, remarkably this does not matter!!

4. To treat such behavior more sophisticated models are required. The resulting instabilities are much weaker, (and still important) than for MHD.

The two Fluid Equations with Asymptotic Assumptions

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \nabla \cdot \underline{B} = 0 \qquad \frac{\partial n}{\partial t} + \nabla \cdot n\underline{u}_e = 0$$

$$\nabla \times \underline{B} = \mu_0 en\left(u_i - u_e\right) \qquad n_e = n_i = n \qquad \frac{\partial n}{\partial t} + \nabla \cdot n\underline{u}_i = 0$$

$$m_i n \frac{d\underline{u}_e}{dt} - en\left(\underline{E} + \underline{u}_i \times \underline{B}\right) + \nabla \cdot \underline{\underline{\overleftrightarrow{P}}}_i = \underline{R}_i$$

$$\left.\frac{3}{2} n \frac{dT_\alpha}{dt} + \underline{\underline{\overleftrightarrow{P}}}_\alpha : \nabla \underline{u}_\alpha + J \cdot \underline{h}_\alpha = Q_\alpha \right\} e$$

$$en\left(\underline{E} + \underline{u}_e \times \underline{B}\right) + +\nabla \cdot \underline{\underline{\overleftrightarrow{P}}}_e = \underline{R}_e$$

Single Fluid Equations

1. Introduce single fluid variable

   $\underline{v} = \underline{u}_i$ the momentum of fluid is carried by ions since $m_i = 0$

   $p = p_e + p_i$ total pressure

   $\rho = m_i n$ mass density

   $\underline{J} = en\left(\underline{u}_i - \underline{u}_e\right)$ current density

2. Use all information!!. This is not trivial!! Initially the unknowns are $\underline{E}$, $\underline{B}$, $\underline{J}$, $\underline{V}$, n, p (19 variables). The finally unknowns are $\underline{E}$, $\underline{B}$, $\underline{J}$, $\underline{V}$, n, p (14 variables)

3. Maxwell equations $\rightarrow$ OK as is in low frequency form

4. Mass conservation

   a. $M_i \times$ ion

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0$$

   b. e (ion-electron) $\rightarrow$ $\nabla \cdot en\left(\underline{u}_i - \underline{u}_e\right)$

   $$= \nabla \cdot \underline{J} = 0$$

   This is automatic from the low frequency Maxwell equations

   $\nabla \times \underline{B} = \mu_0 \underline{J} \rightarrow \nabla \cdot \underline{J} = 0$

5. Momentum Equation (ion + electron)

   a. $$\rho \frac{d\underline{v}}{dt} - en\left(\underline{u}_i - \underline{u}_e\right) \times B + \nabla \cdot \left(\underline{\underline{\overleftrightarrow{P}}}_i + \underline{\underline{P}}_e\right) = \underline{R}_i + \underline{R}_e$$

   $$\underline{J} \times \underline{B} \qquad \nabla \cdot \left[\left(p_i + p_e\right)\underline{\underline{I}} + \underline{\underline{\overleftrightarrow{\Pi}}}_i + \underline{\underline{\overleftrightarrow{\Pi}}}_e\right] \qquad \int d\underline{\tilde{v}}\left[m_e \underline{\tilde{v}}\, c_{ei} + m_i \underline{\tilde{v}}\, c_{ie}\right] = 0$$

b. $\rho\frac{d\underline{v}}{dt} - \underline{J}\times\underline{B} + \nabla p = -\nabla\cdot\left(\overleftrightarrow{\underline{\underline{\Pi}}}_i + \overleftrightarrow{\underline{\underline{\Pi}}}_e\right)$

6. Electron Momentum equation

a. $\underline{E} + \underline{u}_e \times \underline{B} = \frac{\underline{R}_e - \nabla\cdot\underline{\underline{P}}_e}{en}$

$\underline{u}_e = \underline{u}_i - \frac{\underline{J}}{en} = \underline{v} - \frac{\underline{J}}{en}$

b. $\underline{E} + \underline{v}\times\underline{B} = \frac{1}{en}\left[\underline{R}_e - \nabla\cdot\underline{\underline{P}}_e + \underline{J}\times\underline{B}\right]$

7. Energy Equation (ions)

$\overleftrightarrow{\underline{\underline{P}}}_i : \nabla\underline{u}_i$

a. $\frac{3}{2}n\frac{d}{dt}\frac{p_i}{n} + p_i\nabla\cdot\underline{u}_i = Q_i - \nabla\cdot\underline{h}_i - \overleftrightarrow{\underline{\underline{\Pi}}}_i : \nabla\underline{u}_i$

1 2

b. 1: $\frac{3}{2}\frac{dp_i}{dt} - \frac{3}{2}\frac{p_i}{n}\frac{dn}{dt}$

c. 2: $\frac{\partial n}{\partial t} + \nabla\cdot n\underline{v} = 0 = \frac{\partial n}{\partial t} + \underline{v}\cdot\nabla n + n\nabla\cdot\underline{v} \rightarrow \frac{dn}{dt} = -n\nabla\cdot\underline{v}$

$p_i\nabla\cdot\underline{v} = -\frac{p_i}{n}\frac{dn}{dt}$

d. 1+2: $\frac{3}{2}\frac{dp_i}{dt} - \frac{5}{2}\frac{p_i}{n}\frac{dn}{dt} = \frac{3}{2}n^{5/3}\frac{d}{dt}\frac{p_i}{n^{5/3}}$

e. $\frac{d}{dt}\frac{p_i}{\rho^r} = \frac{2}{3\rho^r}\left[Q_i - \nabla\cdot\underline{h}_i - \overleftrightarrow{\underline{\Pi}}_i : \nabla\underline{v}\right]$ $\quad r = 5/3$

8. Energy Equation (electrons)

a. $\frac{d}{dt}\frac{p_e}{\rho^r} = \frac{2}{3\rho^r}\left[Q_e - \nabla\cdot\underline{h}_e - \overleftrightarrow{\underline{\Pi}}_e : \nabla\underline{v} + \frac{\underline{J}}{en}\cdot\nabla\frac{p_e}{\rho^r} + \overleftrightarrow{\underline{\Pi}}_e : \nabla\frac{d}{en}\right]$

from $\frac{d}{dt_e}$ from $\overleftrightarrow{\underline{\Pi}}_e : \nabla u_e$

b. $\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla =$ ion convective derivation

Assumptions Leading to Ideal MHD

1. Philosophy: Ideal MHD is concerned with phenomena occurring on certain length and time scales.

2. Ordering: Using this, we can order all the terms in the one fluid equations. After ignoring small terms, we obtain ideal MHD.

3. Status: At this point only the assumptions $c \to \infty$, $m_e \to 0$ have been used in the equation

Characteristic Length and Time Scales for Ideal MHD

1. $\frac{\partial}{\partial t} \sim \omega \sim \frac{v_{Ti}}{a}$
2. $\frac{\partial}{\partial x} \sim k \sim \frac{1}{a}$
3. $v \sim v_{Ti}$

} macroscopic MHD phenomena

4. $a \to$ macroscopic length

5. $v_{Ti} \to$ macroscopic ion velocity

6. $a/v_{Ti} \to$ corresponding macroscopic time scale

Two Approaches to Ideal MHD

A. Collision dominated plasma: regions limit to ideal MHD

B. Collision free limit: also works but for subtle reasons

Collision Dominated Limit

1. The electrons and ions are assumed collision dominated

2. This is the basic requirement to keep the pressure isotropic. Many collisions keep particle close together. This allows us to divide the plasma into small fluid element and provides a good physical description.

3. There are 2 conditions for a collision dominated plasma

   a. on the time scale of internal there are many collisions, so the plasma is near maxwellion

      - ions: ion-ion coulomb collisions dominate

- electrons: electron-ion, electron-electron collisions are comparable

- ions: $\omega\tau_{ii} \sim \dfrac{v_{Ti}\tau_{ii}}{a} \ll 1$

- electrons: $\omega\tau_{ee} \sim \omega\tau_{ee} \sim \dfrac{v_{Ti}}{a}\tau_{ee} \sim \left(\dfrac{m_e}{m_i}\right)^{1/2} \dfrac{v_{Ti}\tau_{ii}}{a} \ll 1$

- Recall: $\tau_{ee} \sim \tau_{ei} \sim (m_e/m_i)^{1/2}\,\tau_{ii}$ and $\tau_{EQ} \sim (m_i/m_e)^{1/2}\,\tau_{ii}$

- The ion condition is most severe

$$\frac{v_{Ti}\tau_{ii}}{a} \ll 1$$

b. The macroscopic length scale must be much larger than the mean free path for collisions. $\lambda_\alpha = v_{T\alpha}\tau_{\alpha\alpha}$

- ions $\dfrac{\lambda_i}{a} = \dfrac{v_{Ti}\tau_{ii}}{a} \ll 1$ (same as before)

- electrons $\dfrac{\lambda_e}{a} \sim \dfrac{v_{Te}\tau_{ee}}{a} \sim \dfrac{v_{Ti}\tau_{ii}}{a} \ll 1$ (same as ions)

MHD Limit

1. Use the collision dominated assumption to obtain ideal MHD

2. Several additional assumptions will also be required

3. Various moments in the equations are approximated by classical transport theory of Braginskii.

4. Transport coefficients can also be derived in the homework problems

Reduction of 1 Fluid Equation

1. Maxwell Equations – OK

2. Mass conservation – OK

3. Momentum Equation

a. ions: $\overleftrightarrow{\Pi}_{ii} \sim \mu_i\left[2\nabla_{\|}\cdot \underline{u}_{i\|} - \frac{2}{3}\nabla\cdot\underline{u}_i\right] \sim \underbrace{\mu_i}_{\text{viscosity}}\frac{u_i}{a}$

b. electrons: $\overleftrightarrow{\Pi}_{ee} \sim \mu_e \dfrac{u_e}{a}$

c. Note: $\underline{u}_e = \underline{v} - \dfrac{\underline{J}}{en}$

$$\frac{J}{env} \sim \frac{\nabla p}{Benv} \sim \frac{T}{aBev_{Ti}} \sim \frac{r_{ii}}{a} \ll 1$$

assume small gyro radius

d. $\therefore \underline{u}_i \approx \underline{u}_e$: small difference in the flow velocities generate macroscopic current density $\underline{J}$, but $\left|\underline{u}_i - \underline{u}_e\right| \ll v_{Ti}$

e. Ordering:

- $\Pi_{ee} \sim \mu_e \dfrac{u_e}{a} \sim \mu_e \dfrac{v_{Ti}}{a}$
- $\mu_e \sim \left(\dfrac{m_e}{m_i}\right)^{1/2} \mu_i \rightarrow \Pi_e \ll \Pi_i$
- $\dfrac{\Pi_{ii}}{P_i} \sim \dfrac{\mu_i v_{Ti}}{ap_i}$ $\quad \mu_i \sim nT_i\tau_{ii}$ viscosity coefficient
- $\therefore \dfrac{\Pi_{ii}}{p_i} \sim \dfrac{\tau_{ii} v_{Ti}}{a} \ll 1$ collision dominated assumption
- Both Π terms are negligible in momentum equation

f. $\rho \dfrac{d\underline{v}}{dt} = J \times B - \nabla p$ momentum equation

4. Ohms Law

$$\underset{4}{\underline{E} + \underline{v} \times \underline{B}} = \frac{1}{en}\Big[\underset{3}{\underline{R}_e} - \underset{2}{\nabla \cdot \overleftrightarrow{\underline{\underline{P}}}_e} + \underset{1}{\underline{J} \times \underline{B}}\Big]$$

1: Hall effect

2: Electron diamagnetism ω_{re}

3: Resistivity

a. 1 ~ 2

$1\ /\ 4\ \sim J\big/ enV \sim \frac{r_{Li}}{a} \ll 1$ small gyro radius assumptions

b. $\underline{R}e \sim$ resistivity momentum transfer due to collisions

- $\underline{R}_e = en\,\eta\,\underline{d},\ \eta = \frac{m_e}{ne^2\tau_{ei}}$

- $3\ /\ 4 \sim \frac{m_e}{ne^2\tau_{ei}}\frac{J}{v_{Ti}B} \sim \frac{m_e}{e\tau_{ei}B}\left(\frac{r_{ii}}{a}\right) \sim \left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{a}{V_{Ti}\tau_{ii}}\right)\left(\frac{r_{ii}}{a}\right)^2$

c. $\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{a}{v_{Ti}\tau_{ii}}\right)\left(\frac{r_{ii}}{a}\right)^2 \ll 1$

d. The plasma must be larger enough so that resistive diffusion does not play an important role.

5. Energy equation $\left(\underline{v}\cdot\nabla \sim \frac{\partial}{\partial t}\right)$

a. ions: $\Pi_i/p_i \ll 1$

b. electrons: $\Pi_e/p_e \ll 1,\ \left(J/en\cdot\nabla\right)p_e \ll \nabla\cdot p_e,\ \left(\Pi_e\, J/en\right) \ll vp_e$

c. $\frac{d}{dt}\frac{p_i}{\rho^r} = \frac{2}{3\rho^r}\left[Q_i - \nabla\cdot\underline{h}_i\right]$

d. $\frac{d}{dt}\frac{p_e}{\rho^r} = \frac{2}{3\rho^r}\left[Q_e - \nabla\cdot\underline{h}_e\right]$

e. $\underline{h}_i = -\kappa_{\|i}\nabla_{\|}T_i - \kappa_{\perp i}\nabla_{\perp}T_i$

f. $\underline{h}_e = -\kappa_{\|e}\nabla_{\|}T_e - \kappa_{\perp e}\nabla_{\perp}T_e$

(e, f) dominant contribution is from thermal conduction

g. In general $\kappa_{\|} \ggg\ggg\ggg \kappa_{\perp}$

h. $Q_1 = -\frac{n\left(T_i - T_e\right)}{\tau_{eq}} \rightarrow$ equilibration

i. $Q_e = -\frac{n\left(T_e - T_i\right)}{\tau_{eq}} + \frac{J\cdot\underline{R}e}{en} \rightarrow$ equilibration plus ohmic heating

j. Note: cons. of energy $\rightarrow Q_i + Q_e - \underline{J} \cdot \underline{R}e/en = 0$

k. Compare

- $\dfrac{JRe}{en}\Big/\omega p_e = \left(\dfrac{m_e}{m_i}\right)^{1/2}\left(\dfrac{a}{v_{Ti}\tau_{ii}}\right)\left(\dfrac{r_{ii}}{a}\right)^2 \ll 1$

- small ohmic heating in MHD time scale

l. $\therefore \dfrac{d}{dt}\dfrac{p_i}{\rho^r} = \dfrac{2}{3\rho^r}\left[\nabla_n\left(\kappa_{\|_i}\cdot\nabla_\| T_i\right) + n\dfrac{\left(T_e - T_i\right)}{\tau_{EQ}}\right]$

m. $\dfrac{d}{dt}\dfrac{p_e}{\rho^r} = \dfrac{2}{3\rho^r}\left[\nabla_n\left(\kappa_{\|_e}\cdot\nabla_\| T_e\right) + n\dfrac{\left(T_i - T_e\right)}{\tau_{EQ}}\right]$

n. But MHD is a single fluid model - 1 pressure, 1 temperature

o. This occurs if τ_{EQ} is very small, forcing $T_e \approx T_i$

p. Small τ_{EQ} require $\dfrac{nT}{\tau_{EQ}} \ll \omega p$ or $\omega\tau_{EQ} \ll 1$

q. $\left(\dfrac{m_i}{m_e}\right)^{1/2}\dfrac{v_{Ti}\tau_{ii}}{a} \ll 1$

This is more severe than the collision dominated momentum condition energy equilibration $\tau \gg$ momentum equilibration τ.

r. If this is true then

- 1st information $T_e \approx T_i \equiv T/2$

- 2nd information (add equations)

- $\dfrac{d}{dt}\dfrac{p}{\rho^r} = \dfrac{1}{3\rho^r}\nabla_\|\left(\kappa_{\|_i} + \kappa_{\|e}\right)\nabla_\| T$

- But $\kappa_{\|i} \approx \left(m_e/m_i\right)^{1/2}\kappa_{\|e}$, $\kappa_{\|e} \approx nT_e\,\tau_{ei}/m_e$

- Thus $\dfrac{\nabla\cdot\kappa_{\|i}\nabla_\| T}{\omega p} \sim \left(\dfrac{m_i}{m_e}\right)^{1/2}\dfrac{\tau_{ii}v_{Ti}}{a} \ll 1$

This gives Ideal MHD Equation

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{B} = u_0 \underline{J} \qquad n_i = n_e = n$$

$$\frac{\partial p}{\partial t} + \nabla \cdot \rho \underline{v} = 0$$

$$\rho \frac{d\underline{v}}{dt} = \underline{J} \times \underline{B} - \nabla p$$

$$\underline{E} + \underline{V} \times \underline{B} = 0$$

$$\frac{d}{dt} \frac{p}{\rho^r} = 0$$