MAS.160 / MAS.510 / MAS.511 Signals, Systems and Information for Media Technology
Fall 2007

Frequency response

FIR

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$h[k] = \sum_{k=0}^{M} b_k \delta[n-k]$$

$$\downarrow$$

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} h[k] e^{j\hat{\omega}k}$$

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{j\hat{\omega}k}$$

Easy to go from difference equation to frequency response because h[n] finite length and h[n] = [b_0, b_1,…].

IIR

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

$$h[k] \neq \sum_{k=0}^{\infty} b_k \delta[n-k]$$

$$\downarrow$$

X Argh!

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$

Tough to go from diff.eqn. to freq. response because h[n] infinite length, h[n]=f(a_l,b_k) is complicated, and $\mathcal{H}(\hat{\omega})$ may be unbounded.

temporal space - n

complex frequency space- z

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

$$h[k] \neq \sum_{k=0}^{\infty} b_k \delta[n-k]$$

z-transform →

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{\prod_{i=0}^{M}(z - z_{zi})}{\prod_{i=0}^{N}(z - z_{pi})}$$

Fourier transform ↓ @#! Argh! road block

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$

frequency space - ω

← Hurray!

$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

The frequency response is H(z) evaluated on unit circle

Benefits of z-plane and z-transforms:

1. Get around road block by using z-plane and z-transforms.
Compute system function from diff.eq. coefficients, then evaluate on the unit circle to find the frequency response.

2. z-plane (pole/zeros) will tell us if system stable and frequency response exists.

3. By using z-transforms, solution to diff.eq goes from solving convolution in n-space to solving algebraic equations in z-domain (easier).

And lots more…!

Infinite signals

$$x[n] = a^n u[n] \quad \Leftrightarrow \quad X(z) = \sum_{k=0}^{\infty} a^k z^{-k}$$

x[n]=0 n<0
right sided

$$= 1 + az^{-1} + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 \ldots$$

geometric series

$$X(z) = 1 + az^{-1} + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 \ldots$$

$$-\,az^{-1}X(z) = \quad -az^{-1} - \left(az^{-1}\right)^2 - \left(az^{-1}\right)^3 - \left(az^{-1}\right)^4 \ldots$$

$$\left(1 - az^{-1}\right)X(z) = 1$$

$$X(z) = \frac{1}{1 - az^{-1}}$$

$$\left|az^{-1}\right| < 1$$ region of convergence

or $|z| > |a|$

Infinite signals

$$x[n] = -a^n u[-n-1] \Leftrightarrow \quad X(z) = -\sum_{k=-\infty}^{-1} a^k z^{-k}$$

$x[n] = 0 \quad n \geq 0$

left sided

$$= -\tfrac{1}{a}z - \left(\tfrac{1}{a}z\right)^2 - \left(\tfrac{1}{a}z\right)^3 \dots$$

geometric series

$$X(z) = \quad -\tfrac{1}{a}z - \left(\tfrac{1}{a}z\right)^2 - \left(\tfrac{1}{a}z\right)^3 \dots$$

$$-\ az^{-1}X(z) = 1 + \tfrac{1}{a}z + \left(\tfrac{1}{a}z\right)^2 + \left(\tfrac{1}{a}z\right)^3 + \left(\tfrac{1}{a}z\right)^4 \dots$$

$$\left(1 - az^{-1}\right)X(z) = 1$$

$$X(z) = \frac{1}{1 - az^{-1}}$$

$\left|\tfrac{1}{a}z\right| < 1$ region of convergence

or $|z| < |a|$

Infinite series:

$$x[n] = a^n u[n] \quad \Leftrightarrow \quad X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

region of convergence

x[n]=0 n<0
right sided

$$x[n] = -a^n u[-n-1] \quad \Leftrightarrow \quad X(z) = \frac{1}{1 - az^{-1}} \qquad |z| < |a|$$

$x[n] = 0 \quad n \geq 0$

left sided

Finite series:

$$x[n] = a^n \left(u[n-M] - u[n-N]\right) \quad \Leftrightarrow \quad X(z) = \sum_{k=M}^{N-1} a^k z^{-k}$$

$$X(z) = \frac{\left(az^{-1}\right)^M - \left(az^{-1}\right)^N}{1 - az^{-1}}$$

all z region of convergence

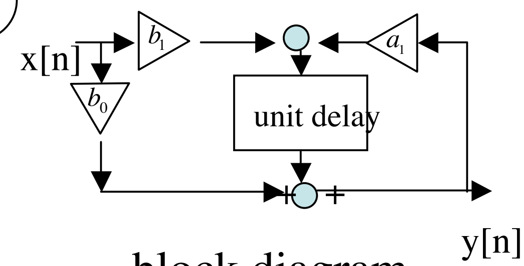
$$\lim_{z \to a} X(z) = N - M$$

Equivalent ways to represent the system

①
$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

difference equation

$\Leftrightarrow$ ②



block diagram

$x[n]=\delta[n]$ $\Updownarrow$

$\Updownarrow$ inspection

③
$$h[n] = y[n]\Big|_{x[n]=\delta[n]} \overset{z}{\Leftrightarrow} H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}} = \frac{\prod_{i=0}^{M}(z-z_{zi})}{\prod_{i=0}^{N}(z-z_{pi})}$$

impulse response sequence

④ system function polynomial

pole-zero locations ⑤

$z = e^{j\omega}$ $\Updownarrow$

⑥
$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

frequency response

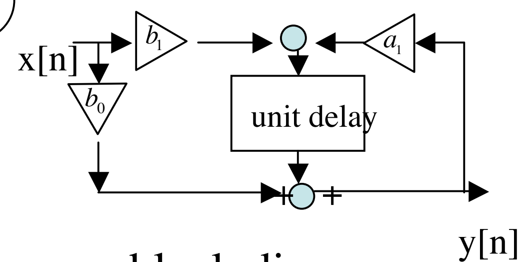
All poles must be inside unit circle for $\mathcal{H}(\omega)$ to converge and the system to be stable. (causal system) (FIR filter always stable)

Equivalent ways to represent the system

①
$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \quad \Leftrightarrow$$

difference equation

②



block diagram

$\Updownarrow$ inspection

$x[n]=\delta[n]$ $\Updownarrow$

③
$$h[n] = y[n]\Big|_{x[n]=\delta[n]} \overset{Z}{\Leftrightarrow} H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{\prod_{i=0}^{M}(z - z_{zi})}{\prod_{i=0}^{N}(z - z_{pi})}$$

impulse response
sequence

④ system function
polynomial

pole-zero
locations ⑤

$z = e^{j\omega}$ $\Updownarrow$

⑥
$$\mathcal{H}(\omega) = H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}}$$

frequency response

The region of convergence must
Contain the unit circle for $\mathcal{H}(\omega)$
to converge and the system
to be stable. (general)
(FIR filter always stable)

Ex.

$$H(z) = \tfrac{1}{3} + \tfrac{1}{3}z^{-1} + \tfrac{1}{3}z^{-2} = \frac{z^2 + z + 1}{3z^2}$$

$$y[n] = H(z)z^n$$

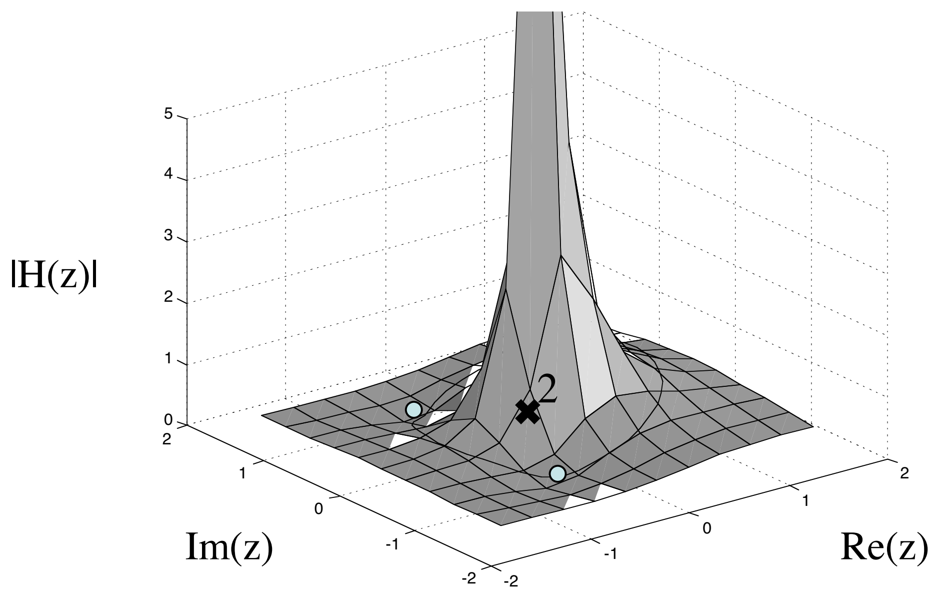
num=0 $\quad H(z) = 0 \quad y[n] = 0 \quad z^2 + z + 1 = 0$ zeros

$z = \tfrac{1}{2}\left(-1 \pm j\sqrt{3}\right) = e^{\pm j2\pi/3}$ roots of numerator

denom=0 $\quad H(z) = \infty \quad y[n] = \infty \quad z^2 = 0$

poles

$z = 0,0$ roots of denominator

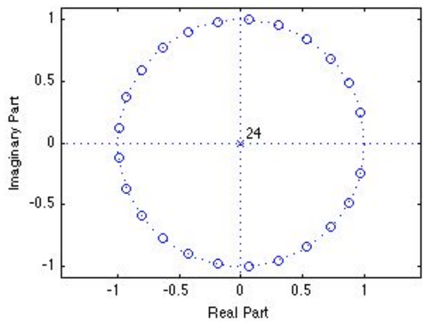


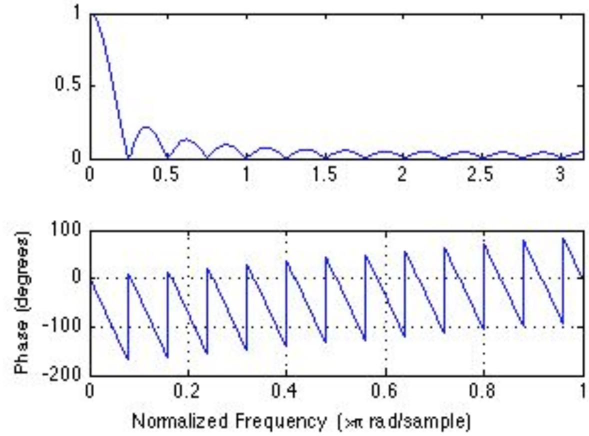
*FIR L point summer/averager
only has zeros on unit circle

FIR filters ~~only have zeros on unit circle, and~~ poles are either at 0 or ∞.

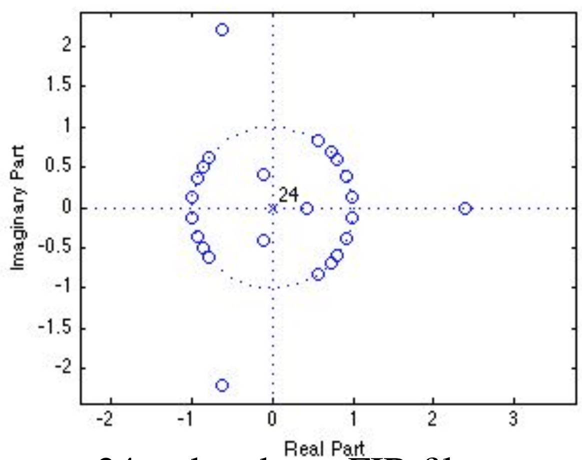
#poles=#zeros

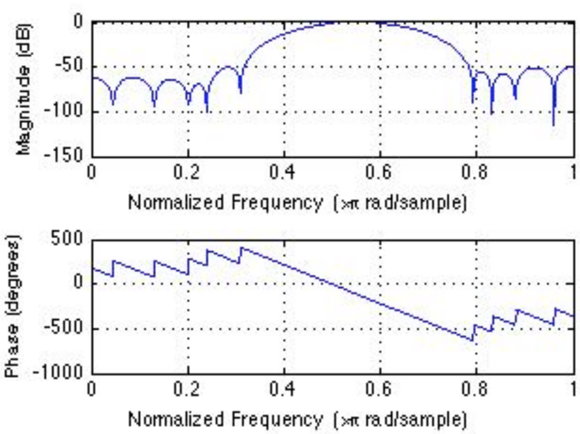
"extra" zero/poles are at z=∞.





25-pt averager lowpass FIR filter
*poles all at zero (or ∞)
*zeros evenly distributed on unit circle
*missing zero at DC (lowpass)





24-pt bandpass FIR filter b=fir1(24, [.45 .65],'bandpass');
*poles all at zero (or ∞)
*zeros not necessarily on unit circle
* Only pole locations affect stability

Ex. $$H(z) = \tfrac{1}{3} + \tfrac{1}{3} z^{-1} + \tfrac{1}{3} z^{-2} = \frac{z^2 + z + 1}{3z^2}$$

$$y[n] = H(z) z^n$$

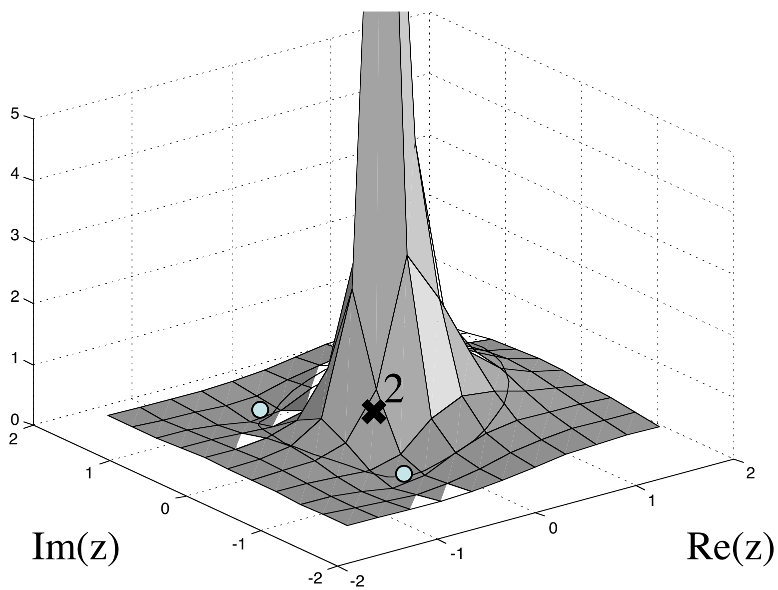
num=0 $H(z) = 0$ $y[n] = 0$ $z^2 + z + 1 = 0$ zeros

$z = \tfrac{1}{2}\left(-1 \pm j\sqrt{3}\right) = e^{\pm j2\pi/3}$ roots of numerator

denom=0 $H(z) = \infty$ $y[n] = \infty$ $z^2 = 0$

poles

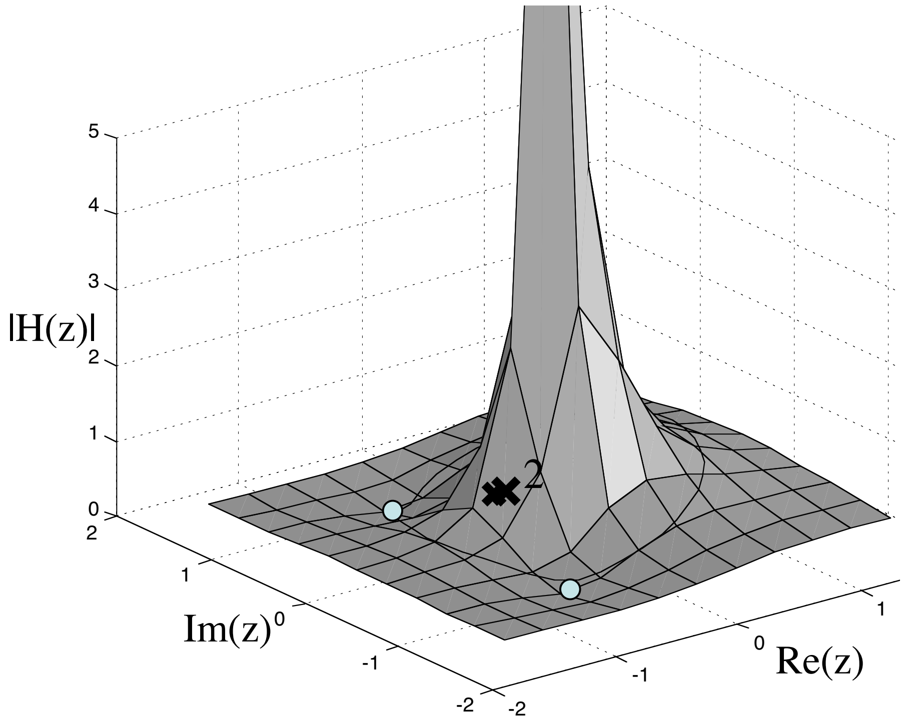
$z = 0,0$ roots of denominator



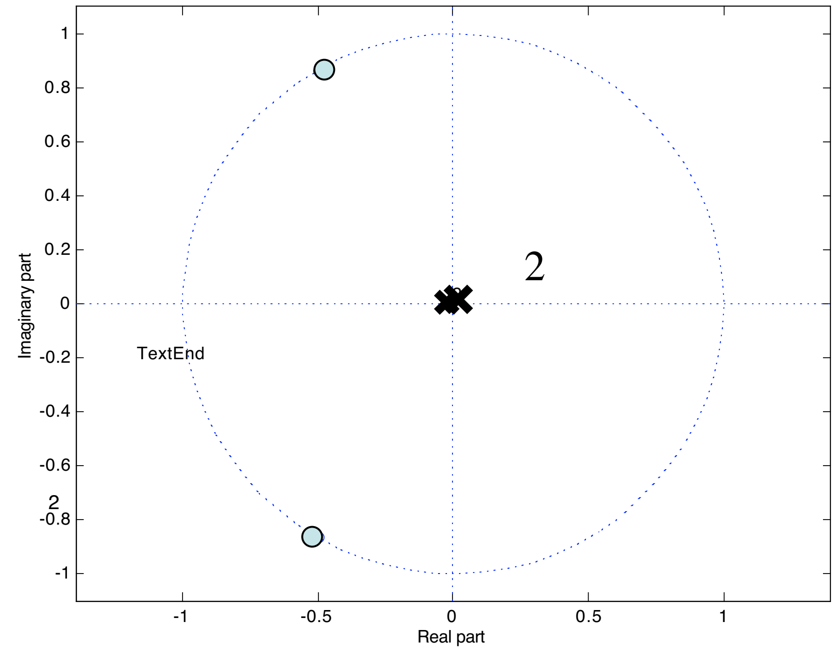
FIR filters have poles
at either at 0 or ∞.

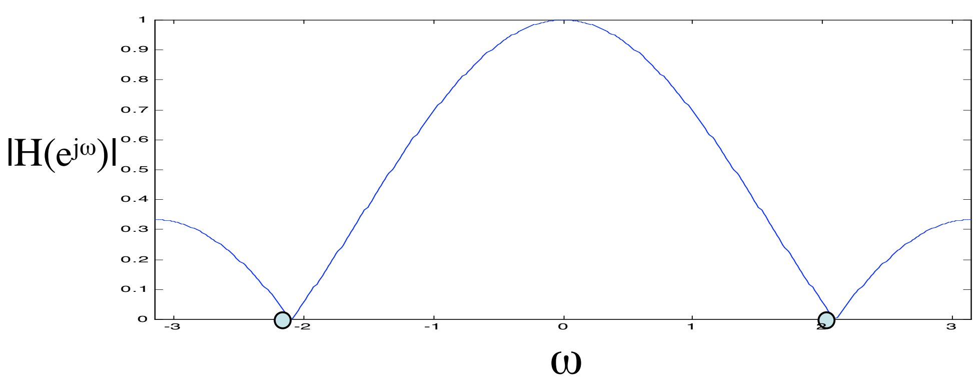
#poles=#zeros
“extra” zero/poles are at z=¥.

system response |H(z)|



pz plot





frequency response

$$\mathcal{H}(\omega) = H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}}$$

The frequency response is
H(z) evaluated on unit circle

Solving impulse response

$$y[n] = x[n] - y[n-2]$$

$$x[n] = \delta[n]$$

iteration

$$y[0] = x[0] - y[-2] = 1 - 0 = 1$$
$$y[1] = x[1] - y[-1] = 0 - 0 = 0$$
$$y[2] = x[2] - y[0] = 0 - 1 = -1$$
$$y[3] = x[3] - y[1] = 0 - 0 = 0$$
$$y[4] = x[4] - y[2] = 0 - (-1) = 1$$
$$\vdots$$

z-transform

$$Y(z) = X(z) - z^{-2}Y(z)$$

$$\left(1 + z^{-2}\right)Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1 + z^{-2}\right)}$$ system function

$$Y(z) = H(z)X(z) = \frac{1}{\left(1 + z^{-2}\right)} X(z)$$

Remember:

$$H(z) = \frac{X(z)}{Y(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$X(z) = \frac{z^2 - \cos(\hat{\omega})z}{z^2 - 2\cos(\hat{\omega})z + 1}$$

$$\Updownarrow$$

$$x[n] = \cos(\hat{\omega}n)u[n]$$

$$\hat{\omega} = \frac{\pi}{2} = \frac{2\pi}{4}$$

$$Y(z) = \frac{1}{\left(1 + z^{-2}\right)} = \frac{z^2}{z^2 + 1}$$

$\Downarrow$ Inverse z-transform (lookup)

$$y[n] = h[n] = \cos\left(\frac{2\pi}{4} n\right)u[n]$$

$$y[n] = \{1, 0, -1, 0, 1, \ldots\}$$

Solve difference equation

$$y[n] = x[n] - y[n-2] \qquad h[n] = \cos\left(\frac{2\pi}{4}n\right)u[n] \quad \text{impulse response}$$

$$x[n] = u[n] \quad \text{step input}$$

iteration

$$y[0] = x[0] - y[-2] = 1 - 0 = 1$$
$$y[1] = x[1] - y[-1] = 1 - 0 = 1$$
$$y[2] = x[2] - y[0] = 1 - 1 = 0$$
$$y[3] = x[3] - y[1] = 1 - 1 = 0$$
$$y[4] = x[4] - y[2] = 1 - 0 = 1$$
$$\vdots$$

convolution

x[n]	1	1	1	1	1	1	1	…
h[n]	1	0	-1	0	1	0	- 1	…
		1	1	1	1	1	1	
			0	0	0	0	0	
1				-1	-1	-1	-1	…
					0	0	0	
						1	1	
0						⋮		
		1	0	0	1	…		

1 1

z-transforms

$$y[n] = x[n] - y[n-2] \qquad x[n] = u[n]$$

$$\Downarrow z \qquad\qquad \Downarrow z$$

$$Y(z) = X(z) - z^{-2}Y(z) \qquad X(z) = \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{\left(1 + z^{-2}\right)} X(z) = H(z)X(z)$$

multiplication

$$Y(z) = \frac{1}{\left(1 + z^{-2}\right)} \frac{1}{\left(1 - z^{-1}\right)}$$

$y[n]$		$Y(z)$
$a^n u[n]$	$\Leftrightarrow$	$\frac{1}{\left(1 - az^{-1}\right)}$
$\cos(\alpha n)u[n]$	$\Leftrightarrow$	$\frac{z(z - \cos\alpha)}{z^2 - (2\cos\alpha)z + 1}$
$\cos\left(\frac{2\pi}{4} n\right)u[n]$	$\Leftrightarrow$	$\frac{1}{\left(1 + z^{-2}\right)}$

Sum of responses of individual single real poles, or complex conjugate pairs of poles.

$\Downarrow$ partial fraction expansion

$$Y(z) = \frac{1/2}{\left(1 - z^{-1}\right)} + \frac{1/2}{\left(1 + z^{-2}\right)} + \frac{1}{2} \frac{z^{-1}}{\left(1 + z^{-2}\right)}$$

$\Downarrow$ Inverse z-transform (lookup)

$$y[n] = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{4} n\right) + \frac{1}{2}\cos\left(\frac{2\pi}{4}(n-1)\right) \Rightarrow$$

n	$y[n]$
0	1/2+1/2+0=1
1	1/2+0+1/2=1
2	1/2-1/2+0=0
3	1/2+0-1/2=0
4	1/2+1/2+0=1
5	1/2+0+1/2=1

Partial fraction expansion

residuals (arrows pointing to A and B)

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)} = \frac{A}{\left(1+2z^{-1}\right)} + \frac{B}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

Sum of responses of individual single real poles, or complex conjugate pairs of poles.

We know:

single pole @ z=a

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

complex conjugate poles

$$\frac{z\left(z-\gamma\cos\alpha\right)}{z^2-\left(2\gamma\cos\alpha\right)z+\gamma^2} \Leftrightarrow \gamma^n\cos\left(\alpha n\right)u[n]$$

$$|z| > |\gamma|$$

Partial fraction expansion

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} = \frac{A}{\left(1+2z^{-1}\right)} + \frac{B}{\left(1-\frac{3}{4}z^{-1}\right)}$$

Sum of responses of individual single real poles, or complex conjugate pairs of poles.

$\Downarrow$ cross multiply

$$Y(z) = \frac{A\left(1-\frac{3}{4}z^{-1}\right) + B\left(1+2z^{-1}\right)}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$\Downarrow$ collect terms

$$Y(z) = \frac{(-\frac{3}{4}A + 2B)z^{-1} + \left(A+B\right)}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

$$Y(z) = \frac{1}{\left(1 + 2z^{-1}\right)\left(1 - \tfrac{3}{4}z^{-1}\right)}$$

$$= \frac{A}{\left(1 + 2z^{-1}\right)} + \frac{B}{\left(1 - \tfrac{3}{4}z^{-1}\right)}$$

$$= \frac{(-\tfrac{3}{4}A + 2B)z^{-1} + \left(A + B\right)}{\left(1 + 2z^{-1}\right)\left(1 - z^{-1}\right)}$$

$$-\tfrac{3}{4}A + 2B = 0 \qquad A + B = 1$$ match coefficients

$$A = \frac{8}{11} \qquad B = \frac{3}{11}$$

$$Y(z) = \frac{8/11}{\left(1 + 2z^{-1}\right)} + \frac{3/11}{\left(1 - \tfrac{3}{4}z^{-1}\right)} = \frac{1}{\left(1 + 2z^{-1}\right)\left(1 - \tfrac{3}{4}z^{-1}\right)}$$

We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

Inverse z-transform

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$$= \frac{8/11}{\left(1+2z^{-1}\right)} + \frac{3/11}{\left(1-\frac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n] = \frac{8}{11}(-2)^n u[n] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n] \qquad |z| > |2|$$

causal, unstable

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

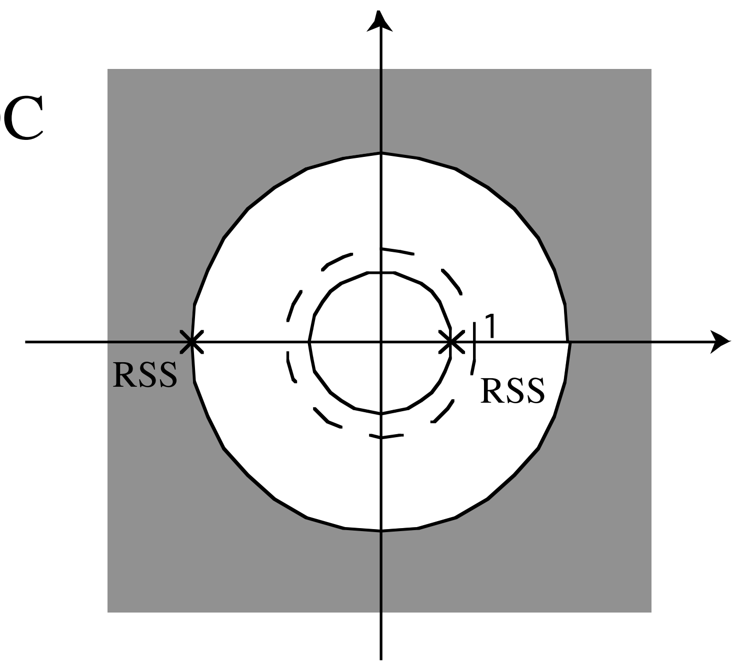
$$|z| > |a|$$

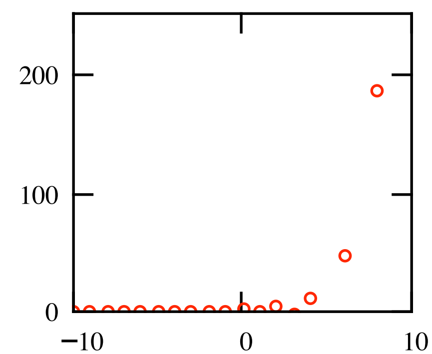
left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

ROC





Inverse z-transform

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$$= \frac{8/11}{\left(1+2z^{-1}\right)} + \frac{3/11}{\left(1-\frac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1] \qquad |z| < \left|\tfrac{3}{4}\right|$$

anticausal, unstable

We know:

right sided sequence

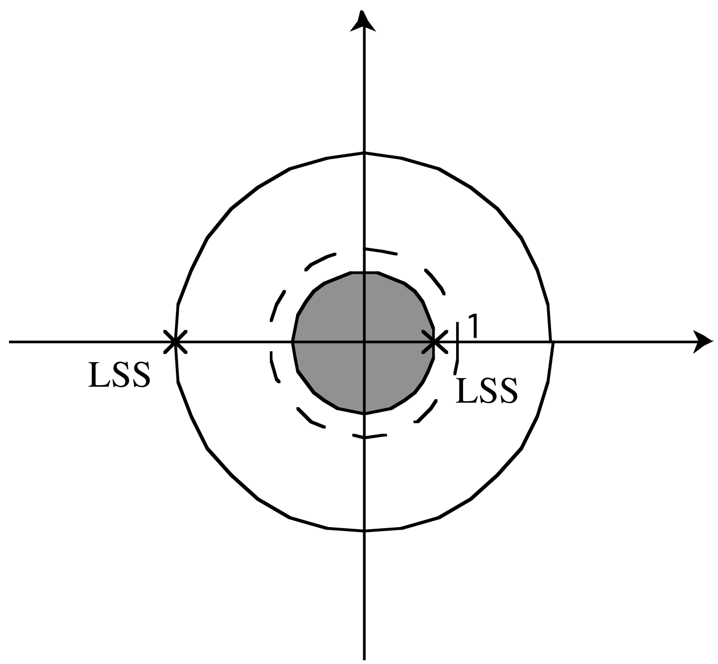
$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

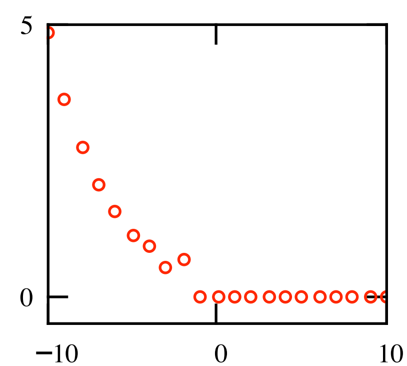
$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$





Inverse z-transform

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$$= \frac{8/11}{\left(1+2z^{-1}\right)} + \frac{3/11}{\left(1-\frac{3}{4}z^{-1}\right)}$$

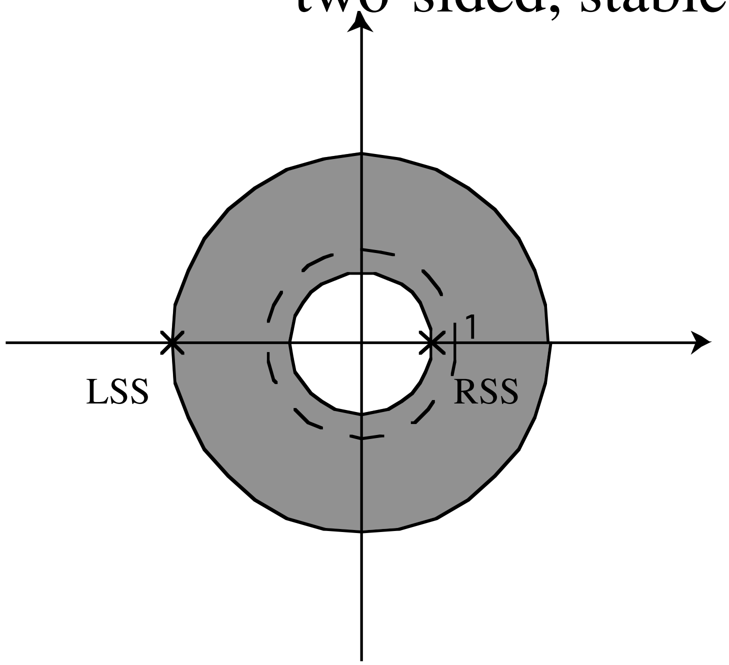
poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n]$$

$$\left|\tfrac{3}{4}\right| < |z| < |2|$$

two-sided, stable



We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

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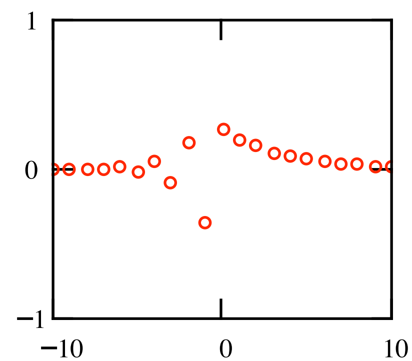


Inverse z-transform

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$$= \frac{8/11}{\left(1+2z^{-1}\right)} + \frac{3/11}{\left(1-\frac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n] = \frac{8}{11}(-2)^n u[n] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1]$$

$$\left|\tfrac{3}{4}\right| < |z| \cap |z| > |2|$$

not possible

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

Inverse z-transform

$$Y(z)=\frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$=\frac{8/11}{\left(1+2z^{-1}\right)}+\frac{3/11}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n]=\frac{8}{11}(-2)^n u[n]+\frac{3}{11}\left(\frac{3}{4}\right)^n u[n]$$ $|z|>|2|$ causal, unstable

$$y[n]=-\frac{8}{11}(-2)^n u[-n-1]-\frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1]$$ $|z|<\left|\tfrac{3}{4}\right|$ anticausal, unstable

$$y[n]=-\frac{8}{11}(-2)^n u[-n-1]+\frac{3}{11}\left(\frac{3}{4}\right)^n u[n]$$ $\left|\tfrac{3}{4}\right|<|z|<|2|$ two-sided, stable

$$y[n]=\frac{8}{11}(-2)^n u[n]-\frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1]$$ $\left|\tfrac{3}{4}\right|<|z|\cap|z|>|2|$ not possible

Partial fraction expansion II

$$Y(z) = \frac{4z + 7.6}{-6z^{-1} + 5 + 4z} = \frac{4z + 7.6}{-6z^{-1} + 5 + 4z} \cdot \frac{z^{-1}}{z^{-1}} = \frac{4 + 7.6z^{-1}}{-6z^{-2} + 5z^{-1} + 4}$$

Must be in terms only of z^{-1}

$$Y(z) = \frac{4 + 7.6z^{-1}}{-6z^{-2} + 5z^{-1} + 4} = \frac{4 + 7.6z^{-1}}{\left(1 + 2z^{-1}\right)\left(4 - 3z^{-1}\right)}$$

factor denominator into form $\left(1 + p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right)$

$$= \frac{1 + 1.9z^{-1}}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4} z^{-1}\right)}$$

$$= \frac{A}{\left(1 + 2z^{-1}\right)} + \frac{B}{\left(1 - \frac{3}{4} z^{-1}\right)}$$

How to find A,B?

"coverup" method

$$Y(z) \cdot \left(1 + 2z^{-1}\right) = \frac{A}{\left(1 + 2z^{-1}\right)} \cdot \left(1 + 2z^{-1}\right) + \frac{B}{\left(1 - \frac{3}{4} z^{-1}\right)} \cdot \left(1 + 2z^{-1}\right)$$

$$Y(z) \cdot \left(1 + 2z^{-1}\right) = A + \frac{B}{\left(1 - \frac{3}{4} z^{-1}\right)} \cdot \left(1 + 2z^{-1}\right)$$

$$Y(z) \cdot \left(1 + 2z^{-1}\right)\Big|_{z=-2} = A + \frac{B}{\left(1 - \frac{3}{4} z^{-1}\right)} \cdot 0 = A$$

Partial fraction expansion II

$$Y(z) = \frac{4 + 7.6z^{-1}}{-6z^{-2} + 5z^{-1} + 4} = \frac{4 + 7.6z^{-1}}{\left(1 + 2z^{-1}\right)\left(4 - 3z^{-1}\right)} = \frac{1 + 1.9z^{-1}}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)}$$

$$= \frac{A}{\left(1 + 2z^{-1}\right)} + \frac{B}{\left(1 - \frac{3}{4}z^{-1}\right)}$$

$$A = Y(z)\left(1 + 2z^{-1}\right)\Big|_{z=-2} = \frac{1 + 1.9z^{-1}}{\left(1 - \frac{3}{4}z^{-1}\right)}\Bigg|_{z=-2} = 0.036$$

$$B = Y(z)\left(1 - \frac{3}{4}z^{-1}\right)\Big|_{z=\frac{3}{4}} = \frac{1 + 1.9z^{-1}}{\left(1 + 2z^{-1}\right)}\Bigg|_{z=\frac{3}{4}} = 0.964$$

$$Y(z) = \frac{0.036}{\left(1 + 2z^{-1}\right)} + \frac{0.964}{\left(1 - \frac{3}{4}z^{-1}\right)}$$

Effects of a zero

zeros: 0,0

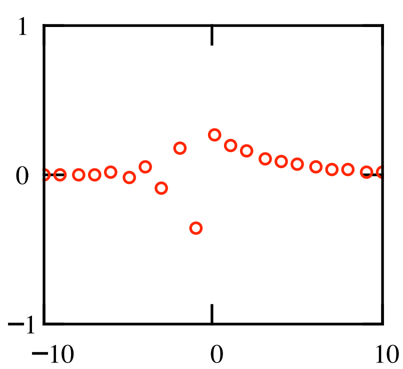
$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

$$= \frac{0.73}{\left(1+2z^{-1}\right)} + \frac{0.27}{\left(1-\frac{3}{4}z^{-1}\right)}$$

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n]$$

$$\left|\tfrac{3}{4}\right| < |z| < |2|$$



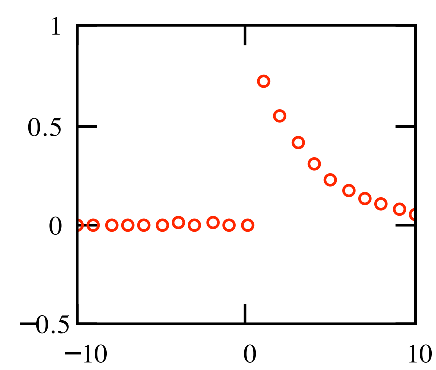
zeros: 0,-1.9

$$Y(z) = \frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$$Y(z) = \frac{0.036}{\left(1+2z^{-1}\right)} + \frac{0.964}{\left(1-\frac{3}{4}z^{-1}\right)}$$

$$y[n] = -0.036(-2)^n u[-n-1] + 0.964\left(\tfrac{3}{4}\right)^n u[n]$$

$$\left|\tfrac{3}{4}\right| < |z| < |2|$$



zero at z=-1.9
close to pole at z=-2,
pole's effect reduced
(0.036 vs. 0.727)

two sided sequences

Partial fraction expansion III

$$Y(z) = \frac{4z + 7.6}{-6z^{-1} + 5 + 4z} = \frac{4z + 7.6}{-6z^{-1} + 5 + 4z} \cdot \frac{z}{z}$$

Must be in terms only of z

$$= \frac{4z^2 + 7.6z}{4z^2 + 5z - 6}$$

$$= \frac{z^2 + 1.9z}{(z+2)(z - \frac{3}{4})}$$

factor into form (z-p1)(z-p2)
Hint: use matlab's **root** command

$$= \frac{Az}{(z+2)} + \frac{Bz}{(z - \frac{3}{4})} + C$$

We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} = \frac{z}{z - a} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} = \frac{z}{z - a} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

Partial fraction expansion III

$$Y(z) = \frac{4z^2 + 7.6z}{4z^2 + 5z - 6} = \frac{z^2 + 1.9z}{(z+2)(z-\frac{3}{4})}$$

$$= \frac{Az}{(z+2)} + \frac{Bz}{(z-\frac{3}{4})} + C$$

$$C = Y(z)\big|_{z=0} = \frac{4z^2 + 7.6z}{4z^2 + 5z - 6}\bigg|_{z=0} = 0$$

$$B = \frac{Y(z)(z-\frac{3}{4})}{z}\bigg|_{z=\frac{3}{4}} = \frac{z^2 + 1.9z}{z(z+2)}\bigg|_{z=\frac{3}{4}} = 0.964$$

$$A = \frac{Y(z)(z+2)}{z}\bigg|_{z=-2} = \frac{z^2 + 1.9z}{z(z-\frac{3}{4})}\bigg|_{z=-2} = 0.036$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

Partial fraction expansion III

$$Y(z) = \frac{4z^2 + 7.6z}{4z^2 + 5z - 6} = \frac{z^2 + 1.9z}{(z+2)\left(z - \frac{3}{4}\right)}$$

$$= \frac{0.036z}{(z+2)} + \frac{0.964z}{\left(z - \frac{3}{4}\right)}$$

We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} = \frac{z}{z - a} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

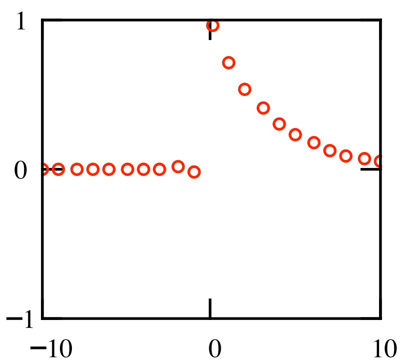
left sided sequence

$$\frac{1}{1 - az^{-1}} = \frac{z}{z - a} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

$$y[n] = -0.036(-2)^n u[-n-1] + 0.964\left(\tfrac{3}{4}\right)^n u[n]$$

$$\left|\tfrac{3}{4}\right| < |z| < |2|$$



two sided sequence

Long Division

$$Y(z) = \frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} = \frac{1+1.9z^{-1}}{-\frac{3}{2}z^{-2}+\frac{5}{4}z^{-1}+1}$$

$$= 1+\tfrac{5}{4}z^{-1}-\tfrac{3}{2}z^{-2}\,\overline{\big)\,1+1.9z^{-1}}$$

Quotient: $1+0.65z^{-1}+0.687z^{-2}+0.116z^{-3}$ right sided sequence

$$\underline{1+1.25z^{-1}-1.5z^{-2}}$$

$$0.65z^{-1}+1.5z^{-2}$$

$$\underline{0.65z^{-1}+0.813z^{-2}-0.975z^{-3}}$$

$$0.687z^{-2}+0.975z^{-3}$$

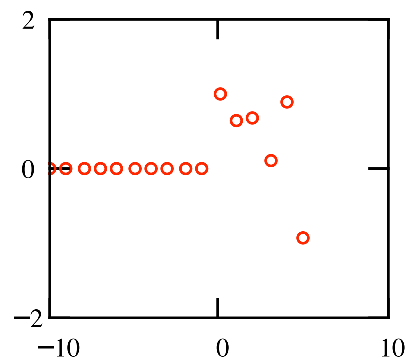
$$\underline{0.687z^{-2}+0.859z^{-3}-1.031z^{-4}}$$

$$0.116z^{-3}+1.031z^{-4}$$

$$y[n] = \delta[n]+0.65\delta[n-1]+0.69\delta[n-2]+0.12\delta[n-3]+\cdots$$

compare

$y[n] = 0.036(-2)^n u[n]+\left(\frac{3}{4}\right)^n 0.964u[n]$ ={1, 0.65, 0.69, 0.12} n=0,1,2,3



Long Division

$$Y(z) = \frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} = \frac{1+1.9z^{-1}}{-\frac{3}{2}z^{-2}+\frac{5}{4}z^{-1}+1}$$

left sided sequence

$$-1.267z - 1.723z^{2} - 2.281z^{3} + \ldots$$

$$= -1.5z^{-2} + 1.25z^{-1} + 1 \,\big)\, 1.9z^{-1} + 1$$

$$\underline{1.9z^{-1} - 1.584 - 1.267z}$$

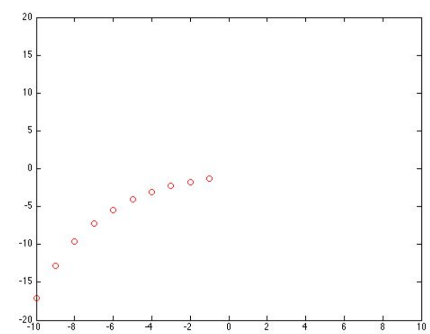
$$2.584 + 1.267z$$

$$\underline{2.584 - 2.154z - 1.723z^{-3}}$$

$$3.421z^{-2} + 0.975z^{-3}$$

$$\underline{3.421z^{-2} - 2.851z^{-3} - 2.281z^{-4}}$$

$$y[n] = -1.27\delta[n+1] - 1.72\delta[n+2] - 2.28\delta[n+3] + \cdots$$



compare

$y[n] = -0.036(-2)^{n}u[-n-1] - \left(\frac{3}{4}\right)^{n} 0.964u[-n-1]$={-1.27, -1.72, -2.28}

n={-1, -2, -3}

Fourier Transforms

Compute spectrum of signals

Fourier Series $$X_k = \frac{2}{T_0}\int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt$$ Periodic in (cont.) time
Discrete freq

DTFT $$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$ Discrete time
Periodic in (cont.) freq

DFT $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n}$$ Discrete & periodic time
Discrete & periodic freq

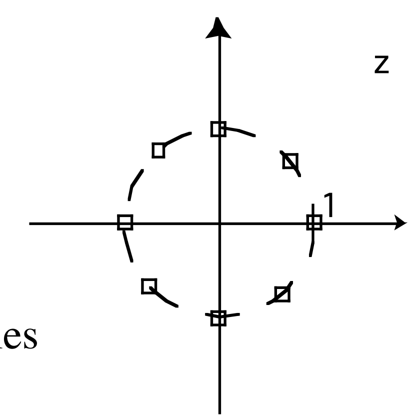
Discrete Fourier Transform (DFT)

Compute spectrum of discrete-time periodic signals

N samples in time domain $\underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}}$ N complex numbers in frequency domain

DFT $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n}$$ analysis

IDFT $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi k/N)n}$$ synthesis



DFT: sample continuous H(ω) (DTFT) at N evenly spaced frequencies

$x[n]$ periodic with period N samples

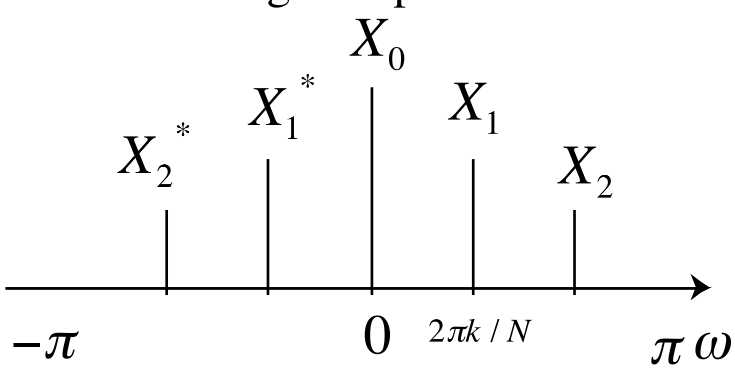
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j(2\pi k/N)n}$$

composed of N frequencies harmonically related

DFT $$X[k] = \sum_{n=0}^{N-1} \underbrace{x[n]e^{-j(2\pi k/N)n}}$$

move X_k to DC

original spectrum



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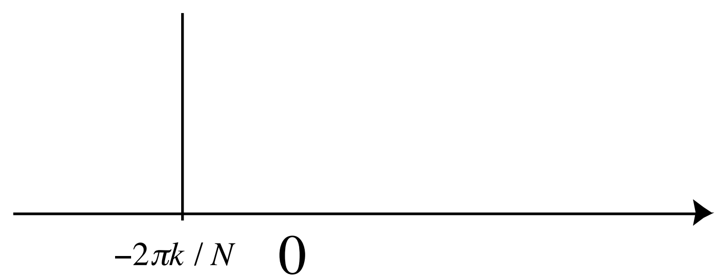

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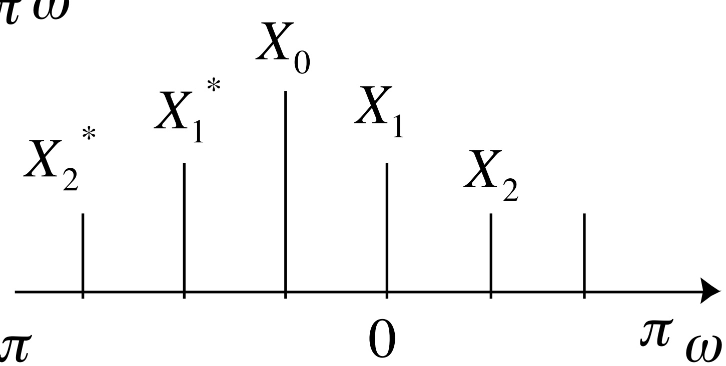


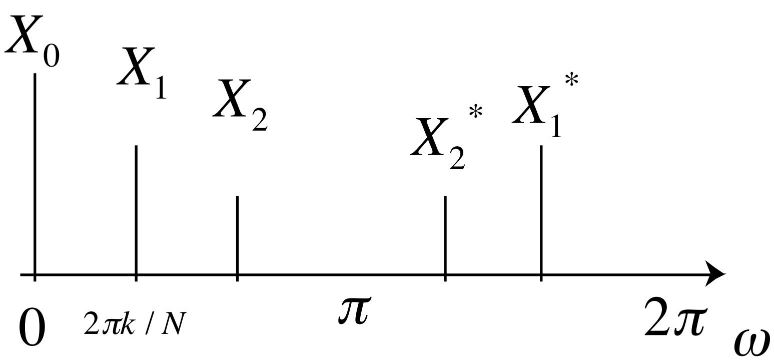
shifted spectrum

$x[n]$ periodic

DFT $$X[k]=\sum_{n=0}^{N-1}\underbrace{x[n]e^{-j(2\pi k/N)n}}$$

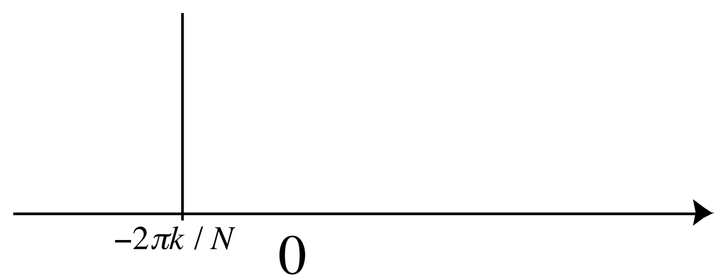
move X_k to DC

original (aliased) spectrum



ω>π aliases of negative frequency components

*



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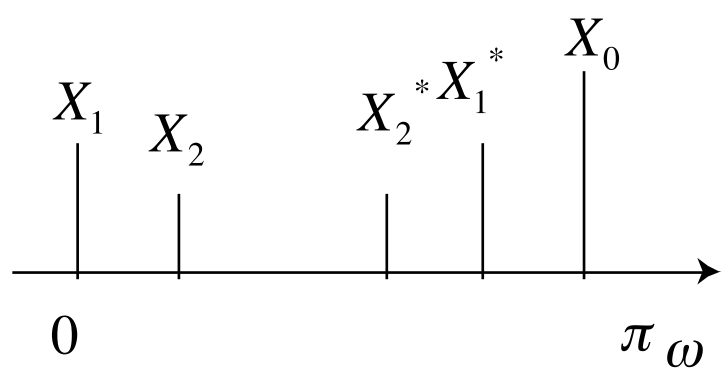


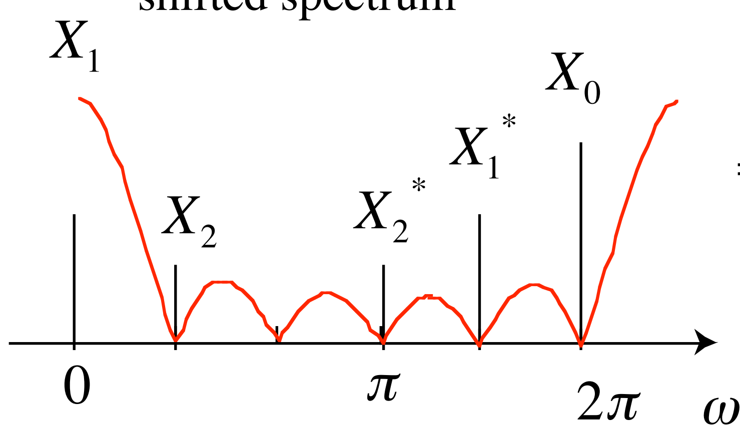
shifted spectrum

$x[n]$ periodic

DFT $\quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n}$

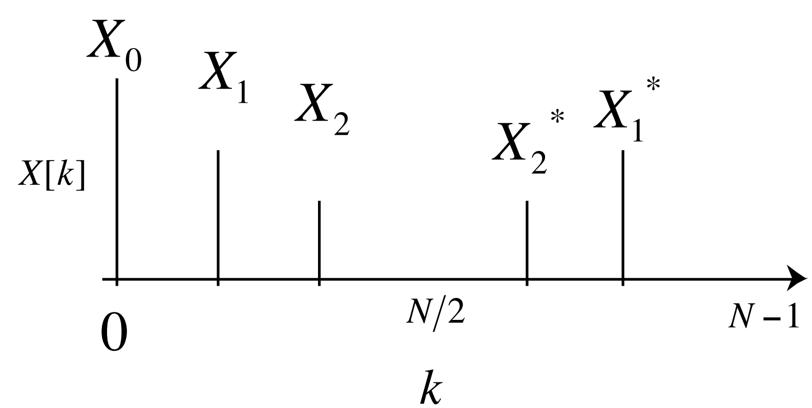
FIR low pass filter to measure DC
(N point running sum, {1,1,1…1})
zeros @ harmonics

shifted spectrum



$\Rightarrow X_1$

discrete spectrum



FIR filter w/ zeros
at harmonics

$x[n]=\{1,\quad 1,\quad 1,\quad 0\}$ Impulse response of 3pt summer

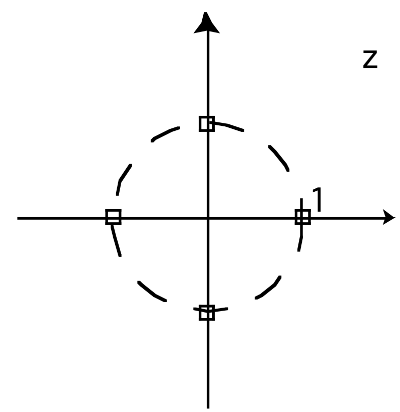
$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j(2\pi k/N)n}=\sum_{n=0}^{3}x[n]e^{-j(2\pi k/4)n}$$

$$=x[0]+x[1]e^{-j(2\pi k/4)}+x[2]e^{-j(2\pi k/4)2}+x[3]e^{-j(2\pi k/4)3}$$

$$=1+e^{-j\frac{\pi}{2}k}+e^{-j\pi k}$$

$$=1+e^{-j\frac{3\pi}{4}k}\left(e^{j\frac{\pi}{4}k}+e^{-j\frac{\pi}{4}k}\right)$$

$$=1+2\left[-\sqrt{2}(1+j)\right]^{k}\cos\left(\tfrac{\pi}{4}k\right)$$



k=0, 1, 2, 3

$$X[k]=\{3,\quad -i,\quad 1,\quad i\}$$

$DC \quad e^{-j(\pi/2)} \quad e^{-j(\pi)} \quad e^{-j\left(\frac{3\pi}{2}\right)}$

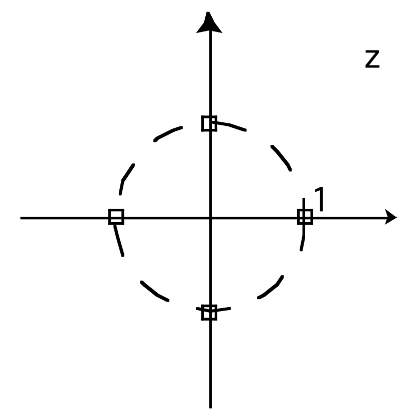
check:
»x=[1 1 1 0];
»X=fft(x)
X =
3.0000 0 - 1.0000i 1.0000 0 + 1.0000i
»fftshift(X)
ans =
1.0000 0 + 1.0000i 3.0000 0 - 1.0000i

Note: 0>ω>2π

only extract limited number of frequencies due to N samples per period

$$x[n] = \{1, \quad 1, \quad 1, \quad 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{3} x[n] e^{-j(2\pi k/4)n}$$



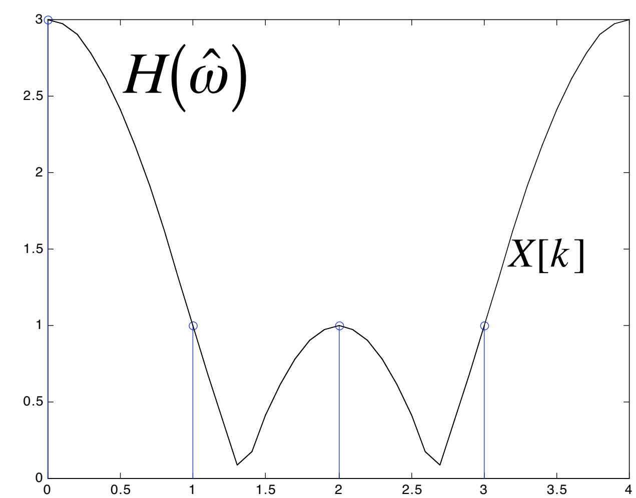
$$= 1 + 2\left[-\sqrt{2}(1+j)\right]^k \cos\left(\tfrac{\pi}{4} k\right)$$

k=0, 1, 2, 3

$$X[k] = \{3, \quad -i, \quad 1, \quad i\}$$

$$DC \quad e^{-j(\pi/2)} \quad e^{-j(\pi)} \quad e^{-j\left(\frac{3\pi}{2}\right)}$$

$$DC \quad e^{-j(\pi/2)} \quad e^{-j\left(\frac{-3\pi}{2}\right)} \quad e^{-j\left(-\frac{\pi}{2}\right)}$$



$$H(\hat{\omega}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$$

Note: 0>ω>2π

only extract limited number of frequencies due to N samples per period

Redo like homework

works okay if you have
x[n]'s =1 or complex exponentials

$$x[n] = \{1, \quad 1, \quad 1, \quad 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{3} x[n] e^{-j(2\pi k/4)n} = \sum_{n=0}^{2} e^{-j(2\pi k/4)n}$$

$$= \frac{1 - e^{-j(2\pi k/4)3}}{1 - e^{-j(2\pi k/4)}} = \frac{1 - e^{-j3\pi k/2}}{1 - e^{-j\pi k/2}}$$

$$= \frac{1 - j^k}{1 - (-j)^k} \qquad k \neq 0$$

Remember

$$\sum_{k=0}^{N-1} a^k = \frac{1 - a^N}{1 - a}$$

$$X[k] = \sum_{n=0}^{2} e^{-j(2\pi 0/4)n} = \sum_{n=0}^{2} e^{-j0} = \sum_{n=0}^{2} 1 = 3$$

k=0

$$X[k] = \{3, -i, 1, i\}$$

k={0,1,2,3}

$$X[1] = \frac{1-j}{1+j}\frac{1-j}{1-j} = \frac{1-2j-1}{2} = -j$$

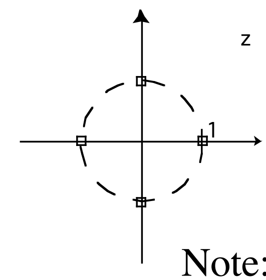
$$X[2] = \frac{1-j^2}{1-(-j)^2} = \frac{1-(-1)}{1-(-1)} = 1$$

$$X[3] = \frac{1-j^3}{1-(-j)^3} = \frac{1-(-j)}{1-j} = \frac{1+j}{1-j}\frac{1+j}{1+j} = \frac{1+2j-1}{1+2} = j$$

Padding

$$x[n] = \{1, \quad 1, \quad 1, \quad 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{3} x[n] e^{-j(2\pi k/4)n} = \sum_{n=0}^{2} x[n] e^{-j(2\pi k/4)n}$$



Note: 0>ω>2π

k=0, 1, 2, 3

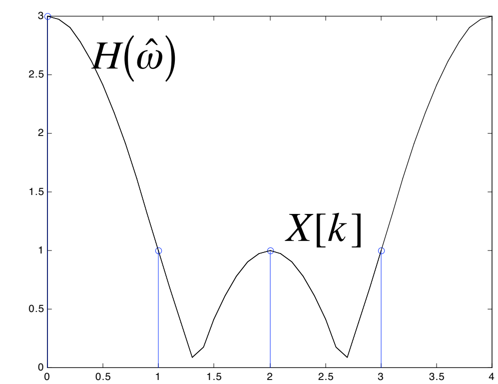
$$X[k] = \{3, \quad -i, \quad 1, \quad i\}$$

$DC \quad \pi/2 \quad \pi \quad \frac{3\pi}{2}$

$DC \quad \pi/2 \quad \pi \quad -\frac{\pi}{2}$

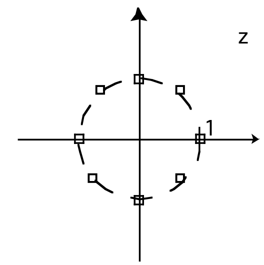
X[k] is sampled version of $H(\hat{\omega})$

N time samples = N frequency samples



$$H(\hat{\omega}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$$

Padding



Pad x[n] to get more samples of $H(\hat{\omega})$

$$x[n] = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$$

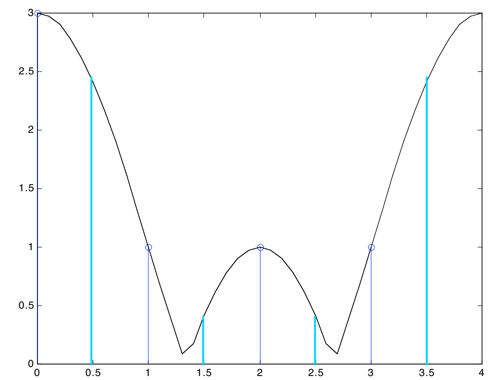
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{7} x[n] e^{-j(2\pi k/8)n} = \sum_{n=0}^{2} x[n] e^{-j(2\pi k/8)n}$$

$$= x[0] + x[1] e^{-j(2\pi k/8)} + x[2] e^{-j(2\pi 2k/8)}$$

k=0, 1, 2, 3, 4, 5, 6, 7

$$X[k] = \{3 \quad 1.7 - j1.7 \quad -i \quad .29 + j2.9 \quad 1 \quad 0.29 - j0.29 \quad i \quad 1.7 + j1.7\}$$

DC $\pi/4$ $\pi/2$ $3\pi/4$ π $-3\pi/4$ $-\frac{\pi}{2}$ $-\pi/4$



$$H(\hat{\omega}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$$

DFT Convolution

$$y[n] * x[n] \overset{\text{DTFT}}{\Leftrightarrow} Z(\hat{\omega}) = Y(\hat{\omega})X(\hat{\omega}) \overset{\text{IDTFT}}{\Leftrightarrow} z[n]$$

sample domain | frequency domain | sample domain

$Y[k]$ sampled version of $Y(\hat{\omega})$

Use DFT to compute Y[k] and X[k]

$$y[n] \otimes x[n] \overset{\text{DFT}}{\Leftrightarrow} Z[k] = Y[k]X[k] \overset{\text{IDFT}}{\Leftrightarrow} z[n]$$

circular convolution

DFT Convolution

Problem:

$z[n]=y[n]*x[n]$

↓ DFT

$Z[k]=Y[k]X[k]$

↓ IDFT

$z'[n]$

$x[n]$: length N
$y[n]$: length N
$z[n]$: length N+N-1

$X[k]$: length N
$Y[k]$: length N
$Z[k]$: length N

$z'[n]$:length N

undersampled frequency
↓
aliased time samples

DFT Convolution

ex.

x[n]=[1 -1 1], y[n]=[1 2 3]

z'[n]=x[n]*y[n]= [1 1 2 -1 3]

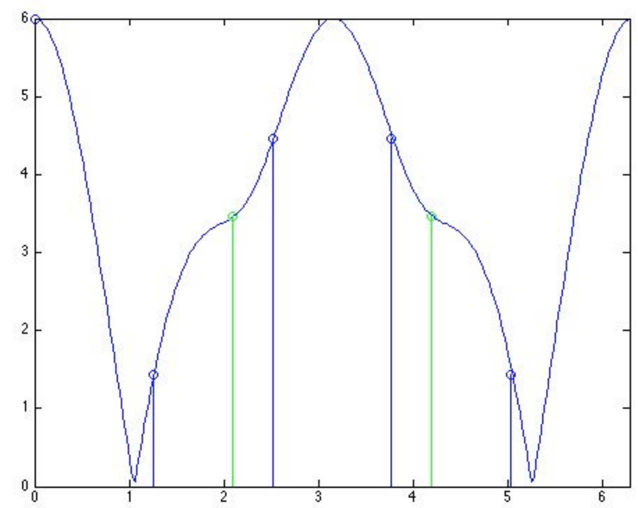
x[n]	1	-1	1		
y[n]	1	2	3		
	1	2	3		
		-1	-2	-3	
			2	3	
z'[n]	1	1	2	-1	3

3pt DFT

$$X[k] = [1 \quad 1 + j1.7321 \quad 1 - j1.7321]$$

$$Y[k] = \left[6 \quad -1.5 + j0.866 \quad -1.5 - j0.866\right]$$

$$Z[k] = X[k]Y[k] = [6 \quad \text{-3 - j1.73} \quad \text{-3 + j1.73}]$$



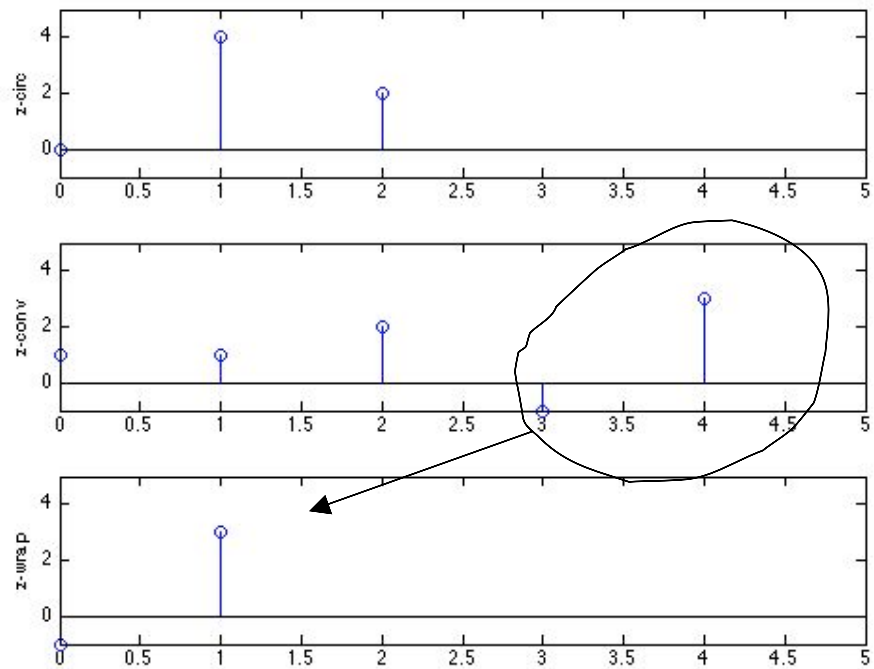
z'[n]=[1 1 2 -1 3]

$$Z'[k] = [6 \quad 1.43 + \text{j}0.139 \quad \text{-}1.93 + \text{j}4.03 \quad \text{-}1.93 \text{ - j}4.03 \quad 1.43 \text{ - j}0.139]$$

$$Z[k] = X[k]Y[k] = [6 \quad -3 - j1.73 \quad -3 + j1.73]$$

↓ 3pt IDFT

$$z[k] = x[n] \otimes y[n] = [0 \quad 4 \quad 2]$$



temporal aliasing
samples wrap

$$Z'[k] = [6 \quad 1.43 + j0.139 \quad -1.93 + j4.03 \quad -1.93 - j4.03 \quad 1.43 - j0.139]$$

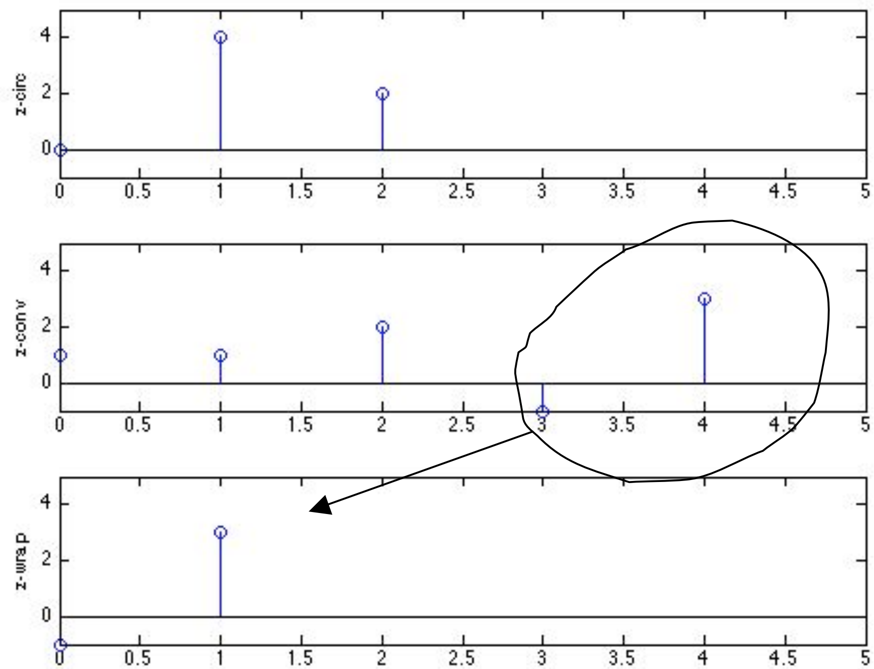
↓ 5pt IDFT

z'[n]=x[n]*y[n]=[1 1 2 -1 3]

$$Z[k] = X[k]Y[k] = [6 \quad -3 - j1.73 \quad -3 + j1.73]$$

↓ 3pt IDFT

$$z[k] = x[n] \otimes y[n] = [0 \quad 4 \quad 2]$$



temporal aliasing
samples wrap

$$Z'[k] = [6 \quad 1.43 + j0.139 \quad -1.93 + j4.03 \quad -1.93 - j4.03 \quad 1.43 - j0.139]$$

↓ 5pt IDFT

z'[n]=x[n]*y[n]=[1 1 2 -1 3]

DFT Convolution

ex.

x[n]=[1 -1 1 0 0], y[n]=[1 2 3 0 0]

To avoid temporal aliasing, pad signals so lengths are 2N-1

$$X[k] = [1 \;\; -0.118 + j0.363 \;\; 2.12 + j1.54 \;\; 2.12 - j1.54 \;\; -0.118 - j0.363]$$

$$Y[k] = \left[6 \;\; -0.809 - j3.67 \;\; 0.309 + j1.68i \;\; 0.309 - j1.68 \;\; -0.809 + j3.67\right]$$

$$Z[k] = [6 \;\; 1.43 + j0.139 \;\; -1.93 + j4.03 \;\; -1.93 - j4.03 \;\; 1.43 - j0.139]$$

$$z[n] = x[n] \otimes y[n] = [1 \quad 1 \quad 2 \quad -1 \quad 3]$$

If len(x)=N, len(y)=M, then pad so lengths are N+M-1

How does a Fast Fourier Transform (FFT) work?

2 Point DFT

DFT $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n}$ N=2

$= x[0]e^{-j(2\pi k/4)0} + x[1]e^{-j(2\pi k/2)1}$

$= x[0] + x[1]e^{-j(\pi k)}$

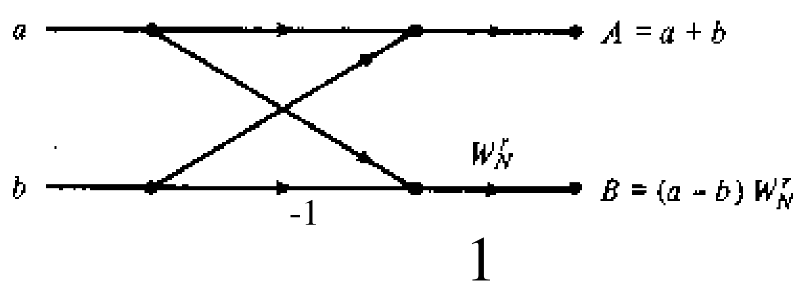
FFT is an efficient way of calculating a DFT.

$X_2[0] = x[0] + x[1]$ 1 1

$X_2[1] = x[0] - x[1]$ 1 -1

coefficients block

FFT butterfly



$W_2^k = e^{-j(2\pi k/2)}$

$W_2^0 = 1$

N^2=4 mult

N^2-N=2 adds

4 Point DFT

DFT $$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j(2\pi k/N)n}$$ N=4

$$=x[0]e^{-j(2\pi k/4)0}+x[1]e^{-j(2\pi k/4)1}+x[2]e^{-j(2\pi k/4)2}+x[3]e^{-j(2\pi k/4)3}$$

$$=x[0]+x[1]e^{-j\left(\frac{\pi}{2}k\right)}+x[2]e^{-j(\pi k)}+x[3]e^{-j\left(\frac{3\pi}{2}k\right)}$$

$$X[0]=x[0]+x[1]+x[2]+x[3]$$ running sum

$$X[1]=x[0]+x[1]e^{-j\left(\frac{\pi}{2}\right)}+x[2]e^{-j(\pi)}+x[3]e^{-j\left(\frac{3\pi}{2}\right)}$$
$$=x[0]-jx[1]-x[2]+jx[3]$$

$$X[2]=x[0]+x[1]e^{-j(\pi)}+x[2]e^{-j(2\pi)}+x[3]e^{-j(3\pi)}$$
$$=x[0]-x[1]+x[2]-x[3]$$

$$X[3]=x[0]+x[1]e^{-j\left(\frac{3\pi}{2}\right)}+x[2]e^{-j(3\pi)}+x[3]e^{-j\left(\frac{9\pi}{2}\right)}$$
$$=x[0]+jx[1]-x[2]-jx[3]$$

$$X[4]=x[0]+x[1]e^{-j(2\pi)}+x[2]e^{-j(4\pi)}+x[3]e^{-j(6\pi)}$$
$$=x[0]+x[1]+x[2]+x[3]$$ alias of X[0]

N^2=16 mult
N^2-N=12 adds

4 Point DFT

DFT $$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j(2\pi k/N)n} \qquad N=4$$

$$= x[0]+x[1]e^{-j\left(\frac{\pi}{2}k\right)}+x[2]e^{-j(\pi k)}+x[3]e^{-j\left(\frac{3\pi}{2}k\right)}$$

$$X[0]=x[0]+x[1]+x[2]+x[3]$$ running sum

$$X[1]=x[0]-jx[1]-x[2]+jx[3]$$

$$X[2]=x[0]-x[1]+x[2]-x[3]$$

$$X[3]=x[0]+jx[1]-x[2]-jx[3]$$

$$X[4]=x[0]+x[1]+x[2]+x[3]$$ alias of X[0]

$$X[5]=x[0]-jx[1]-x[2]+jx[3]$$ alias of X[1]

periodic in frequency

only extract limited number of frequencies due to N samples per period

DFT

N=4

$$X_4[0] = x[0] + x[1] + x[2] + x[3]$$
$$X_4[1] = x[0] - jx[1] - x[2] + jx[3]$$
$$X_4[2] = x[0] - x[1] + x[2] - x[3]$$
$$X_4[3] = x[0] + jx[1] - x[2] - jx[3]$$

1 1 1 1

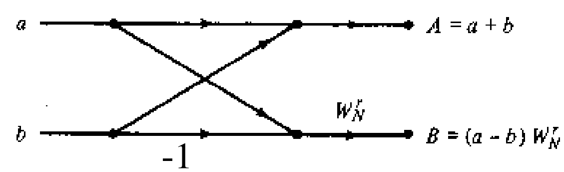
1 -j -1 j

1 -1 1 -1

1 j -1 -j

N=2

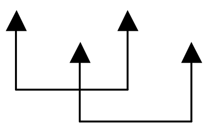
$$X_2[0] = x[0] + x[1] \quad 1 \;\; 1$$
$$X_2[1] = x[0] - x[1] \quad 1 \;\; -1$$



even: X[0],X[2]

1 1 1 1

1 -1 1 -1



odd: X[1],X[3]

1 -j -1 j

1 j -1 -j

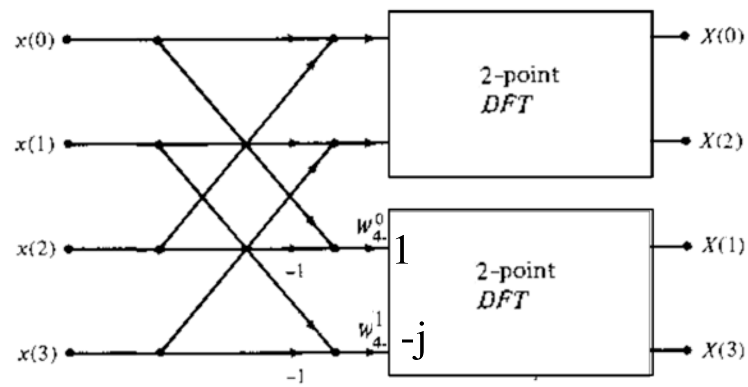
$$X_4[0] = \left[x[0] + x[2]\right] + \left[x[1] + x[3]\right] = X_2[0]_{even}$$
$$X_4[2] = \left[x[0] + x[2]\right] - \left[x[1] + x[3]\right] = X_2[1]_{even}$$

$$W_4^k = e^{-j(2\pi k/4)}$$

Looks like a 2pt FFT of combined signals

Looks like a 2pt FFT of combined signals

recursive



$$X_4[1] = \left[x[0] - x[2]\right] + {}^{-}j\left[x[1] - x[3]\right] = X_2[0]_{odd}$$
$$X_4[3] = \left[x[0] - x[2]\right] - {}^{-}j\left[x[1] - x[3]\right] = X_2[1]_{odd}$$

FFT

fewer multiplies/adds

$N/2\log_2 N = 4$ mult

DFT

N=4

$$X_4[0] = x[0] + x[1] + x[2] + x[3]$$
$$X[1] = x[0] - jx[1] - x[2] + jx[3]$$
$$X[2] = x[0] - x[1] + x[2] - x[3]$$
$$X[3] = x[0] + jx[1] - x[2] - jx[3]$$

1 1 1 1

1 -j -1 j

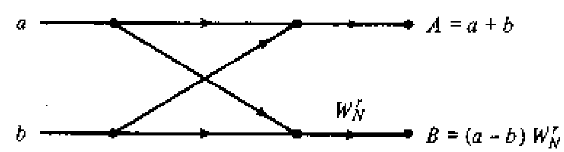
1 -1 1 -1

1 j -1 -j

N=2

$$X_2[0] = x[0] + x[1]$$ 1 1

$$X_2[1] = x[0] - x[1]$$ 1 -1



even: X[0],X[2]

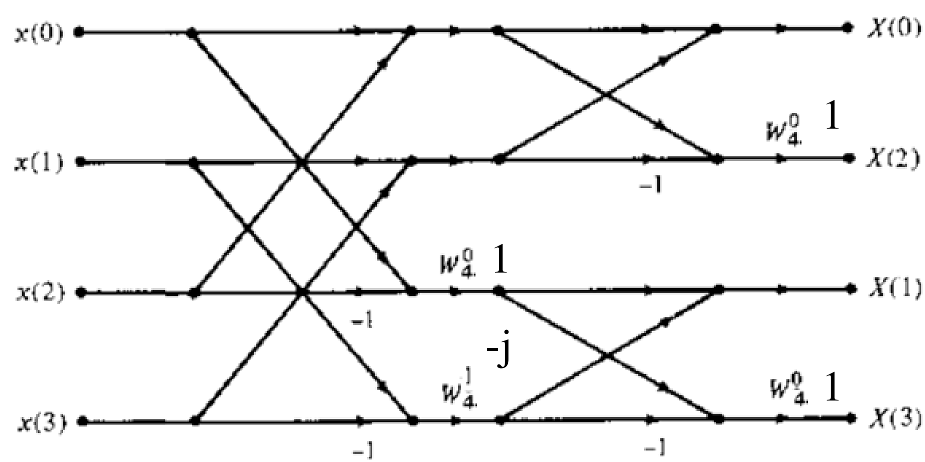
1 1 | 1 1

1 -1 | 1 -1

odd: X[1],X[3]

1 -j | -1 j

1 j | -1 -j



4pt FFT butterfly

$N/2\log_2 N$=4 mult

$N\log_2 N$=8 add

DFT $$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j(2\pi k/N)n}$$ N=8

$$= x[0]e^{-j(2\pi k/4)0} + x[1]e^{-j(2\pi k/8)1} + x[2]e^{-j(2\pi k/8)2} + x[3]e^{-j(2\pi k/8)3}$$
$$x[4]e^{-j(2\pi k/8)4} + x[5]e^{-j(2\pi k/8)5} + x[6]e^{-j(2\pi k/8)6} + x[7]e^{-j(2\pi k/8)7}$$

$$= x[0] + x[1]e^{-j\left(\frac{\pi}{4}k\right)} + x[2]e^{-j\left(\frac{\pi}{2}k\right)} + x[3]e^{-j\left(\frac{3\pi}{4}k\right)}$$
$$x[4]e^{-j(\pi k)} + x[5]e^{-j\left(\frac{5\pi}{4}k\right)} + x[6]e^{-j\left(\frac{6\pi}{4}k\right)} + x[7]e^{-j\left(\frac{7\pi}{4}k\right)}$$

$$X[0] = x[0]+x[1]+x[2]+x[3]+x[4]+x[5]+x[6]+x[7]$$ running sum

$$X[1] = x[0]+\tfrac{\sqrt{2}}{2}(1-j)x[1]-jx[2]-\tfrac{\sqrt{2}}{2}(1+j)x[3]$$
$$-x[4]+\tfrac{\sqrt{2}}{2}(-1+j)x[5]+jx[6]+\tfrac{\sqrt{2}}{2}(1+j)x[7]$$

$$X[2] = x[0]-jx[1]-x[2]+jx[3]+x[4]-jx[5]-x[6]+jx[7]$$

$$X[3] = x[0]+\tfrac{\sqrt{2}}{2}(-1-j)x[1]+jx[2]+\tfrac{\sqrt{2}}{2}(1-j)x[3]$$
$$-x[4]+\tfrac{\sqrt{2}}{2}(1+j)x[5]-jx[6]+\tfrac{\sqrt{2}}{2}(-1+j)x[7]$$

$$\vdots$$

N^2=64mult

N^2-N=56 add

DFT $$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j(2\pi k/N)n} \qquad N=8$$

$$= x[0]+x[1]e^{-j\left(\frac{\pi}{4}k\right)}+x[2]e^{-j\left(\frac{\pi}{2}k\right)}+x[3]e^{-j\left(\frac{3\pi}{4}k\right)}$$

$$x[4]e^{-j(\pi k)}+x[5]e^{-j\left(\frac{5\pi}{4}k\right)}+x[6]e^{-j\left(\frac{6\pi}{4}k\right)}+x[7]e^{-j\left(\frac{7\pi}{4}k\right)}$$

	$x[n]$							
$X[k]$	1	1	1	1	1	1	1	1
	1	$\frac{\sqrt{2}}{2}(1-j)$	-j	$\frac{\sqrt{2}}{2}(-1-j)$	-1	$-\frac{\sqrt{2}}{2}(1-j)$	j	$\frac{\sqrt{2}}{2}(1+j)$
	1	-j	-1	j	1	-j	-1	j
	1	$\frac{\sqrt{2}}{2}(-1-j)$	j	$\frac{\sqrt{2}}{2}(1-j)$	-1	$-\frac{\sqrt{2}}{2}(-1-j)$	-j	$\frac{\sqrt{2}}{2}(-1+j)$
	1	-1	1	-1	1	-1	1	-1
	1	$\frac{\sqrt{2}}{2}(-1+j)$	-j	$\frac{\sqrt{2}}{2}(1+j)$	-1	$-\frac{\sqrt{2}}{2}(-1+j)$	j	$\frac{\sqrt{2}}{2}(-1-j)$
	1	j	-1	-j	1	j	-1	-j
	1	$\frac{\sqrt{2}}{2}(1+j)$	j	$\frac{\sqrt{2}}{2}(-1+j)$	-1	$-\frac{\sqrt{2}}{2}(1+j)$	-j	$\frac{\sqrt{2}}{2}(1-j)$

N^2=64mult
N^2-N=56 add

DFT

N=8

even: X[0],X[2],X[4],X[6]

$x[n]$ (columns), $X[k]$ (rows)

1	1	1	1	1	1	1	1
1	-j	-1	j	1	-j	-1	j
1	-1	1	-1	1	-1	1	-1
1	j	-1	-j	1	j	-1	-j

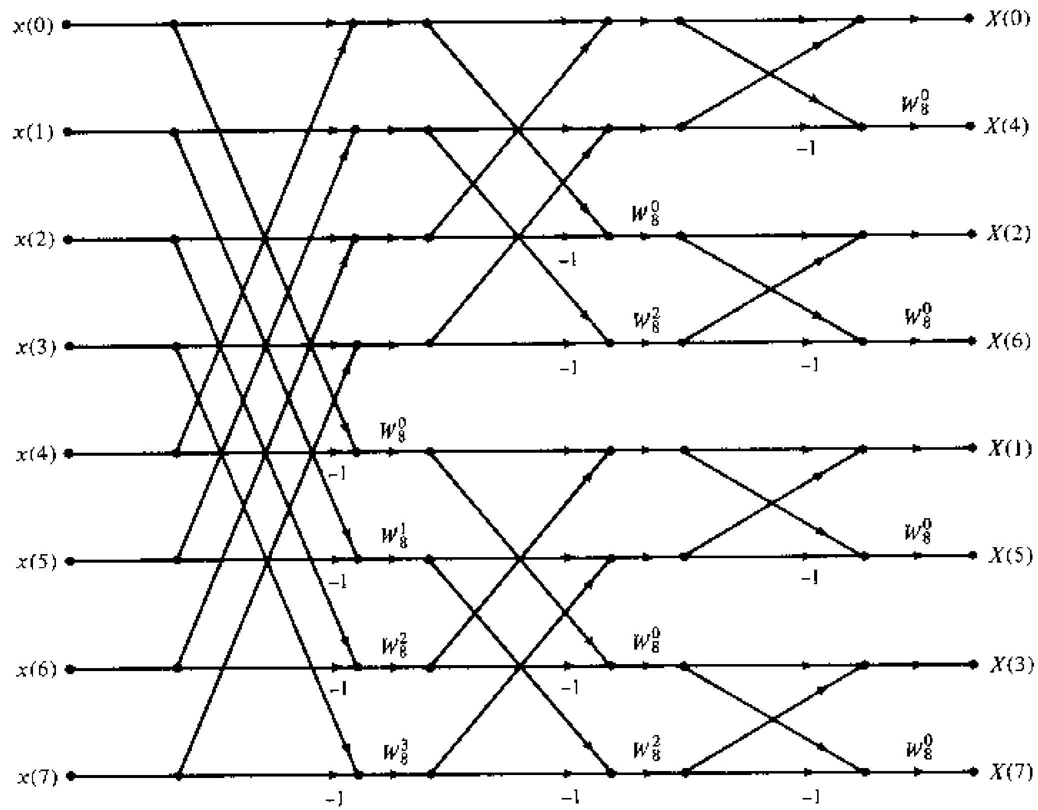
N=4

1	1	1	1
1	-j	-1	j
1	-1	1	-1
1	j	-1	-j

odd: X[1],X[3],X[5],X[7]

1	$\frac{\sqrt{2}}{2}(1-j)$	-j	$\frac{\sqrt{2}}{2}(-1-j)$	-1	$-\frac{\sqrt{2}}{2}(1-j)$	j	$\frac{\sqrt{2}}{2}(1+j)$
1	$\frac{\sqrt{2}}{2}(-1-j)$	j	$\frac{\sqrt{2}}{2}(1-j)$	-1	$-\frac{\sqrt{2}}{2}(-1-j)$	-j	$\frac{\sqrt{2}}{2}(-1+j)$
1	$\frac{\sqrt{2}}{2}(-1+j)$	-j	$\frac{\sqrt{2}}{2}(1+j)$	-1	$-\frac{\sqrt{2}}{2}(-1+j)$	j	$\frac{\sqrt{2}}{2}(-1-j)$
1	$\frac{\sqrt{2}}{2}(1+j)$	j	$\frac{\sqrt{2}}{2}(-1+j)$	-1	$-\frac{\sqrt{2}}{2}(1+j)$	-j	$\frac{\sqrt{2}}{2}(1-j)$

8pt FFT



$$W_8^{\,k} = e^{-j(2\pi k/8)}$$

$N/2\log_2 N$=12 mult
$N\log_2 N$=24 adds

$$y[n] = x[n] - y[n-2]$$

$\Downarrow$ z-transform

The signal has same z-transform as system. The signal is the impulse response of the system

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1+z^{-2}\right)} = \frac{z^2}{z^2+1}$$ system function

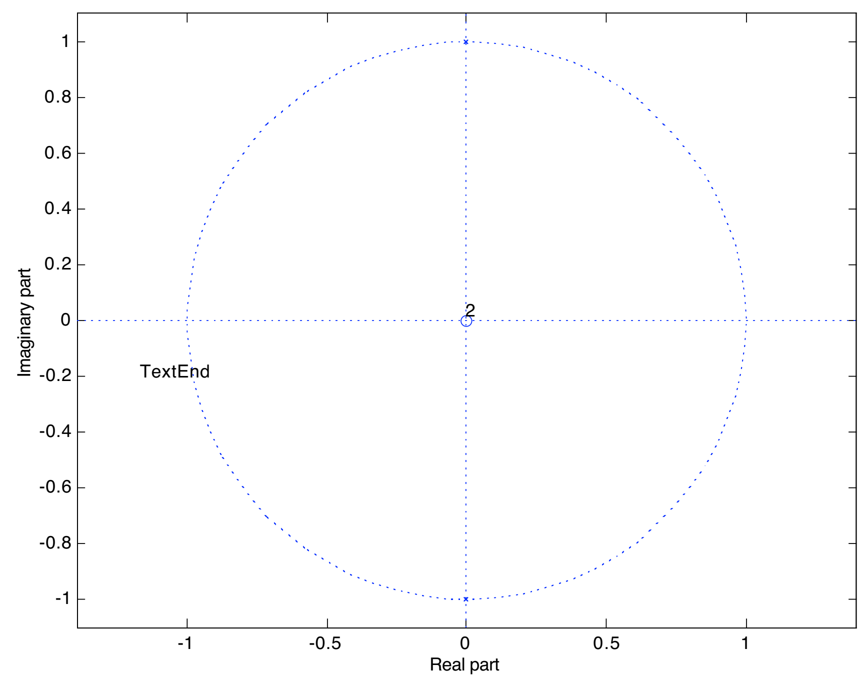
$$zeros = roots(z^2) = 0,0$$
$$poles = roots(z^2+1) = \pm j$$

Poles: values of z for input $x[n] = z^n$ where output $y[n] = H(z)z^n \to \infty$
Zeros: values of z for input $x[n] = z^n$ where output $y[n] = H(z)z^n \to 0$

$$y[n] = h[n] = \cos\left(\frac{2\pi}{4}n\right)u[n]$$ signal

$\Downarrow$ z-transform

$$Y(z) = \frac{1}{\left(1+z^{-2}\right)} = \frac{z^2}{z^2+1}$$



Poles: z locations of $y[n] = z^n$

Zeros: related to the magnitude and phase of $y[n] = z^n$

The closer a zero is to a pole, the smaller the effect the pole.

DFT Convolution

ex.

x[n]=[1 -1 1], y[n]=[1 2 3]

z[n]=x[n]*y[n]= [1 1 2 -1 3]

x[n]	1	-1	1		
y[n]	1	2	3		
	1	2	3		
		-1	-2	-3	
			1	2	3
z[n]	1	1	2	-1	3

3pt DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{2} x[n] e^{-j(2\pi k/3)n}$$

1

$$= x[0]e^{-j(2\pi 0/3)k} + x[1]e^{-j(2\pi 1/3)k} + x[2]e^{-j(2\pi 2/3)k}$$

$$= 1 - e^{-j(2\pi/3)k} + e^{-j(4\pi/3)k}$$

$$X[0] = 1 - e^{-j(2\pi/3)0} + e^{-j(4\pi/3)0} = 1 - 1 + 1 = 1$$

$$X[1] = 1 - e^{-j(2\pi/3)1} + e^{-j(4\pi/3)1} = 1 + j1.7321$$

$$X[2] = 1 - e^{-j(2\pi/3)2} + e^{-j(4\pi/3)2} = 1 - 1.7321i$$

$$X[k] = [1 \quad 1 + j1.7321 \quad 1 - j1.7321]$$

3pt IDFT $Z[k] = X[k]Y[k] = [6 \quad -3 - j1.73 \quad -3 + j1.73]$

$$z[n] = \frac{1}{N}\sum_{k=0}^{N-1} Z[k]e^{-j(2\pi k/N)n} = \frac{1}{3}\sum_{k=0}^{2} Z[k]e^{-j(2\pi k/3)n}$$

$$= \left(Z[0]e^{-j(2\pi 0/3)k} + Z[1]e^{-j(2\pi 1/3)k} + Z[2]e^{-j(2\pi 2/3)k}\right)/3$$

$$= \left(6 + (-3 - j1.73)e^{-j(2\pi/3)k} + (-3 + j1.73)e^{-j(4\pi/3)k}\right)/3$$

$$Y[0] = 1 + 2e^{-j(2\pi/3)0} + 3e^{-j(4\pi/3)0} = 1 + 2 + 3 = 6$$

$$Y[1] = 1 + 2e^{-j(2\pi/3)1} + 3e^{-j(4\pi/3)1} = -1.5 + j0.866$$

$$Y[2] = 1 + 2e^{-j(2\pi/3)2} + 3e^{-j(4\pi/3)2} = -1.5 - j0.866$$

$$Y[k] = \left[6 \quad -1.5 + j0.866 \quad -1.5 - j0.866\right]$$

$$X[k] = [1 \quad 1 + j1.7321 \quad 1 - j1.7321]$$

3pt DFT y[n]=[1 2 3]

$$Y[k] = \sum_{n=0}^{N-1} Y[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{2} y[n] e^{-j(2\pi k/3)n}$$

$$= y[0]e^{-j(2\pi 0/3)k} + y[1]e^{-j(2\pi 1/3)k} + y[2]e^{-j(2\pi 2/3)k}$$

$$= 1 - 2e^{-j(2\pi/3)k} + 3e^{-j(4\pi/3)k}$$

$$Y[0] = 1 + 2e^{-j(2\pi/3)0} + 3e^{-j(4\pi/3)0} = 1 + 2 + 3 = 6$$

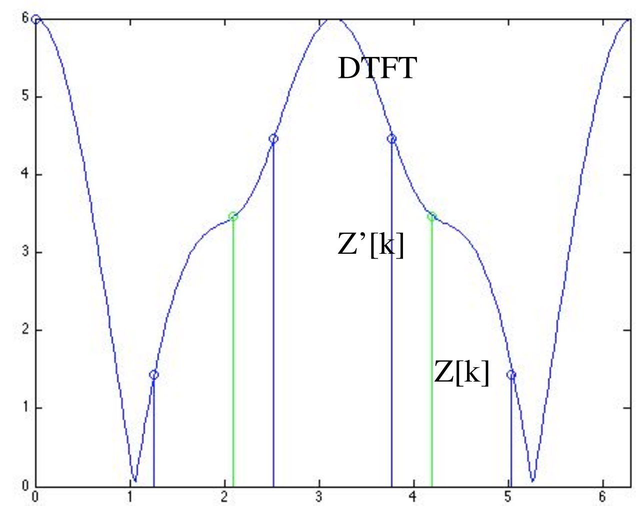
$$Y[1] = 1 + 2e^{-j(2\pi/3)1} + 3e^{-j(4\pi/3)1} = -1.5 + j0.866$$

$$Y[2] = 1 + 2e^{-j(2\pi/3)2} + 3e^{-j(4\pi/3)2} = -1.5 - j0.866$$

$$Y[k] = [6 \quad -1.5 + j0.866 \quad -1.5 - j0.866]$$

$$X[k] = [1 \quad 1 + j1.7321 \quad 1 - j1.7321]$$

$$Z[k] = X[k]Y[k] = [6 \quad -3 - j1.73 \quad -3 + j1.73]$$



z'[n]=[1 1 2 -1 3]

$$Z'[k] = [6 \quad 1.43 + j0.139 \quad -1.93 + j4.03 \quad -1.93 - j4.03 \quad 1.43 - j0.139]$$

Infinite signals

$$x[n] = -a^n u[-n-1] \Leftrightarrow \quad X(z) = -\sum_{k=-\infty}^{-1} a^k z^{-k} = -\sum_{k=1}^{\infty} a^{-k} z^k$$

$x[n] = 0 \quad n \geq 0$

left sided

$$= -\sum_{k=1}^{\infty} \left(\tfrac{1}{a} z\right)^k = 1 - \sum_{k=0}^{\infty} \left(\tfrac{1}{a} z\right)^k$$

$$= 1 - \frac{1}{1 - \left(\tfrac{1}{a} z\right)} = \frac{1 - \left(\tfrac{1}{a} z\right) - 1}{1 - \left(\tfrac{1}{a} z\right)}$$

$$= \frac{-\left(\tfrac{1}{a} z\right)}{1 - \left(\tfrac{1}{a} z\right)} = \frac{-1}{\left(\tfrac{1}{a} z\right)^{-1} - 1} = \frac{1}{1 - az^{-1}}$$

$$X(z) = \frac{1}{1 - az^{-1}}$$

$\left|\tfrac{1}{a} z\right| < 1$ region of convergence

or $|z| < |a|$