Lecture 10: The High Beta Tokamak Con'd and the High Flux Conserving Tokamak

Properties of the High β Tokamak

1. Evaluate the MHD safety factor:

$$\frac{\Psi_0}{a^2 B_0} = \frac{1}{2q_*}\left[\rho^2 - 1 + \nu\left(\rho^3 - \rho\right)\cos\theta\right]$$

$$\frac{B_\theta}{\in B_0} = \frac{1}{q_*}\left[\rho + \frac{\nu}{2}\left(3\rho^2 - 1\right)\cos\theta\right]$$

2. The safety factor on axis is given by

   a. $q_0 = \Delta_0 B_\phi\left[\Psi_{rr}\Psi_{\theta\theta}\right]^{-1/2}$ (exact)

   b. $q_0 = q_*\left[\dfrac{3}{\eta\left(2+\eta\right)}\right]^{1/2}$

   $\eta = \left(1 + 3\nu^2\right)^{1/2}$

   c. Note $q_0 < q_*$

3. The safety factor at the plasma edge is given by

   a. $q_a = \dfrac{1}{2\pi}\displaystyle\int\left(\frac{rB_\phi}{RB_\theta}\right)_S d\theta \approx \frac{1}{2\pi}\int\frac{aB_0}{RB_\theta\left(a,\theta\right)}d\theta = \frac{\in B_0}{2\pi}\int_0^{2\pi}\frac{d\theta}{B_\theta\left(a,\theta\right)}$

   b. $q_a = \dfrac{\in B_0}{2\pi}\dfrac{q_*}{\in B_0}\displaystyle\int_0^{2\pi}\frac{d\theta}{1+\nu\cos\theta}$

   c. $q_a = \dfrac{q_*}{\left(1-\nu^2\right)^{1/2}}$

4. Note that

   a. $q_a > q_*$

   b. for $\nu \to 0$ $\quad q_a \to q_* \sim \dfrac{1}{I}$

c. as $\nu \to 1 \quad q_a \to \infty$?

d. as $\nu \to 1 \quad q_* \propto \dfrac{1}{I}$ by definition: $q_a \not\propto 1/I$

5. What is the significance of $\nu \to 1$. Clearly $\nu \le 1$ for real solutions

6. As $\nu \to 1$

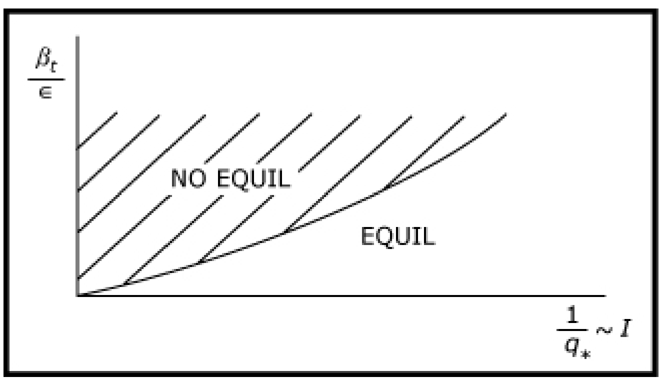
a. $\in \beta_p \to 1$

b. $\dfrac{\beta_t}{\in} \to \dfrac{1}{q_*^2} \quad \left(\dfrac{\beta_t}{\in} = \dfrac{\nu}{q_*^2} \right)$

c. $\dfrac{\Delta_a}{a} \to \dfrac{1}{3}$

7. In the high β tokamak there is an equilibrium β limit

$$\frac{\beta_t}{\in} < \frac{1}{q_*^2}$$



8. The significance of $\nu \to 1$ can be understood by solving the Grad-Shafranov equation outside the plasma

9. Outside the plasma we solve

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \hat{\psi}_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \hat{\psi}_0}{\partial \theta^2} = 0 \qquad \text{(no current, no pressure)}$$

$$\hat{\psi}_0(a,\theta) = 0 \qquad \text{(continuity of flux)}$$

$$\hat{B}_\theta(a,\theta) = B_\theta(a,\theta) = (\in B_0/q_*)[1 + \nu\cos\theta] \qquad \text{(no surface currents)}$$

10. The solution is given by

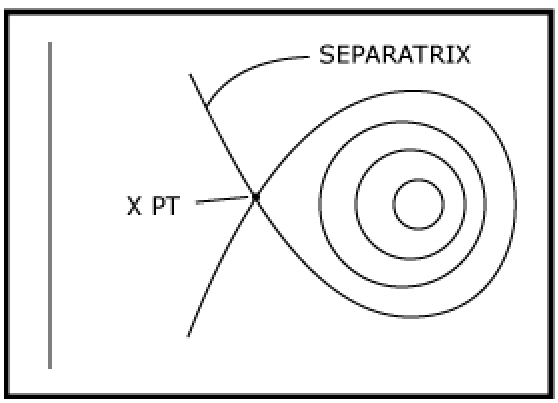
$$\hat{\psi} = c_1 + c_2 \ln r + c_3 r \cos\theta + \frac{c_4}{r}\cos\theta$$

$$\frac{\hat{\psi}(r,\theta)}{a^2 B_0} = \frac{1}{q_*}\left[\underset{I}{\ln\rho} + \underset{B_v}{\frac{\nu}{2}\left(\rho\right.} - \underset{\text{Dvam.}}{\left.\frac{1}{\rho}\right)}\cos\theta\right]$$

$$\frac{\hat{B}_\theta}{\in B_0} = \frac{1}{q_*}\left[\frac{1}{\rho} + \frac{\nu}{2}\left(1 + \frac{1}{\rho^2}\right)\cos\theta\right]$$

$$\frac{\hat{B}_r}{\in B_0} = \frac{1}{q_*}\frac{\nu}{2}\left(1 - \frac{1}{\rho^2}\right)\sin\theta$$

11. The vacuum field has a separatrix: $\hat{B}_r(r_s,\theta_s) = \hat{B}_\theta(r_s,\theta_s) = 0$



12. Choose $\theta = \pi$ or 0. This makes $\hat{B}_r = 0$

    a. Only $\theta = \pi$ has the possibility of a real solution for r_s, satisfying

    $$\hat{B}_\theta(r_s,\theta_s) = 0$$

    b. At $\theta_s = \pi$

    $$\hat{B}_\theta(r_s,\theta_s) = \frac{1}{q_*}\left[\frac{1}{\rho_s} - \frac{\nu}{2}\left(1 + \frac{1}{\rho_s^2}\right)\right] = 0$$

    c. Solve for ρ_s

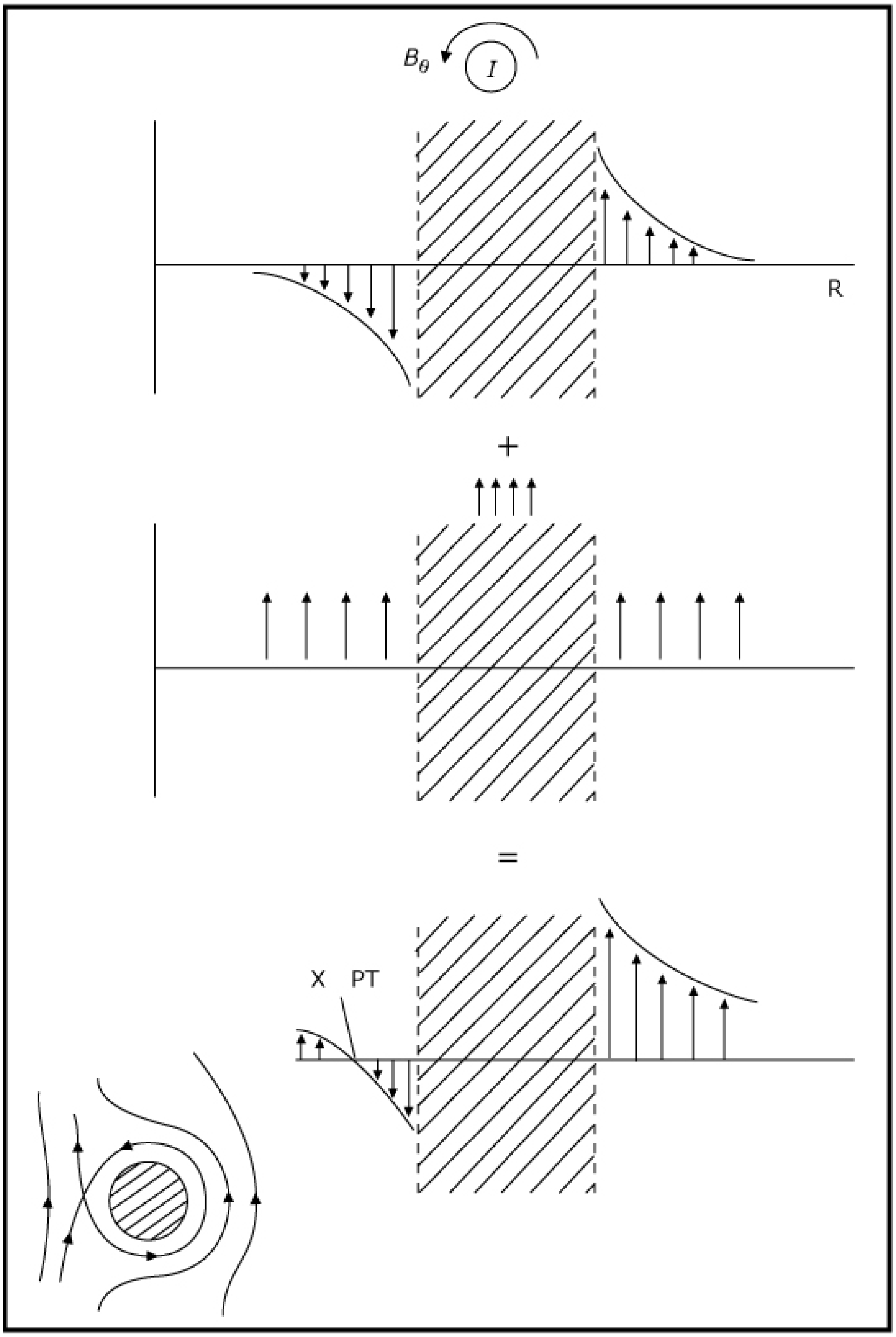
    $$\rho_s = \frac{1}{\nu}\left[1 + \left(1 - \nu^2\right)^{1/2}\right]$$ radius of the separatrix X point

13. For low β $(\nu \ll 1)$, $\rho_s \approx 2/\nu$: the X point is far from the plasma

For $\nu \sim 1, \rho_s \sim 1$: the X point is near the plasma

For $\nu = 1, \rho_s = 1$: the X point moves onto the plasma surface

14. Physical picture of the separatrix and X point

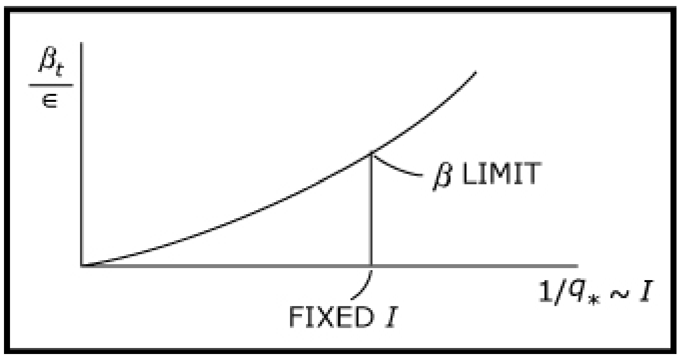


15. The equilibrium β limit corresponds to the situation where the separatrix moves onto the plasma surface

16. At fixed I, the β limit given by $\beta_t \le \epsilon^2 / q_*^2$

17. At fixed I, the only way to hold higher pressure is to increase the vertical field. Eventually, the separatrix moves onto the plasma surface

18.



19. Calculation of the vertical field

a. $$\hat{B}_\theta = \frac{\epsilon B_0}{q_*}\left[\frac{1}{\rho} + \frac{\nu}{2}\left(1 + \frac{1}{\rho^2}\right)\cos\theta\right]$$

$$\hat{B}_r = \frac{\epsilon B_0}{q_*}\frac{\nu}{2}\left(1 - \frac{1}{\rho^2}\right)\sin\theta$$

b. Far from the plasma

$$\hat{B}_\theta = \frac{\epsilon B_0}{q_*}\frac{\nu}{2}\cos\theta$$

$$\hat{B}_r = \frac{\epsilon B_0}{q_*}\frac{\nu}{2}\sin\theta$$

c. $$B_v = \hat{B}_\theta \cos\theta + \hat{B}_r \sin\theta = \frac{\epsilon B_0 \nu}{2 q_*}$$

d. Note: B_v increases with ν

$$B_v = \frac{\mu_0 I}{4\pi R_0}\beta_p \qquad \text{(high } \beta\text{)}$$

$$B_v = \frac{\mu_0 I}{4\pi R_0}\left[\beta_p + \frac{l_i - 3}{2} + \ln\frac{8R_0}{a}\right] \quad \text{(ohmic)}$$

dominates at high $\beta_p \sim \frac{1}{\in}$

Summary of the High β Tokamak

1. Ordering

   $q \sim 1$

   $\beta_t \sim \in$

   $\beta_p \sim 1/\in$

   $\Delta_a/a \sim 1$

2. There is an equilibrium β_t limit when the separatrix moves onto the plasma surface
3. This will always occur at fixed I and β_t increases

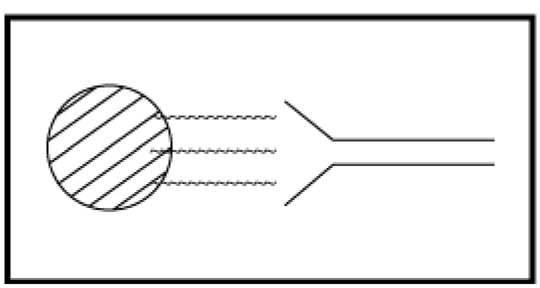
Flux Conserving Tokamak

The Equilibrium β Limit

1. Is there really an equilibrium β_t limit in a tokamak?
2. A more realistic treatment shows that such a limit need not exist
3. This corresponds to the flux conserving tokamak equilibrium (FCT)
4. Paradoxically, the FCT is a special case of the HBT equilibrium just discussed

What is Flux Conservation?

1. Consider a tokamak with a large external heating source (rf, neutral beams)



2. a. The plasma absorbs energy

   b. The temperature rises

   c. β_t rises

   e. Poloidal currents are induced

3. Assume the heating time is slow compared to the ideal MHD inertial time

   MHD: $\tau_M \sim a/v_{ti}$

   Heating: $\tau_H \sim T/(\partial T/\partial t)$

   $\tau_H \gg \tau_M$

4. The plasma evolution can be thought of as a series of quasistatic equilibria, each one satisfying the Grad-Shafranov equation

   $$\rho\frac{d\mathbf{v}}{dt} = \mathbf{J}\times\mathbf{B} - \nabla p$$

   neglect when $\tau_H \gg \tau_M$

5. Assume the heating time is fast compared to the resistive diffusion time

   Resistive time $\tau_D \sim \dfrac{a^2\mu_0}{\eta}$

   $\tau_D \gg \tau_H$

6. If we neglect resistive diffusion, then during the heating process the plasma behaves electrically, like a perfect conductor

7. The FCT assumptions $\tau_D \gg \tau_H \gg \tau_M$ imply that the free functions $p(\psi), F(\psi)$ must satisfy certain constraints

8. a. In general *p, F* are determined by the transport evolution

   b. For the FCT *p, F* are determined by the FCT "transport prescription"

FCT Prescription for $p(\psi)$

1. Assume we start with an ohmically heated tokamak before auxiliary power is added

   $p(\psi, t=0) = p_\Omega(\psi)$ initial pressure distribution

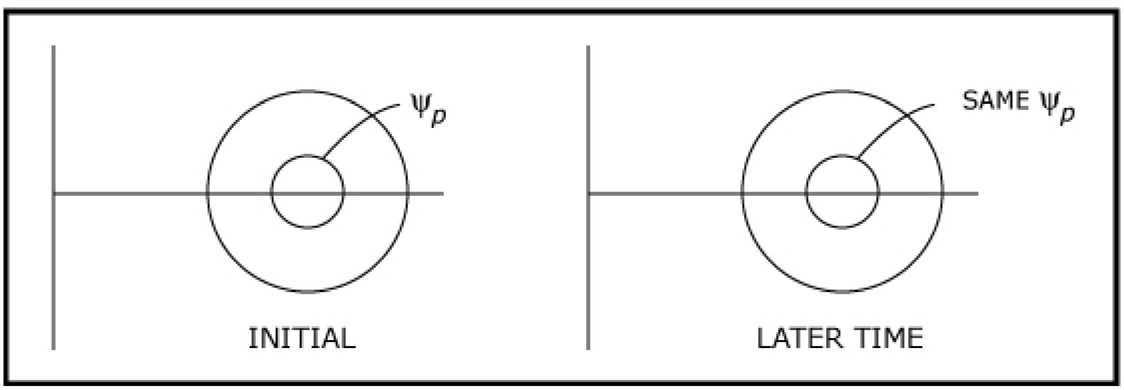
2. At any time later in the heating sequence

   a. $p(\psi,t) = \underbrace{W(\psi,t)}_{\text{modeled from heating calculations}} p_\Omega(\psi)$

   b. Often $W(\psi,t) = W(t)$, corresponding to a slow increase in the magnitude of p due to heating

FCT Prescription for $F(\psi)$ (The Critical Issue)

1. Since the plasma acts like a perfect conductor, the toroidal and poloidal fluxes must be conserved. This is the FCT constraint



2. Consider a given poloidal flux surface ψ_p initially and at a later time

3. For flux conservation, the toroidal flux contained within the surface $\psi_p = \text{const}$ must remain the same as the plasma evolves. There is no diffusion of flux. This is the FCT constraint. We must choose $F(\psi)$ so this property is preserved.

4. Calculate $\psi_t = \psi_t(\psi,t), \psi_p = 2\pi\psi$

   $$\psi_t = \int B_\phi(r,\theta)\, r dr d\theta$$

5. Let us write ψ_t as a function of $F(\psi,t)$

6. Change variables

a. $r, \theta \to \psi^{'}\left(r, \theta^{'}\right), \theta^{'}$

$\theta^{'} = \theta$

$\psi = \psi\left(r, \theta\right)$

b. $d\psi^{'} d\theta^{'} = \begin{vmatrix} \dfrac{\partial \psi^{'}}{\partial r} & \dfrac{\partial \psi^{'}}{\partial \theta} \\ \dfrac{\partial \theta^{'}}{\partial r} & \dfrac{\partial \theta^{'}}{\partial \theta} \end{vmatrix} dr\, d\theta = \dfrac{\partial \psi^{'}}{\partial r} dr\, d\theta = RB_\theta dr\, d\theta$

7. Then

a. $\psi_t\left(\psi, t\right) = \int_0^\psi d\psi^{'} \int_0^{2\pi} d\theta^{'} \left(\dfrac{rB_\phi}{RB_\theta}\right)_{\psi^{'}, \theta^{'}}$

$$= 2\pi \int_0^\psi d\psi^{'} q\left(\psi^{'}, t\right)$$

b. $\dfrac{\partial \psi_t}{\partial \psi} = 2\pi q\left(\psi, t\right)$

8. If $\psi_t\left(\psi, t\right)$ is to remain unchanged during the heating sequence

$$\frac{\partial \psi_t}{\partial t} = 0$$

then $q\left(\psi, t\right)$ must be the same for each quasistatic equilibrium

9. Thus, we must choose $F\left(\psi, t\right)$ so that

$$q\left(\psi, t\right) = q_\Omega\left(\psi\right)$$

initial ohmic q profile

10. We can now relate $F\left(\psi, t\right)$ to $q_\Omega\left(\psi\right)$

$$q\left(\psi, t\right) = q_\Omega\left(\psi\right) = \frac{1}{2\pi} \int d\theta \left(\frac{rB_\phi}{RB_\theta}\right)_S = \frac{F\left(\psi, t\right)}{2\pi} \int_0^{2\pi} \frac{r d\theta}{R\left(\partial \psi / \partial r\right)}$$

11. Solving for F we find that FCT Grad-Shafranov equation becomes

$$\Delta^* \psi = -R^2 \frac{d}{d\psi}\left(\mu_0 W p_\Omega\right) - \frac{1}{2}\frac{d}{d\psi}\left[\frac{q_\Omega}{\frac{1}{2\pi}\int \frac{r d\theta}{R\left(\partial\psi/\partial R\right)}}\right]^2$$

This is an exact form, using no expansions

12. It is a nonlinear partial-integro-differential equation

13. In general, it must be solved numerically

14. It can be solved approximately by variational techniques

15. In class we shall calculate an “industrial strength” solution to the FCT equation