22.01 Fall 2015, Problem Set 5 (Analytical Version)

Due: November 3rd, 11:59PM on Stellar

November 14, 2015

Complete all the assigned problems, and do make sure to show your intermediate work. Please upload your full problem set in PDF form on the Stellar site. Make sure to upload your work at least 15 minutes early, to account for computer/network issues.

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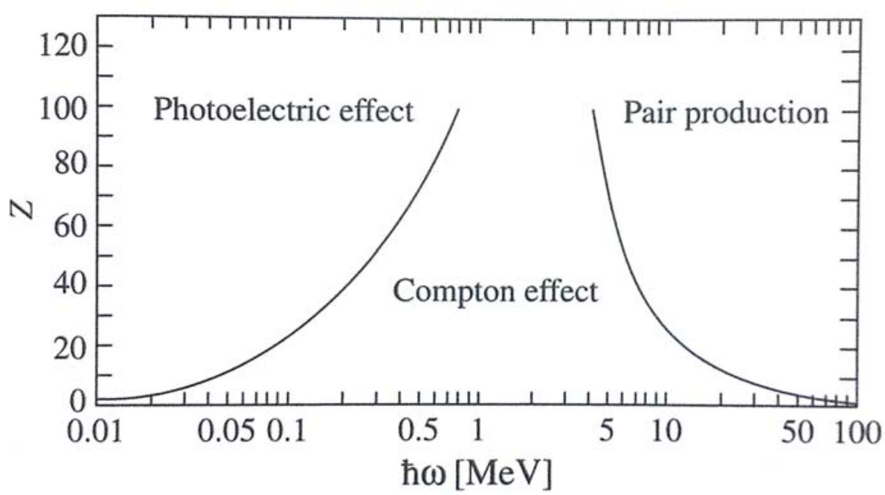
1 Conceptual Questions

Consider the three methods by which photons can interact with electrons in matter: Compton scattering, the photoelectric effect, and pair production.

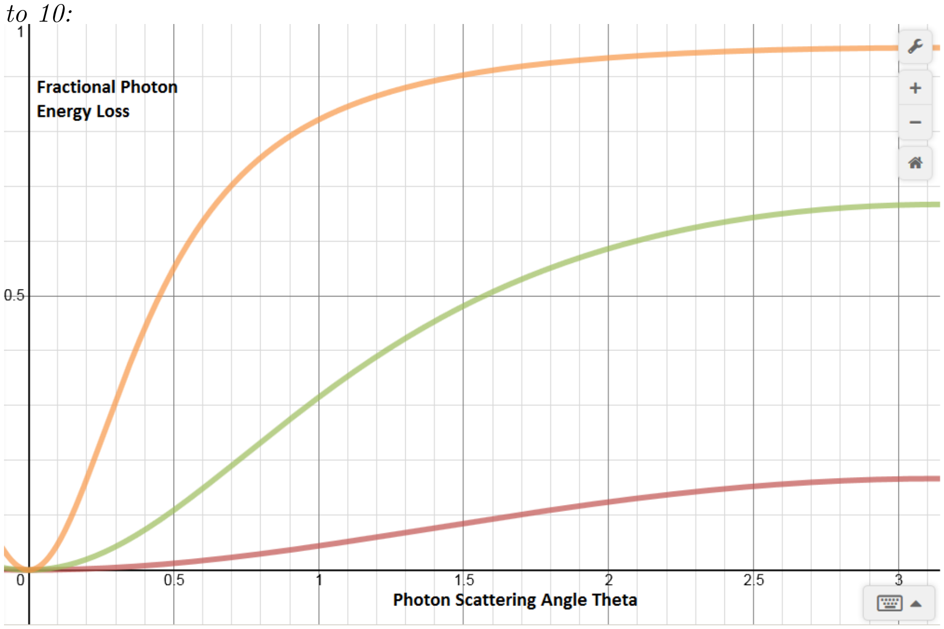
1. At which photon energies are each of these effects the most prominent? In other words, which of these effects can be neglected at which energies?
   The photoelectric effect is most prominent at lower energies, while pair production happens the most at higher energies, above 1.022 MeV. These two effects are also more prevalent for high-Z materials, as they are more electron dense. Compton scattering fills in the remainder of the medium-energy, more low-Z space. See the figure below as a partial explanation:



2. In figure 10.5, why is the energy of the Compton edge of Compton electrons not equal to the energy of the incoming photon, and at what photon scattering angle is this Compton edge produced? Explain the origin of this energy difference.
   This energy difference accounts for the fact that the photon cannot transfer all of its energy in a scattering collision with an electron, unless it were to have infinite energy. This is determined by Equation (10.8) in Yip:

$$T_{e^-} = \hbar\omega \frac{\alpha\,(1 - cos\theta)}{1 + \alpha\,(1 - cos\theta)}; \qquad \alpha = \frac{\hbar\omega}{m_{e^-}c^2} \tag{1}$$

This means that higher energy photons can transfer a fractionally higher amount of their energy in a Compton scattering process, because as the energy of the photon becomes larger ($\hbar\omega \to \infty$) then $\alpha \to \infty$ and $T_{e^-} \to \hbar\omega$. Below is a graph showing Equation (10.8) of Yip for three values of α, from 0.1 to 1 to 10:



3. In figure 10.17, which electron energy shell transitions (give the numbers of the levels involved) are responsible for the discontinuities in the attenuation coefficient? Which of the three photon interaction methods is responsible for these discontinuities?
The K-lines represent electron transitions down to the first (most bound) energy shell, while the L-lines represent electron transitions down to the second (2nd most bound) energy shell. The L1, L2, L3... lines represent the shell number transitions from $3 \to 2$, $3 \to 2$, and $3 \to 2$ respectively. These electron shell transition energies are determined by the following formula:

$$E = \frac{hc}{\lambda} = Ry\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) \tag{2}$$

The photoelectric effect, or the direct ejection of electrons by photon absorption, is principally responsible, although Compton scattering may play a minor role at these energies, as depending on the scattering angle a Compton scatter may be enough to eject an electron. Photoelectic emission is by far the dominant process. Pair production is not possible at these energies, as even the K-edge resides at about 0.1 MeV, ten times too low for pair production to take place.

4. In figure 10.13, why does the pair production cross section become non-zero abruptly at energies above $\frac{\hbar\omega}{2m_e c^2}$?
Because the photon must have enough energy to create a pair of particles, an electron and a positron, with equal rest masses. Once it does, the reaction is possible, and therefore its cross section becomes non-zero.

5. Explain why it is more likely to see a single-escape pair production peak from a NaI detector (figure 10.18), while it is more likely to see a double-escape pair production peak from a semiconductor detector (figure 10.19).
Because the NaI detector is physically larger, it is more likely that at least one of the annihilation photons from pair production will be absorbed in the detector rather than escaping. This will add its 511 keV of energy to the total charge accumulated. Semiconductor detectors tend to be quite small, so chances are that both of the annihilation photons produced will escape the detector.

2 Analytical Questions

1. By treating the photon in figure 10.2 as a particle with energy $\hbar ck$ and momentum $\hbar k$, and by conserving x-momentum, y-momentum, and total energy, show the origins (re-derive) equations 10.6 through 10.9.
   We start with the momentum and energy conservation equations:

$$\hbar k = \hbar k' + p \tag{3}$$

$$\hbar ck = \hbar ck' + T \tag{4}$$

where k is the initial wavevector of the photon, and p is the ending momentum of the electron, given by the formula $cp = \sqrt{T\left(T + 2m_e c^2\right)}$. We start by conserving total momentum, and isolating the electron momentum on one side (which we have an expression for):

$$p_e = \hbar k - \hbar k' \tag{5}$$

Squaring both sides we obtain:

$$p_e^2 = \hbar^2 k^2 + \hbar^2 k'^2 - 2\hbar^2 kk' cos\theta \tag{6}$$

by recognizing that momenta are vector quantities. Now we take energy conservation into account. We can rearrange the energy T of the electron from the electron momentum-energy relation:

$$p_e^2 = \frac{T\left(T + 2m_e c^2\right)}{c^2} \tag{7}$$

And we can use the energy conservation equation to substitute in for T:

$$T = \hbar ck - \hbar ck' \tag{8}$$

to yield:

$$p_e^2 = \frac{\left(\hbar ck - \hbar ck'\right)\left(\hbar ck - \hbar ck' + 2m_e c^2\right)}{c^2} = \frac{\hbar^2 \cancel{c^2} k^2 - \hbar^2 \cancel{c^2} kk' + 2\hbar ck m_e \cancel{c^2} - \hbar^2 \cancel{c^2} kk' + \hbar^2 \cancel{c^2} k'^2 - 2\hbar ck' m_e \cancel{c^2}}{\cancel{c^2}} \tag{9}$$

3

$$p_e^2 = \hbar^2 k^2 - 2\hbar^2 kk' + 2\hbar ck m_e + \hbar^2 k'^2 - 2\hbar ck' m_e \tag{10}$$

Now we equate Equations 6 and 10 to eliminate the momentum of the electron:

$$\cancel{\hbar^2 k^2} + \cancel{\hbar^2 k'^2} - 2\hbar^2 kk' cos\theta = \cancel{\hbar^2 k^2} - 2\hbar^2 kk' + 2\hbar ck m_e + \cancel{\hbar^2 k'^2} - 2\hbar ck' m_e \tag{11}$$

$$-2\hbar^2 kk' cos\theta = -2\hbar^2 kk' + 2\hbar ck m_e - 2\hbar ck' m_e \tag{12}$$

$$\cancel{2}\hbar^{\cancel{2}} kk'\left(1 - cos\theta\right) = \cancel{2}\cancel{\hbar} c m_e \left(k - k'\right) \tag{13}$$

$$\hbar kk'\left(1 - cos\theta\right) = m_e c\left(k - k'\right) \tag{14}$$

$$\frac{\hbar}{m_e c} kk'\left(1 - cos\theta\right) = k - k' \tag{15}$$

$$\frac{\hbar}{m_e c}\left(1 - cos\theta\right) = \frac{1}{k'} - \frac{1}{k} \tag{16}$$

Finally we multiply each side of the equation by 2π:

$$\frac{h}{m_e c}\left(1 - cos\theta\right) = \frac{2\pi}{k'} - \frac{2\pi}{k} \tag{17}$$

and we recognize that $k = \frac{2\pi}{\lambda}$:

$$\frac{h}{m_e c}\left(1 - cos\theta\right) = \lambda' - \lambda \tag{18}$$

Next, we turn to Equation 10.7 in Yip. We note in the reading that $\alpha = \frac{\hbar\omega}{m_e c^2}$, the ratio of the photon's incoming energy to the elctron rest mass energy. We then take Equation 18 and divide by one of the wavelengths, λ:

$$\frac{h}{m_e c\lambda}\left(1 - cos\theta\right) = \frac{\lambda'}{\lambda} - 1 \tag{19}$$

Next we recognize that $\lambda = \frac{c}{\nu}$, and $\omega = 2\pi\nu$. This gives us:

$$\frac{h\nu}{m_e c^2}\left(1 - cos\theta\right) = \frac{c\nu}{c\nu'} - 1 \tag{20}$$

$$\frac{\hbar}{m_e c^2}\left(1 - cos\theta\right) = \frac{\omega}{\omega'} - 1 \tag{21}$$

$$\frac{\omega}{\omega'} = 1 + \frac{\hbar}{m_e c^2}\left(1 - cos\theta\right) = 1 + \alpha\left(1 - cos\theta\right) \tag{22}$$

$$\frac{\omega'}{\omega} = \frac{1}{1 + \alpha\left(1 - cos\theta\right)} \tag{23}$$

Next, we turn to Equation 10.8 in Yip. We take Equation 4:

$$T = \hbar ck - \hbar ck' = \hbar\omega - \hbar\omega' \tag{24}$$

and we substitute in Equation 23 for ω':

$$T = \hbar\omega - \hbar\omega' = \hbar\omega - \hbar\frac{\omega}{1 + \alpha\left(1 - cos\theta\right)} \tag{25}$$

We multiply both sides of the first term on the right by $\frac{1+\alpha(1-cos\theta)}{1+\alpha(1-cos\theta)}$:

$$T = \hbar\omega\frac{1 + \alpha\left(1 - cos\theta\right)}{1 + \alpha\left(1 - cos\theta\right)} - \hbar\omega\frac{1}{1 + \alpha\left(1 - cos\theta\right)} \tag{26}$$

$$T = \hbar\omega\left[\frac{1 + \alpha\left(1 - cos\theta\right)}{1 + \alpha\left(1 - cos\theta\right)} - \frac{1}{1 + \alpha\left(1 - cos\theta\right)}\right] \tag{27}$$

$$T = \hbar\omega\frac{\alpha\left(1 - cos\theta\right)}{1 + \alpha\left(1 - cos\theta\right)} \tag{28}$$

Finally, we turn to Equation 10.9 in Yip. Since we want to relate the two angles, θ for the photon and ϕ for the electron, we conserve the total x- and y-momenta:

$$\hbar k - \hbar k' cos\theta = p_e cos\phi \tag{29}$$

$$\hbar k' sin\theta = p_e sin\phi \tag{30}$$

Next, we divide Equation 29 by Equation 30:

$$\frac{\cancel{\hbar}k - \cancel{\hbar}k' cos\theta}{\cancel{\hbar}k' sin\theta} = \frac{k - k' cos\theta}{k' sin\theta} = cot\left(\phi\right) \tag{31}$$

From Equation 23, we have the following relation:

$$\frac{\omega'}{\omega} = \frac{1}{1 + \alpha\left(1 - cos\theta\right)} \tag{32}$$

which yields:

$$\omega'\left(1 + \alpha\left(1 - cos\theta\right)\right) = \omega \tag{33}$$

Substituting this into Equation 31, and noting that $\omega = ck$, we get:

$$\frac{\frac{\omega'}{c}\left(1+\alpha\left(1-cos\theta\right)\right)-\frac{\omega'}{c}cos\theta}{\frac{\omega'}{c}sin\theta} = \frac{1+\alpha\left(1-cos\theta\right)-cos\theta}{sin\theta}cot\left(\phi\right) \tag{34}$$

$$\frac{1+\alpha-\alpha cos\theta-cos\theta}{sin\theta} = \frac{\left(1+\alpha\right)\left(1-cos\theta\right)}{sin\theta} = cot\left(\phi\right) \tag{35}$$

Finally we note the following trigonometric identity:

$$\frac{1-cos\left(\theta\right)}{sin\left(\theta\right)} = tan\left(\frac{\theta}{2}\right) \tag{36}$$

So we arrive at:

$$\left(1+\alpha\right)tan\left(\frac{\theta}{2}\right) = cot\left(\phi\right) \tag{37}$$

2. Derive a relation between the minimum and maximum values of the Klein-Nishina cross section as a function of incoming photon energy. Graph the angle of minimum scattering probability as a function of incoming photon energy. The angle of maximum scattering probability is always at zero degrees... with one exception. What is it?
For this problem, we will use the unpolarized cross section, as we didn't discuss any implications of polarization in class. This is Equation 10.16 in Yip:

$$\frac{d\sigma_C}{d\Omega} = \frac{r_e^2}{2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-sin^2\theta\right) \tag{38}$$

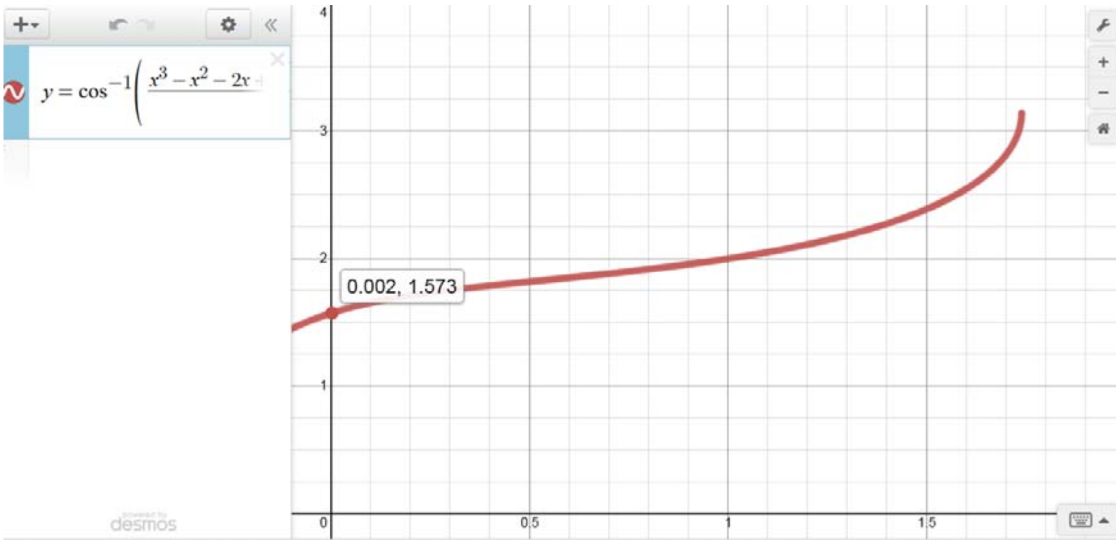
We know that the maximum value of the cross section is always at $\theta = 0$, since $\sin^2\theta$ is always positive, and this term is always subtracted, and $\sin^2 0 = 0$. Now we just have to find an expression for the minimum value of the cross section. Our strategy is as follows:
1) Substitute $\frac{\omega'}{\omega}$ into Equation 38 to get something purely in terms of α and θ
2) Take the derivative of this function with respect to θ
3) Set this derivative equal to zero, and solve for θ_{min} in terms of α, which is a measure of the photon energy
4) Plus this value of θ_{min} into Equation 38 to get the value of $\frac{d\sigma_C}{d\Omega}_{min}$
Once again we can use Equation 23 to substitute for $\frac{\omega'}{\omega}$, using Equation 10.19 in Yip:

$$\frac{d\sigma_C}{d\Omega} = \frac{r_e^2}{2}\left(1+cos^2\theta\right)\left(\frac{1}{1+\alpha\left(1-cos\theta\right)}\right)^2\left[1+\frac{\alpha^2\left(1-cos\theta\right)^2}{\left(1+cos^2\theta\right)\left(1+\alpha\left(1-cos\theta\right)\right)}\right] \tag{39}$$

Next we differentiate this cross section with respect to angle and set it equal to zero, to find the angle where the cross section reaches a minimum. This was done numerically, using Maple 15 on Athena:;:

$$\frac{\left(\frac{d\sigma_C}{d\Omega}\right)}{d\theta} = 0 \Longrightarrow \theta_{min} = cos^{-1}\left[\frac{\alpha^3-\alpha^2-2\alpha\pm\sqrt{-3\alpha^4+6\alpha^3+10\alpha^2+4\alpha+1}-1}{\alpha^3-2\alpha^2-2\alpha}\right] \tag{40}$$

Graphing this function, we obtain a plot of α vs. θ_{min} using the positive square root, as the negative option gives an imaginary solution:



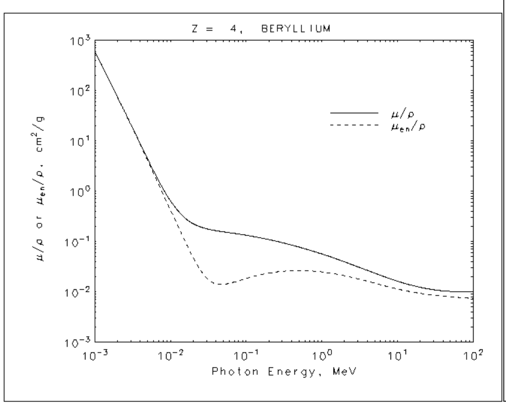
As a sanity check, this actually makes sense. The value of the angle of $\frac{d\sigma_C}{d\Omega}_{min}$ at $\alpha = 0$ is $\frac{\pi}{2}$, which can be seen in Figure 10.4 of Yip. This also follows mathematically, as in the limit of very low photon energy, the cross section reduces to:

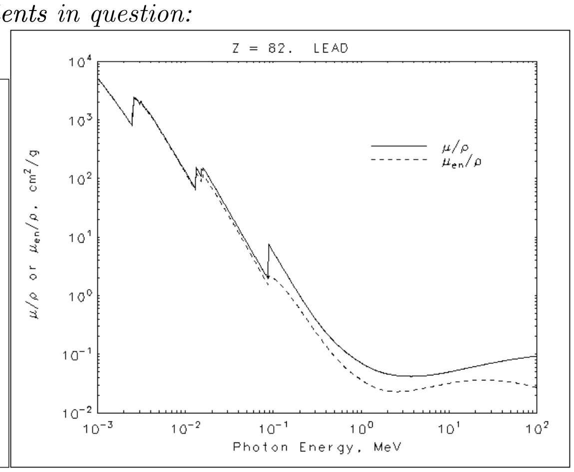
$$\frac{d\sigma_C}{d\Omega} \approx \frac{r_e^2}{2}\left(1 + cos^2\theta\right) \tag{41}$$

which is symmetric about $\theta = \frac{\pi}{2}$. The angle continues to increase with increasing photon energy, as Figure 10.4 shows, until one hits approximately $\alpha = 1.74$, where the function is undefined. This means that there is no minimum, and the cross section is continuously decreasing. This can also be seen in Figure 10.4. When $\alpha = 1$ ($\hbar\omega = 0.51\,MeV$), there is a defined, yet shallow minimum. Once we reach $\hbar\omega = 2.56\,MeV$ ($\alpha = 5$), the function is continuously decreasing.

3. For these problems, refer to the NIST table of x-ray attenuation coefficients.

   (a) Explain the qualitative differences in the attenuation coefficients of beryllium and lead in a quantitative manner, at the following energies: $E_\gamma < 100$keV, $E_\gamma = 1$MeV, $E_\gamma = 100$MeV. By this, we mean compare relative values of the relevant scattering cross sections, and explain any discrepancies between these and the relative values of the attenuation coefficients.
   Here are the two mass attenuation coefficients in question:



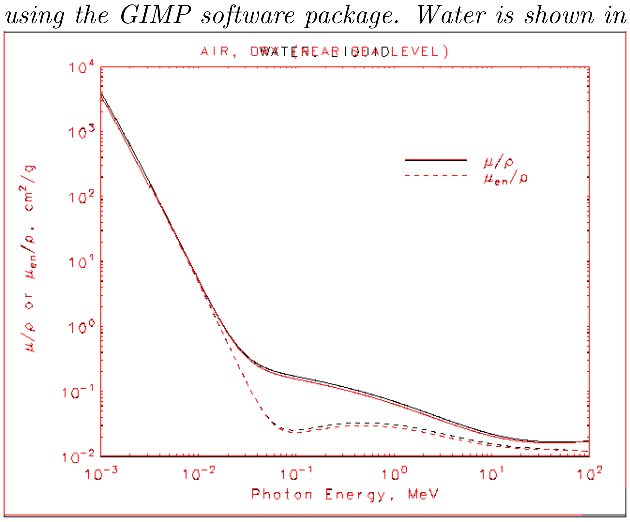


At energies below 100keV, the value of lead's mass attenuation is much higher due to the $\sigma \propto Z^{4-5}$ dependence of the photoelctric effect cross section. This means that higher Z materials undergo the photoelectric effect more often than lighter ones, even without accounting for differences in density. The sharp edges are x-ray emission lines, and are discussed in part (b).

At energies of roughly 1MeV, beryllium's cross section is lower, because all photon-electron interactions depend strongly on the electron density of the material, which is itself determied by Z on a per-atom basis. In addition, the local minimum is observed right around 1MeV for lead, as that's when the photoelectric effect dies off and Compton scattering takes over. For beryllium, that process occurred at lower energies.
At energies around 100MeV, the radiative losses in lead are far higher than those for beryllium, as the cross section for pair production is roughly $\sigma \propto Z^2$.
In the end, the answer to all three energies in this question is Z-dependence of cross sections.

(b) What is the origin of the discontinuities in the attenuation coefficient for lead? Why is there more than one step change within close proximity at some places?
The step discontinuities in lead s mass attenuation coefficient are due to the K-, L-, and M-transition lines from electron ejection at these energies. If one were to eject a level-1 electron for example, one would see a higher level electron fall down to the 1st energy level. As soon as the incoming photon has enough energy to make this process happen, this additional energy loss mechanism "turns on." The multiple steps in close proximity are for different transitions, like the $3 \to 2$ or $4 \to 2$ transitions in the L-line area.

(c) For which energies is the *attenuation coefficient* in water higher than that in air? What about the *mass attenuation coefficient*?
The ***attenuation coefficient*** *is* ***always*** *higher in water than in air, because water is 1000 times denser than air. The* ***mass attenuation coefficient****, which normalizes energy loss to density, is almost always almost equal. This is shown by overlaying the two mass attenuation coefficients using the GIMP software package. Water is shown in black, while air is shown in red:*



3 Applied Questions

1. How thick would a lead apron have to be to shield you from 99.9% of the x-rays from a dental exam, assuming they are generated from a ^{60}Co source?
For this problem, we use the photon attenuation equation:

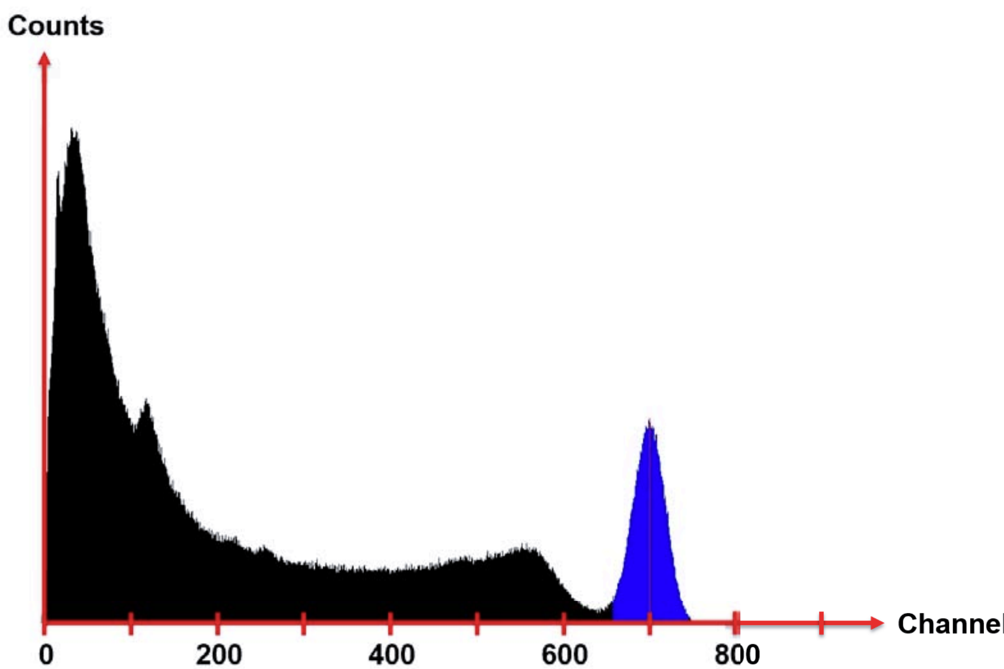
$$\frac{I}{I_0} = 0.001 = e^{-\left(\frac{\mu}{\rho}\right)\rho x} \tag{42}$$

Looking up the decay of ^{60}Co on the KAERI table of nuclides, the most likely scenario is for beta decay to release two gamma rays, one at an energy of 1.173MeV and another at an energy of 1.333MeV. Let's just say that they're both 1.25MeV gamma rays. Using the NIST table of mass attenuation coefficients (see Problem 2.3.a), we obtain a mass attenuation coefficient in lead of $\left(\frac{\mu}{\rho}\right) = 0.05876\,\frac{cm^2}{g}$. The density of lead was found to be 11.3 $\frac{g}{cm^3}$. Solving for x in this equation we get a thickness of 10.4 cm.

That would be inordinately heavy to wear, showing that you likely don't attenuate that many x-rays at a real dentist appointment.

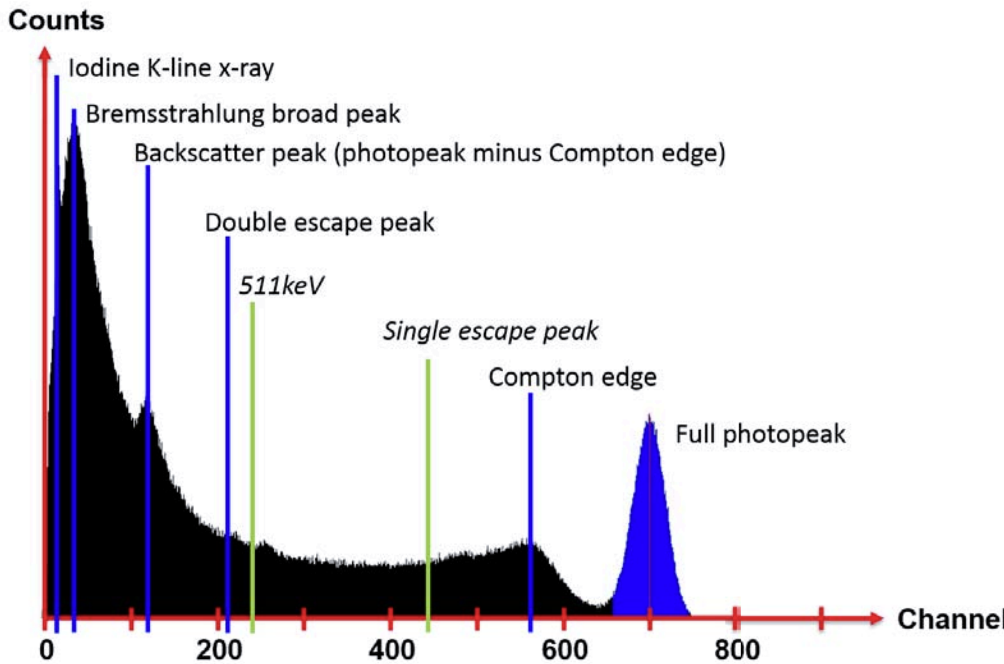
2. These ^{60}Co gamma rays will generate x-rays in the lead apron. Which interaction mechanism is responsible for this, and what percent of the x-rays will your clothing and skin absorb before they enter your body?
   At these energies, pair production is technically possible, but quite unlikely. Therefore, ionization by Compton scattering and the photoelectric effect is responsible, with high-Z material like lead making the ***photoelectric effect*** *most responsible. It is also most likely that inner-shell electrons will be ejected, making these K-line transitions. Using the NIST tables of x-ray attenuation coefficients for lead, one finds that the K-line is at an energy of 88keV. The mass attenuation coefficient at this energy (conveniently stated on the table) is* $7.863\,\frac{cm^2}{g}$. *Let's assume that you are wearing a T-shirt, about 2mm thick, and it's made of 100% cotton. Cotton is a hydrocarbon (cellulose) based material, and the closest material to cotton on the NIST table is likely polystyrene (latex). Latex has a mass attenuation coefficient of about* $0.168\,\frac{cm^2}{g}$ *at 88keV, while we will approximate cotton's density at one third its theoretical density of* $1.52\,\frac{g}{cm^3}$, *because it is made of open woven fibers. Using these values, we get an attenuation of just 1.7% of the lead K-line x-rays due to the T-shirt.*
   Turning to values for human skin, we assume that your skin is also 2mm thick, and is made of NIST ICRU-44 soft tissue. Its density should be about the same as water ($1\,\frac{g}{cm^3}$), *and its mass attenuation coefficient at 88keV is about* $0.175\,\frac{cm^2}{g}$. *Using these numbers, your skin stops about 3.4% of the x-rays remaining after they pass through your T-shirt.*
   This gives a total of 95% x-ray transmission into your body, or a total of 5% x-ray absorption by your clothes and skin.

3. For the ^{40}K gamma spectrum shown, answer the following questions:



   (a) Label all visible peaks and major features, their energies, and explain their origins.
   Potassium-40 gives of 1.461MeV gamma rays when it undergoes electron capture (EC), so the blue peak is the 1.461MeV photopeak. This is a result of ALL processes which result in the full energy of the incoming gamma ray being absorbed in the detector, including:
   -Photoelectric effect ejection and subsequent electron ionization chains
   -Compton scattering, where the scattered photon is also absorbed by any of the allowed mechanisms, and so is its next photon, and so on

-Pair production, where both 511keV gamma rays are absorbed by Compton scattering or the photoelectric effet
Knowing that Channel 700 lies squarely on the photopeak energy, we can divide 1.461MeV by 700 channels to get a width of 2.087keV per channel. This will allow us to identify the remaining peaks. Below is a chart showing the labelled peaks, plus locations where peaks would have been expected to be found:



The lowest energy peak is likely an iodine K-line, as a typical scintillator material is NaI. <u>This one doesn't count for credit, it's just informational.</u>
The next broad peak is likely a combination of noise and bremsstrahlung from the electrons changing direction in the detector material.
The next peak is a backscatter peak, of energy almost equal to the photopeak minus the Compton edge. This is due to photons Compton scattering in the materials around the detector, and only the detection of the ***scattered*** *photon by the detector. In essence Compton scattering in the materials surrounding the detector produces an "inverse bathtub" of extra peaks, of which only the most likely (backscattered) ones are present.*
The next peak is a double escape peak, very weak, representing the event where pair production takes place and both annihilation photon escape.
The next peak, which isn't present, would be from 511keV photons produced by pair production in the materials surrounding the detector.
The next peak, also not present, would have been the single escape peak for pair production.
The next broad peak is the Compton edge, representing the largest amount of energy able to be transferred to an electron.
The final peak is the full photopeak, a sum of all the possible scenarios in which all the incoming photon's energy is absorbed.

(b) Identify where the locations of any missing peaks should be, their energies, and explain their origins.
(See part (a) for full solution)

(c) How do you explain the energy difference between the full energy gamma peak and the Compton edge?
Photons of non-zero energy are not able to transfer all their energy to a Compton electron in a

scattering process. The maximum amount of energy transfer is given by the following equation:

$$T = \hbar\omega \frac{\alpha\,(1 - cos\theta)}{1 + \alpha\,(1 - cos\theta)}; \quad \theta = \pi\ (backscattering) \tag{43}$$