Massachusetts Institute of Technology

Subject 6.651J/8.613J/22.611J — R. Parker
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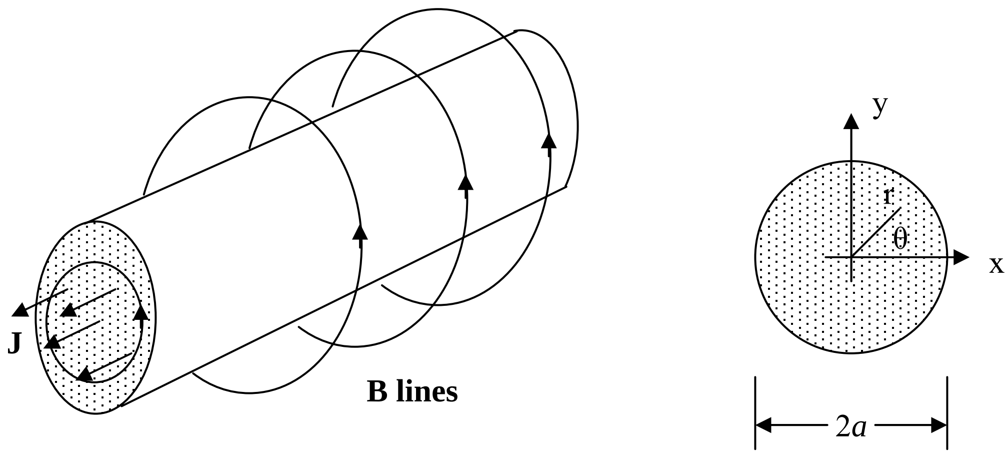
Problem Set 6

Problem 1.

In this problem, we'll analyze the m=0 stability of the straight z-pinch shown below. The plasma is assumed to obey the usual z-pinch equilibrium, $(p + B_{\theta 0}^2 / 2\mu_0)' + B_{\theta 0}^2 / \mu_0 r = 0$ and there is no equilibrium flow. Modes having the form $\vec{v}_1 = \vec{V}(r)\exp(ikz + i\omega t)$ will be examined for their stability. For simplicity, we'll make the assumption that the plasma is incompressible, i.e., $\nabla \bullet \vec{v} = 0$, and assume the simple boundary condition $V_r(a) = 0$ where a is the column radius.



a) The perturbed magnetic field is given by

$$\vec{B}_1 = \frac{1}{i\omega}\nabla \times (\vec{v}_1 \times \vec{B}_0) = \frac{1}{i\omega}\Big((\nabla \bullet \vec{B}_0)\vec{v}_1 + \vec{B}_0 \bullet \nabla \vec{v}_1 - (\nabla \bullet \vec{v}_1)\vec{B}_0 - \vec{v}_1 \bullet \nabla \vec{B}_0\Big).$$

Using this relation and the incompressibility assumption, show that

$$\vec{B}_1 = \hat{\theta}\frac{1}{i\omega}\left(\frac{B_\theta}{r} - B_\theta'\right)V_r$$

b) The momentum equation can be written in the form

$$i\omega\rho_0\mu_0\vec{V} = \nabla \bullet (\vec{B}_0\vec{B}_1 + \vec{B}_1\vec{B}_0) - \nabla(\mu_0 p_1 + \vec{B}_0 \bullet \vec{B}_1)$$

By evaluating the terms on the right, show that the momentum equations become

$$i\omega\rho_0\mu_0 V_r = -2\frac{B_\theta B_{1\theta}}{r} - \left(B_\theta B_{1\theta} + \mu_0 p_1\right)'$$

$$i\omega\rho_0\mu_0 V_z = -ik\left(B_\theta B_{1\theta} + \mu_0 p_1\right)$$

c) Combine the results of parts a) and b) together with the constraint $\nabla \bullet \vec{V} = 0$ to get a single, second order differential equation for $\vec{V}(r)$. Put your result in the form

$$\left(\rho_0 \frac{1}{r}(rV_r)'\right)' + k^2 f(r,\omega^2)V_r = 0$$

d) By multiplying the result obtained in part c) by rV_r' and integrating from $r = 0$ to $r = a$ (where V_r is assumed to vanish), show that the plasma will be unstable if

$$B_\theta' - \frac{B_\theta}{r} > 0$$

for $0 < r < a$.

Interpreting the above condition in terms of particle drifts leads to an apparent paradox since this requires the net particle drift (grad *B* plus curvature) to be in the –*z* direction, which would be stabilizing according to the picture developed in class. The paradox can be resolved by calculating the energy involved in the perturbation. One can show that this is proportional to

$$\frac{2B_\theta^2}{r} + \mu_0 p'$$

It is interesting to note that *both* of these terms arise from field line curvature; thus arguments based only on "bad" curvature are incomplete and one must consider stabilizing as well as destabilizing effects of curvature to properly examine stability of MHD equilibria. (Nevertheless, a region of bad curvature is *necessary* for MHD instability.)

e) If $\frac{2B_\theta^2}{r} + p'\mu_0 < 0$ for $0 < r < a$, the energy argument shows that the plasma will be unstable. Show that this condition is equivalent to that found in part d).