22.615, MHD Theory of Fusion Systems
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Lecture 19

1. Stability of the straight tokamak
    1. pressure driven modes (Suydams Criterion)
    2. internal modes
    3. external modes
2. Tokamak Ordering

$$\in \equiv a/R_0 \ll 1 \qquad \frac{B_\theta}{B_z} \sim \in$$

$$q \sim 1 \qquad \frac{2\mu_0 p}{B_z^2} \sim \in^2 \text{ or } \in$$

3. Suydams Criterion

$$p' \sim \frac{p}{a}$$

$$rB_z^2\left(\frac{q'}{q}\right)^2 \sim \frac{B_z^2}{a}$$

$$\therefore \frac{8\mu_0 p'}{\iota B_z^2\left(q'/q\right)^2} \sim \frac{\mu_0 p}{B_z^2} \sim \in, \in^2$$

1. Over most of the plasma the destabilizing term in Suydams criterion is much smaller than the stabilizing contribution.

   $\therefore$ $\boxed{\text{Suydams criterion satisfies over most of the plasma}}$

2. Exception:

   near r=0 $\quad p'(r) \approx p''(0)r$

$$q(r) \approx q(0) + q''(0)\frac{r^2}{2} \approx q(0)$$

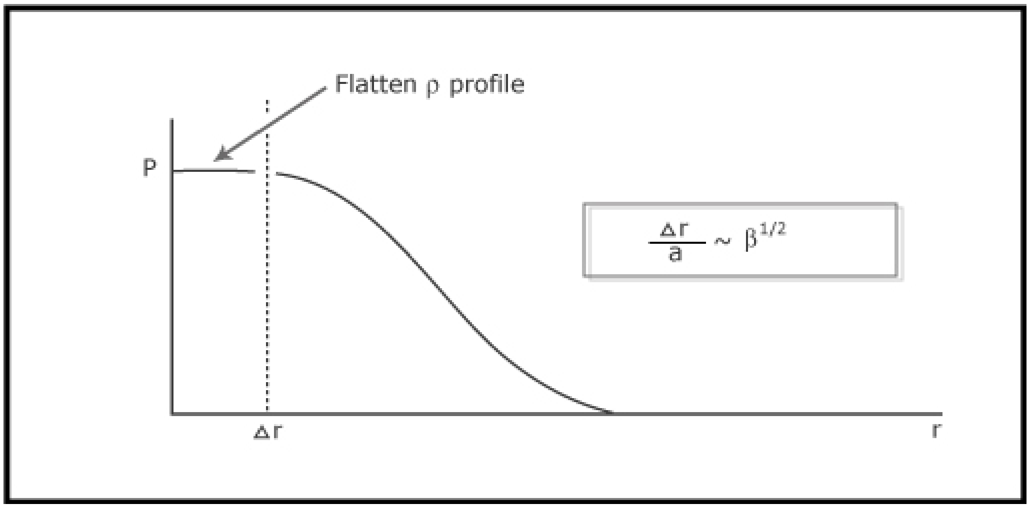
$$q'(r) \approx q''(0)r$$

$$rB_z^2\,\frac{q'^2}{q^2} + 8\mu_0 p' > 0$$

$$\boxed{\left[\left(\frac{B_z q''}{q}\right)^2\right]_0 r^3 + 8\mu_0\left[p''\right]_0 r > 0}$$

dominates near r=0

3. Resolutions: straight case



4. Resolution: Toroidal Case

   a. In toroidal case there are important modifications to Suydams criterion: Mercier criterion. These corrections can eliminate the need for flattening the p profile

   b. Simple, low β circular limit of Mercier criterion

$$rB_z^2\left(\frac{q'}{q}\right)^2 + 8\mu_0 p'\left(1-q^2\right) > 0$$

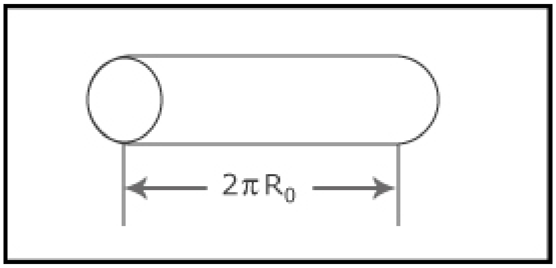
toroidal correction

   c. For $q(0) > 1$, pressure term is stabilizing: average curvature is formable.

5. Conclusion**:**

> Localized interchange modes are not very important in a straight tokamak because β is very small. Near r=0, we need either flattening (straight) or $q(0) > 1$ (toroidal)

Internal Modes in a Straight Tokamak



$L_z = 2\pi R_0$ $\qquad$ $m \sim 1$ poloidal wave number

$$a/R_0 \equiv \in \qquad \lambda = \frac{2\pi}{k} = \frac{2\pi R_0}{n} \rightarrow k = -\frac{n}{R_0}$$

$B_\theta / B_z \sim \in$ $\qquad$ $n \sim 1$ toroidal wavenumber

1. Use this ordering to simplify f and g

a. $f = \dfrac{rF^2}{k_0^2}$

$$k_0^2 = k^2 + \frac{m^2}{r^2} = \frac{n^2}{R_0^2} + \frac{m^2}{r^2} \approx \frac{m^2}{r^2}$$

$$F = kB_z + \frac{mB_\theta}{r} = -\frac{nB_z}{R_0} + \frac{mB_\theta}{r} = \frac{mB_z}{R_0}\left[-\frac{n}{m} + \frac{B_\theta B_0}{rB_z}\right]$$

$$= \frac{mB_z}{R_0}\left[\frac{1}{q} - \frac{n}{m}\right] \approx \frac{mB_0}{R_0}\left[\frac{1}{q} - \frac{n}{m}\right]$$

$$\therefore f = \frac{r^3}{m^2}\frac{m^2B_0^2}{R_0^2}\left(\frac{1}{q} - \frac{n}{m}\right)^2 = \frac{r^3B_0^2}{R_0^2}\left(\frac{1}{q} - \frac{n}{m}\right)^2 \sim \in^2 \left(aB_0^2\right)$$

b. $g_1 = \dfrac{2k^2\mu_0 p'}{k_0^2} = \dfrac{2n^2}{R_0^2}\dfrac{r^2}{m^2}p' = 2\left(\dfrac{n}{m}\dfrac{r}{R_0}\right)^2 p' \sim \dfrac{\in^2 \beta B_0^2}{a}$ (small)

$$g_3 = \frac{2k^2}{rk_0^4}\left(k^2B_z^2 - \frac{m^2B_\theta^2}{r^2}\right) = \frac{2n^2B_0^2r^3}{R_0^4m^2}\left(\frac{n^2}{m^2} - \frac{1}{q^2}\right) \sim \frac{\in^4 B_0^2}{a} \text{ (small)}$$

$$g_2 = \frac{k_0^2r^2 - 1}{k_0^2r^2}rF^2 \approx \left(m^2 - 1\right)\frac{rB_0^2}{R_0^2}\left(\frac{1}{q} - \frac{n}{m}\right)^2 \sim \frac{\in^4 B_0^2}{a}$$

2. Therefore

$$\boxed{\frac{\delta W_F}{2\pi^2 R_0/\mu_0} \approx \frac{B_0^2}{R_0^2}\int r\,dr\left(\frac{n}{m}-\frac{1}{q}\right)^2\left[r^2\xi'^2+\left(m^2-1\right)\xi^2\right]}$$

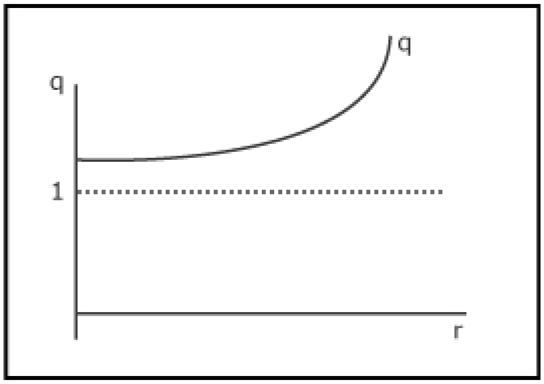
Stability of Internal Modes

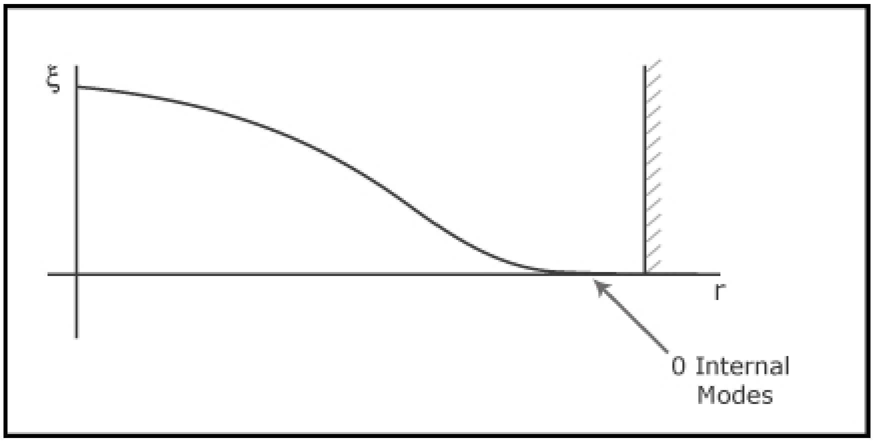
1. $m \geq 2 \rightarrow$ stable, both terms positive.

2. $m = 1 \qquad nq(r) > 1 \qquad (n=1\text{ worst})$

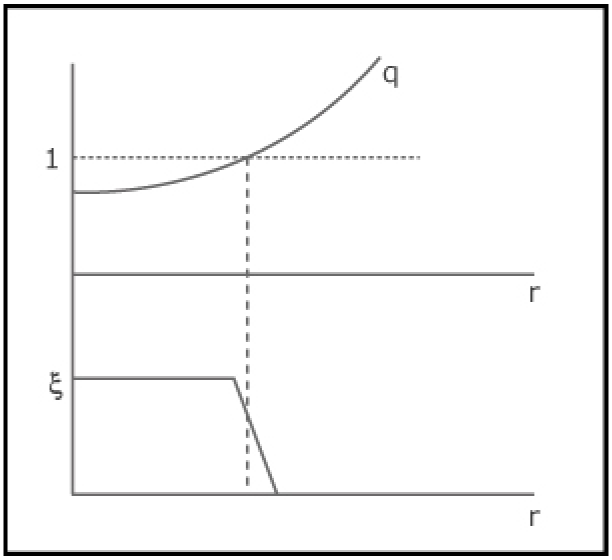
$$1-\frac{1}{q} \neq 0$$

$$I \propto \left(\frac{1}{q}-1\right)^2 \xi'^2 > 0$$

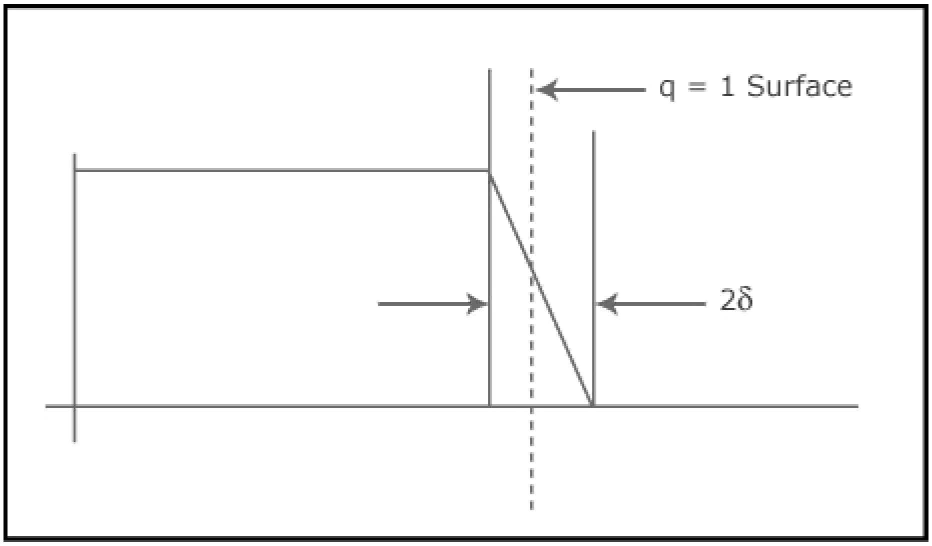




3. m=1 q(r) < 1 somewhere



use the following trial function



a. $\xi = 1 \qquad 0 < r < r_s - \delta$

$$= \frac{1}{2}\left(1 - \frac{x}{\delta}\right) \qquad r_s - \delta < r < r_s + \delta$$

$$= 0 \qquad r > r_s + \delta$$

$$\frac{1}{q(r)} = \frac{1}{q(r_s)} - \frac{q^{'}(r_s)(r - r_s)}{q^2(r_s)}$$

$$= 1 - q^{'}(r - r_s)$$

b. $$\frac{\delta W_F}{2\pi^2 R_0/\mu_0} = \frac{B_0^2}{R_0^2}\int_{-\delta}^{\delta} r\,dr\left[1-1+q'x\right]^2 r^2 \xi'^2$$

$$= \frac{B_0^2}{R_0^2}\left(r^3 q'^2\right)_{r_s} \underbrace{\int (x)^2\left(-\frac{1}{2\delta}\right)^2 dx}_{\frac{1}{6}\delta}$$

$$= \frac{B_0^2}{6R_0^2}\left(r^2 q'^2\right)_{r_s}\delta$$

c. $\delta W_F \to 0$ as $\delta \to 0$

d. with an $m=1$ resonant surface in the plasma, the system is marginally stable in leading order; i.e. if $q(0) < 1$

e. to test stability for this case we must calculate δW to next order for the $m=1$ mode.

Calculate Next Order δW for m=1, n=1 Mode

1. $$f = \frac{r^3 m^2 B_z^2}{R_0^2\left(m^2 + n^2 r^2/R_0^2\right)}\left(n - \frac{1}{q}\right)^2$$

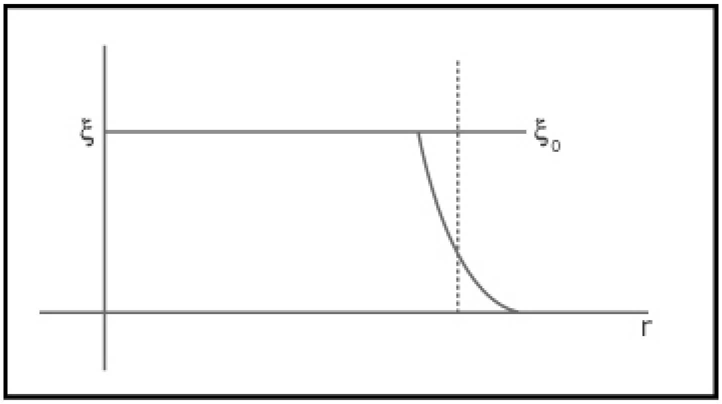
2. $$g = \frac{m^2 - 1 + k^2 r^2}{k_0^2 r^2} rF^2 + \frac{2k^2 \mu_0 p'}{k_0^2} + \frac{2k^2}{r k_0^4}\left(kB_z - \frac{mB_\theta}{r}\right)F$$

$$\approx \frac{n^2 r^2}{R_0^2}\left[\mu_0 2p' + \frac{rB_0^2}{R_0^2}\left(\frac{1}{q} - n\right)\left(\frac{1}{q} - n - 2n - \frac{2}{q}\right)\right.$$

$$= \frac{n^2 r^2}{R_0^2}\left[2\mu_0 p' - \frac{rB_0^2}{R_0^2}\left(\frac{1}{q} - n\right)\left(3n + \frac{1}{q}\right)\right. \sim \in^4$$

3. $$\frac{\delta W_F}{2\pi^2 R_0/\mu_0} = \int dr\left(f\xi'^2 + g\xi^2\right)$$

Use same trial function as before



Summary of Internal Modes in a Straight Tokamak

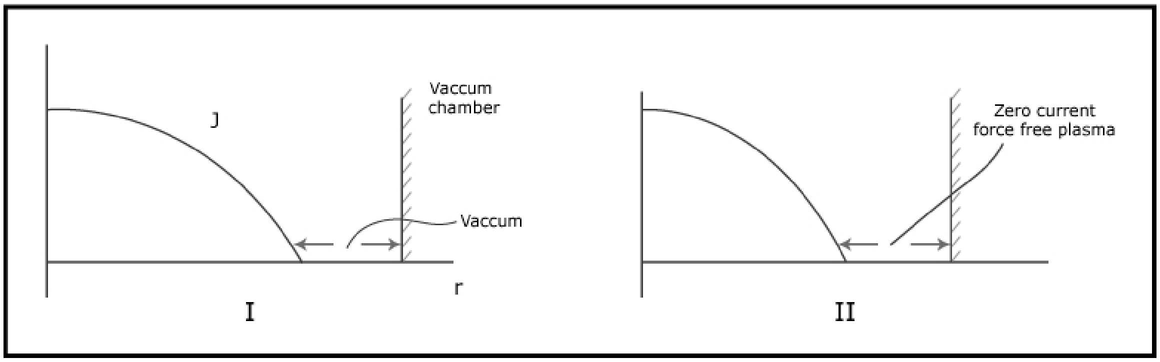
1. $m \geq 2$ stable
2. $m=1$, $n=1$ worst case for $n=1$, requires $q(0) > 1$ for stability
3. internal modes do not limit β, or I $\left(q(a)\right)$, but clamp $q(0) \approx 1$ by sawtooth oscillations
4. To show stability we needed to calculate $\delta W = \underset{\substack{\| \\ 0}}{\in^2 \delta W_2} + \underset{\substack{\uparrow \\ \text{const.}}}{\in^4 \delta W_4}$

Consider now External Modes

1. Vacuum is force free fields
2. $m=1$ Kruskal Shafranov limit
3. High m external kinks

Subtle Issues For External Modes

Vacuum as force free Plasma



1. cold, but highly conducting plasma surrounds core - more realistic than vacuum

2. Is there any difference in stability in these 2 cases.

   I $\sigma = 0$

   $\Big>$ might anticipate big difference

   II $\sigma = \infty$

3. But! Vac. $\delta W_v = \frac{1}{2}\int \left|\underline{\hat{B}}_1\right|^2 d\underline{r}$

   FFP $\delta W_{FFP} = \frac{1}{2}\int d\underline{r}\left[\left|\underline{\hat{B}}_1\right|^2 + \gamma p\left|\nabla \cdot \xi\right|^2 + \underline{\xi}_\perp^* \cdot \left(\underline{J} \times \underline{\hat{B}}_1\right) + \left(\xi \cdot \nabla p\right)\nabla \cdot \underline{\xi}_\perp^*\right]$

   in FFP $J = p = 0$ in equilibrium

   $\delta W_{FFP} = \frac{1}{2}\int d\underline{r}\left|\underline{\hat{B}}_1\right|^2$

   Thus, FFP same as Vac. ⟶ might anticipate no difference in stability since δW's are the same for each.

4. How do we calculate $\delta W_v, \delta W_{FFP}$. Minimizing condition is

   $\nabla \times \underline{\hat{B}}_1 = \nabla \cdot \underline{\hat{B}}_1 = 0$ "vacuum" fields

   Vac: BC. $\underline{n} \cdot \underline{\hat{B}}_1\Big|_{S_w} = 0 \qquad \underline{n} \cdot \underline{\hat{B}}_1\Big|_{S_p} = \underline{n} \cdot \nabla \times \underline{\xi_\perp} \times \underline{B}\Big|_{S_p}$

   FFP $\underline{n} \cdot \underline{\hat{B}}_1\Big|_{S_w} = 0 \qquad \underline{n} \cdot \underline{\hat{B}}_1\Big|_{S_p} = \underline{n} \cdot \nabla \times \underline{\xi_\perp} \times \underline{B}\Big|_{S_p}$

   <u>and</u> $\underline{\hat{B}}_1 = \nabla \times \left(\underline{\xi} \times \underline{\hat{B}}\right)$

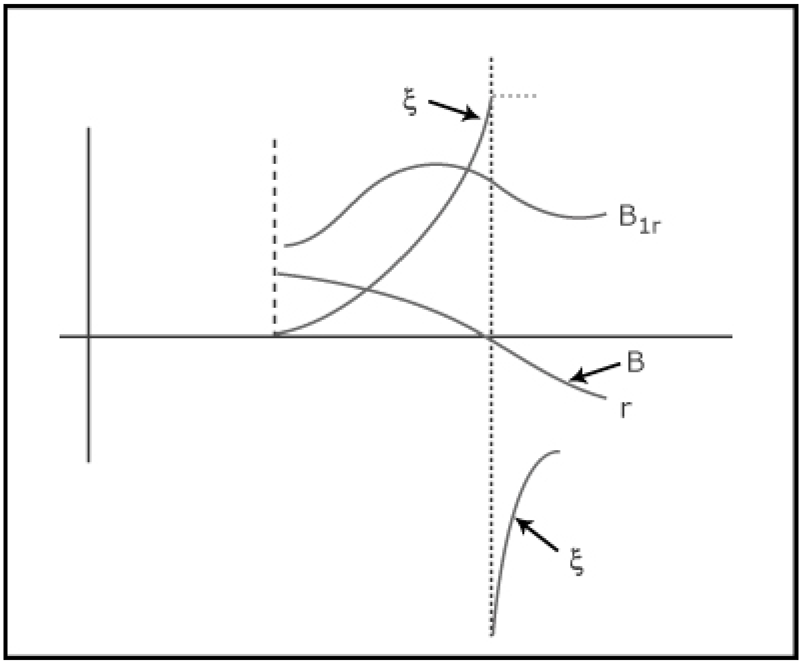
5. In the FFP we must check that a well behaved $\underline{\hat{B}}_1$ always gives rise to well behaved $\underline{\xi}$. This is an <u>additional</u> constraint that can make the FFP more stable

6. Example: cylindrical screw pinch

   $B_{1r} + \iota F\xi \to \xi = -\frac{\iota B_{1r}}{F}$

   a. if k, m are such that $F \neq 0$ in FFP region then ξ is well behaved and $\delta W_v = \delta W_{FFP}$

b. Usually, however $F = 0$ in FFP for external modes. Then, ξ is unbounded ⟶ leads to infinite energy. This is not an allowable displacement



c. Calculation must be redone with new boundary condition $\hat{\underline{B}}_{1r}\left(r_s\right) = 0$. Thus is an additional constraint, which is equivalent to placing a conducting wall at $r = r_s$

external ⟶ internal mode with wall at singular surface.

d. $\therefore$ FFP is more stable than Vac if $F\left(r_s\right) = 0$ in FFP region.

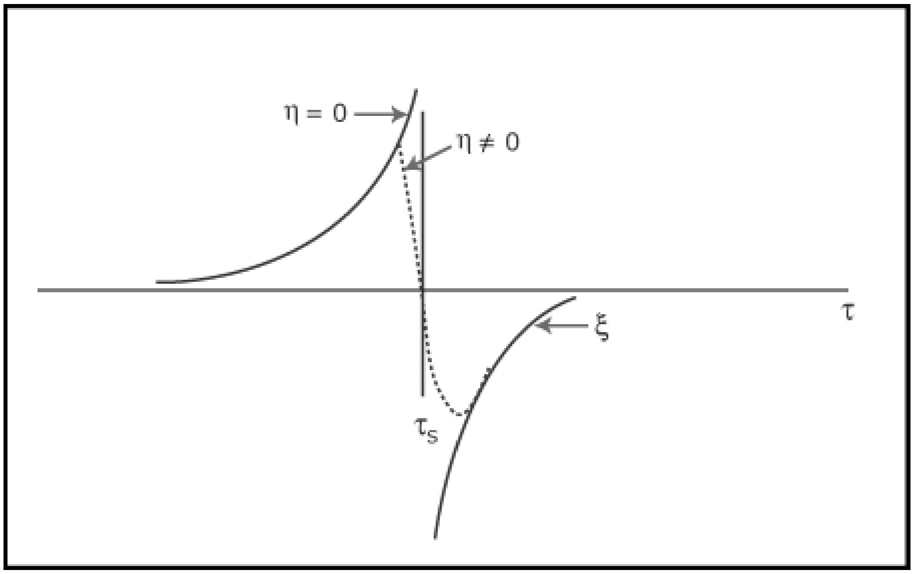
7. But !! most realistic case is neither vacuum nor FFP, but a plasma with a small resistivity

In that case $\delta W_\eta = \frac{1}{2}\int \left|\hat{B}_1\right|^2 d\underline{r}$

and $\frac{\partial \hat{B}_1}{\partial t} = \nabla v\left(\underline{v}\times\underline{B} - \eta\underline{J}\right) \rightarrow \hat{\underline{B}}_1 = \nabla\times\left(\underline{\xi}\times B\right) - \frac{\iota\eta}{\omega}\nabla\times\nabla\times\hat{\underline{B}}_1$

Careful analysis choose that $\underline{\xi}$ is bounded at the resonant surface.

$\therefore$ Stability boundary is the same as Vacuum case, but growth rate is smaller, depending upon resistivity



Summary

Vacuum: certain stability boundary, growth rate $\sim v_T/R$

Ideal FFP: same stability boundary, growth rate if $\underline{k} \cdot \underline{B} \neq 0$

much more stable $(\gamma = 0)$ if $\underline{k} \cdot \underline{B} = 0$

Resistive FFP: same boundary as vacuum but

$$\gamma \sim \gamma_{MHD} \left(\frac{\tau_{MHD}}{\tau_{RES}}\right)^{\nu} \qquad 0 < \nu < 1$$