22.611J , 6.65J , 8.613J PS#5
Solution Key

1 The ion polarization drift is just

$$\vec{v}_\perp = \frac{m\dot{\vec{E}}_\perp}{qB_0^2} = \frac{m\dot{\vec{E}}_\perp}{eB_0^2} = \frac{im\omega\vec{E}_\perp}{eB_0^2}$$

(See HW# for derivation)

so, using

$$\vec{J}_\perp = \vec{\sigma}\cdot\vec{E}_\perp = n_i\vec{v}_\perp$$

$$\vec{\sigma}\cdot\vec{E}_\perp = \frac{in_im\omega\vec{E}_\perp}{eB_0^2}$$

$$\Rightarrow \boxed{\overline{\overline{\sigma}} = \frac{i\rho\omega\overline{\overline{I}}}{eB_0^2}}$$ in SI units

(The Alfvén wave derivation then just follows the class notes)

Discussion

- Polarization effects are in MHD. (We just derived it!)
- There's no conflict w/ ohm's law ($\vec{E}+\vec{u}\times\vec{B}=0$) because $\vec{u}$ is the center of mass velocity of the fluid.... (hence $\vec{u}$ can be zero while $\vec{J}$ is finite). Therefore, all the polarization effects are in $\vec{J}$.
- In MHD, $\vec{J}\neq n_i\vec{u}$!
- the ions & electrons can still form independent currents, as long as $\vec{u}$ is not affected; these currents are reflected in $\vec{J}$. →

1 Con't

- In other words, MHD gives one order higher $\vec{J}$ than $\vec{u}$!

→ The dominant effect on $\vec{u}$ is $\vec{E} \times \vec{B}$ .... but 2nd order effects like the polarization drift end up in $\vec{J}$.

2) Wave Packets & Group Velocity

– first, show that

$$A(x) = \underbrace{Re\left(e^{ik_0 x} e^{-\frac{x^2}{2\sigma^2}}\right)}_{1} = \underbrace{Re\left[\frac{\sigma}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} dk\, e^{ikx} e^{\left(-\frac{\sigma^2(k-k_0)^2}{2}\right)}\right]}_{2}$$

We can get 1 from 2 by:

let $k' = -k_0 + k$

$dk' = dk$

then, 2 :.

$$2 = Re\left[\frac{\sigma}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} dk'\, e^{i(k'+k_0)x} e^{\left(-\frac{\sigma^2 k'^2}{2}\right)}\right]$$

$$= Re\left[\frac{\sigma}{(2\pi)^{\frac{1}{2}}} e^{ik_0 x}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}\left(k'^2 - \frac{ik'2x}{\sigma^2} + Y^2\right)} e^{+\frac{\sigma^2}{2}Y^2} dk'\right.$$

(Use Y to complete the square)

$$2 = Re\left[\frac{\sigma}{(2\pi)^{\frac{1}{2}}} e^{ik_0 x}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}(k' + 1/\sigma^4)^2} e^{\frac{\sigma^2}{2\sigma^4}} dk'\right]$$

since $k'^2 - \frac{ik'2x}{\sigma^2} + Y^2 = (k' - Y)^2$

then $+\cancel{2}\cancel{k'}Y = +\frac{i\cancel{2}\cancel{k'}x}{\sigma^2}$

$Y = ix/\sigma^2$, $Y^2 = -x^2/\sigma^4$

$$2 = Re\left[\frac{\sigma}{(2\pi)^{\frac{1}{2}}} e^{ik_0 x} e^{\frac{-x^2}{2\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}(k' + 1/\sigma^4)^2} dk'\right.$$

let $\chi = k' + 1/\sigma^4$

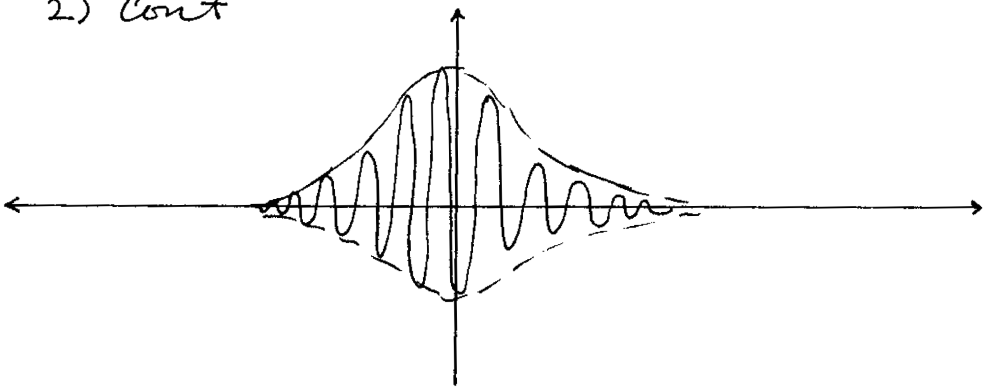
$d\chi = dk'$

then,

$$2 = Re\left[\frac{\cancel{\sigma}}{\cancel{(2\pi)^{\frac{1}{2}}}} e^{ik_0 x} e^{\frac{-x^2}{2\sigma^2}} \frac{\cancel{\sqrt{\pi}\sqrt{2}}}{\cancel{\sigma}}\right]$$

$$\boxed{2 = 1 = Re\left[e^{ik_0 x} e^{-x^2/2\sigma^2}\right]}$$

2) Con't



Approximating $\omega(k) \simeq \omega(k_0) + \frac{\partial \omega}{\partial k}\Big|_{k=k_0}(k-k_0)$ in

$$A(x) = Re\left[\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \exp(ikx - i\omega(k)t)\exp\left(-\frac{\sigma^2(k-k_0)^2}{2}\right)\right]$$

$$\cong Re\left[\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, e^{\left(ikx - i\omega(k_0)t - i\frac{\partial\omega}{\partial k}\big|_{k_0} t\,(k-k_0)\right)}\exp\left(-\frac{\sigma^2(k-k_0)^2}{2}\right)\right]$$

$$\cong Re\left[\frac{\sigma}{\sqrt{2\pi}}\, e^{-i\omega(k_0)t + i\frac{\partial\omega}{\partial k}t k_0}\underbrace{\int_{-\infty}^{\infty} dk\, e^{ikx - i\frac{\partial\omega t}{\partial k}\big|_{k_0} k}\, e^{\left(-\frac{\sigma^2(k-k_0)^2}{2}\right)}}_{A}\right]$$

Lets look at (A) in more detail:

$A(x,t) \propto$ (A) ...

→ Now, $e^{ikx - ikt\frac{\partial\omega}{\partial k}\big|_{k=k_0}}$ factor ~~is exactly~~ comes from

$$A\left(x - \left(\frac{\partial\omega}{\partial k}\right)\Big|_{k_0} t,\ 0\right)$$

→ In other words, exactly the same profile at $A(x, t=0)$, but moving at $\boxed{v_g = +\left(\frac{\partial\omega}{\partial k}\right)\Big|_{k_0}}$

⇒ which is the group velocity!

2) Extra Credit

if we've

$$\omega(k) \simeq \omega_0 + \frac{\partial \omega}{\partial k}\Big|_{k_0}(k-k_0) + \frac{\partial^2 \omega}{\partial k^2}\Big|_{k=k_0}(k-k_0)^2$$

then, since

$$A(x,t) = Re\left[\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \exp(ikx - i\omega t)\exp\left(-\frac{\sigma^2(k-k_0)^2}{2}\right)\right]$$

$$\Rightarrow Re\left[\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \exp\left(ikx - i\omega_0 t - \frac{i\partial\omega}{\partial k}(k-k_0)t\right)\right.$$

$$\left. \times \exp\left(\frac{\left(-\sigma^2 \; i\frac{\partial^2\omega}{\partial k^2}\Big|_{k=k_0} t\right)(k-k_0)^2}{2}\right)\right]$$

→ which is the same as before, except now, the wave packet has a $i\frac{\partial^2 \omega}{\partial k^2}\Big|_{k=k_0} t$ dispersion factor added to σ^2!

→ hence, w/ increasing t,

$\left(\sigma^2 + i\frac{\partial^2\omega}{\partial k^2}\Big|_{k=k_0} t\right)$ becomes larger and larger, increasing the packet size.

3) Ion Acoustic Waves:

→ Start w/ equation of motion

$$m_i n_i \frac{d\vec{v}_i}{dt} = e n_i \vec{E} - \vec{\nabla} P_i = -e n_i \vec{\nabla}\tilde{\phi}_1 - \vec{\nabla} n_i T_i$$

(Since $\vec{E}_1 = -\vec{\nabla}\tilde{\phi}_1$)

→ $\bar{v} = \cancel{v_0}^{\,0} + \tilde{v}_1$

$n_e \sim n_0\left(1 + \frac{e\phi_1}{T_e}\right)$

$n_i \sim n_e$

$\vec{E} = \tilde{E}_1$

$$-i\omega m_i (n_0 \tilde{v}_i) = -e n_0 i k \tilde{\phi} - i k \tilde{n}_i T_i$$

$$+\omega m_i \cancel{n_0} \tilde{v}_i = +e \cancel{n_0} k \tilde{\phi} + k \cancel{n_0} \frac{e\tilde{\phi}_1}{T_e} T_i$$

$$\omega = \frac{e k \tilde{\phi} + k e \tilde{\phi}\, T_i/T_e}{m_i \tilde{v}_i}$$

→ Now, use equation of Continuity to solve for $\tilde{v}_i$:

$$\frac{\partial n_i}{\partial t} + \bar{\nabla}\cdot(n_i \vec{v}_i) = 0$$

$$-\cancel{i}\omega \cancel{n_0}\left(1 + \frac{e\tilde{\phi}}{T_e}\right) + \cancel{i}k\cancel{n_0}\tilde{v}_i = 0$$

$$\tilde{v}_i = \frac{\omega\left(1 + \frac{e\tilde{\phi}}{T_e}\right)}{k}$$

$$\omega = \frac{e k^2 \tilde{\phi}\,(1 + T_i/T_e)}{m_i \omega (1 + e\tilde{\phi}/T_e)}$$

$$\omega^2 = \frac{k^2 (1 + T_i/T_e)}{m_i\left((e\tilde{\phi})^{-1} + 1/T_e\right)} = \frac{k^2 (T_e + T_i)}{m_i\left(\frac{T_e}{e\tilde{\phi}} + \cancel{T_e}1\right)} \simeq \frac{k^2 (T_e + T_i)}{m_i}$$

but $T_e \gg e\tilde{\phi}$

3) Con't

$$\omega^2 \sim k^2 \left[\frac{(T_e + T_i)}{m_i} \right]$$

$$C_{IA}^2 = \left[\frac{T_e + T_i}{m_i} \right]$$

$$\boxed{C_{IA} \sim \left(\frac{2T_e}{m_i} \right)^{\frac{1}{2}}}$$

if ions 've no temperature
($T_i = 0$, as implied by Part b))

then

$$C_{IA} \sim \left(\frac{T_e}{m_i} \right)^{\frac{1}{2}}$$