MAS.160 / MAS.510 / MAS.511 Signals, Systems and Information for Media Technology
Fall 2007

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

FIR

$$y[n] = \sum_{k=0}^{M} h[k] x[n-k]$$

convolution

$$x[n] = Ae^{j\phi} e^{j\hat{\omega} n}$$

Complex exponential input

$$\downarrow$$

$$y[n] = \sum_{k=0}^{M} h[k] Ae^{j\phi} e^{j\hat{\omega}(n-k)}$$

$$= \left(\sum_{k=0}^{M} h[k] e^{-j\hat{\omega} k} \right) Ae^{j\phi} e^{j\hat{\omega} n}$$

$$= \mathcal{H}(\hat{\omega}) Ae^{j\phi} e^{j\hat{\omega} n}$$

let

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} h[k] e^{j\hat{\omega} k}$$

$\mathcal{H}(\hat{\omega})$ frequency response

Ex. $y[n]=\frac{1}{3}x[n]+\frac{1}{3}x[n-1]+\frac{1}{3}x[n-2]$ FIR

$$h[n]=\frac{1}{3}\delta[n]+\frac{1}{3}\delta[n-1]+\frac{1}{3}\delta[n-2]$$

$$\begin{aligned}
\mathcal{H}(\hat{\omega}) &= \sum_{k=0}^{2} h[k]e^{-j\hat{\omega}k} \\
&= h[0]e^{-j\hat{\omega}0}+h[1]e^{-j\hat{\omega}1}+h[2]e^{-j\hat{\omega}2} \\
&= \tfrac{1}{3}+\tfrac{1}{3}e^{-j\hat{\omega}}+\tfrac{1}{3}e^{-j\hat{\omega}2} \\
&= \tfrac{1}{3}\left(1+e^{-j\hat{\omega}}+e^{-j2\hat{\omega}}\right) \\
&= \tfrac{1}{3}e^{-j\hat{\omega}}\left(e^{j\hat{\omega}}+1+e^{-j\hat{\omega}}\right) \\
&= \tfrac{1}{3}e^{-j\hat{\omega}}\left(1+2\cos\hat{\omega}\right)
\end{aligned}$$

$$\mathcal{H}(\hat{\omega}) = \tfrac{1}{3} e^{-j\hat{\omega}} (1 + 2\cos\hat{\omega})$$
$$= \tfrac{1}{3} (1 + 2\cos\hat{\omega})(\cos\hat{\omega} - j\sin\hat{\omega})$$

$$\left|\mathcal{H}(\hat{\omega})\right| = \tfrac{1}{3}\left|(1 + 2\cos\hat{\omega})\right|$$

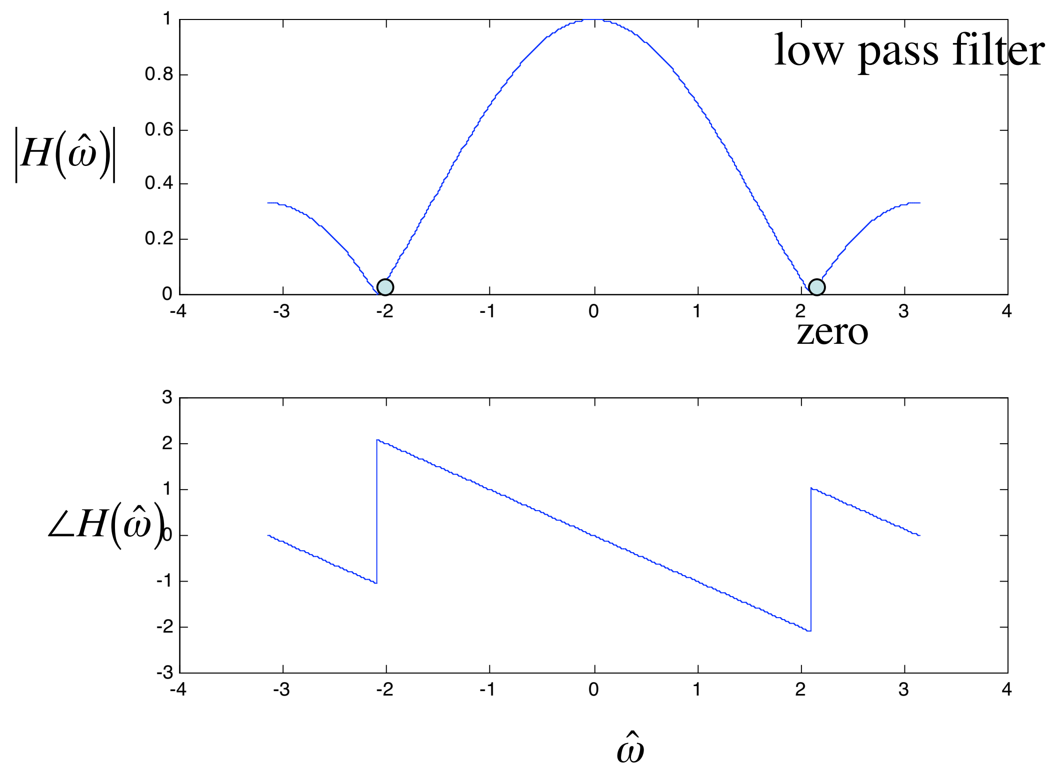
Note: $\left|\mathcal{H}\left(\tfrac{2\pi}{3}\right)\right| = 0$

$\angle\mathcal{H}(\hat{\omega}) = -\hat{\omega}$ linear phase

$$= \begin{cases} -\hat{\omega} & 2\pi/3 \le \hat{\omega} < 0 \\ -\hat{\omega} + \pi & \pi \le \hat{\omega} < 2\pi/3 \end{cases}$$

$$\angle\mathcal{H}(-\hat{\omega}) = -\angle\mathcal{H}(\hat{\omega})$$

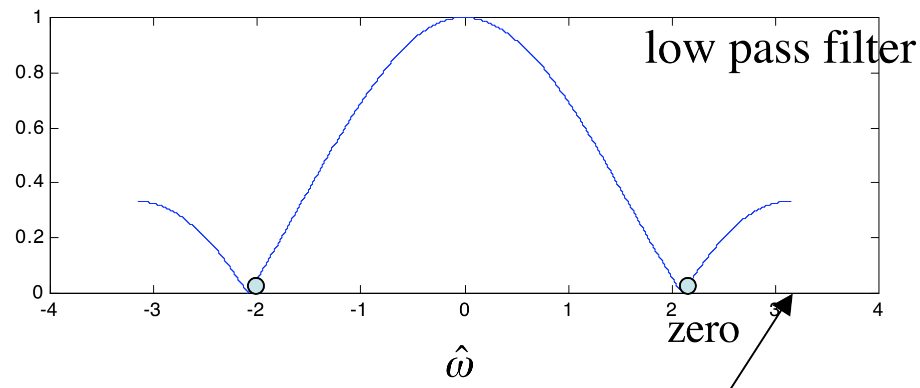
principal value of phase fn



Normalized vs. actual frequency

$$\left|\mathcal{H}(\hat{\omega})\right| = \tfrac{1}{3}\left|(1 + 2\cos\hat{\omega})\right|$$

Note: $\left|\mathcal{H}\left(\tfrac{2\pi}{3}\right)\right| = 0$



$\hat{\omega} = \tfrac{2\pi}{3}$ normalized frequency

$f = ?$ actual frequency (Hz)

$$-\pi < \hat{\omega} < \pi$$

$$f\big|_{\hat{\omega}=\pi} = \frac{\pi f_s}{2\pi} = \frac{f_s}{2} \quad \text{Nyquist}$$

$$\hat{f} = \frac{f}{f_s}$$

$$\hat{\omega} = 2\pi\hat{f} = 2\pi\frac{f}{f_s}$$

$$f = \frac{\hat{\omega} f_s}{2\pi}$$

$$\hat{\omega} = \tfrac{2\pi}{3},\ f_s = 8000\,Hz$$

$$f = \frac{\tfrac{2\pi}{3}8000}{2\pi} = \frac{8000}{3}Hz = 1667\,Hz$$

Convolution / solving the difference equation

$$x[n] = 3 + 3\cos(0.6\pi n)$$ input

$$y[n] = \tfrac{1}{3}x[n] + \tfrac{1}{3}x[n-1] + \tfrac{1}{3}x[n-2]$$ FIR filter

sample domain

$$y[n] = 1 + \cos(0.6\pi n) + 1 + \cos(0.6\pi(n-1)) + 1 + \cos(0.6\pi(n-2))$$

$$= 3 + \cos(0.6\pi n)$$

$$+ \cos(0.6\pi)\cos(0.6\pi n) + \sin(0.6\pi)\sin(0.6\pi n)$$

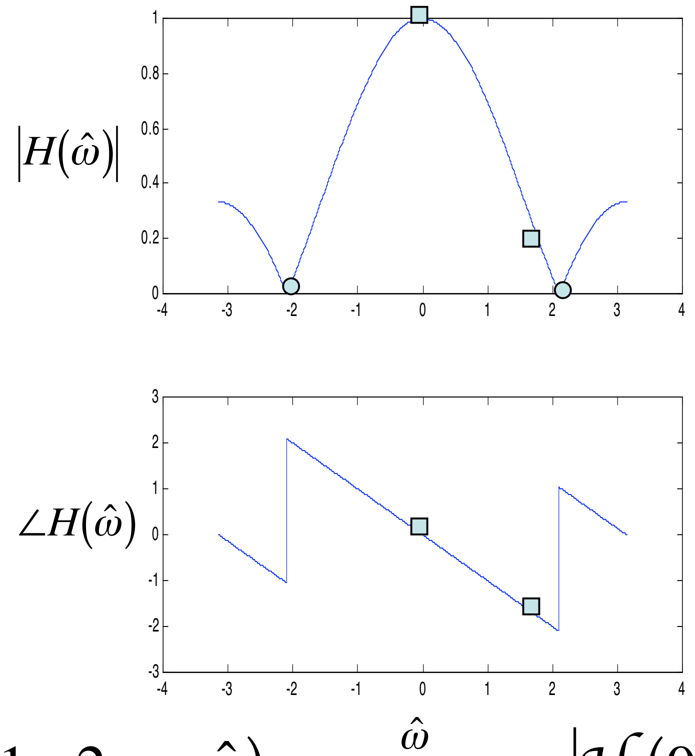
$$+ \cos(1.2\pi)\cos(0.6\pi n) + \sin(1.2\pi)\sin(0.6\pi n)$$

$$\vdots$$

$$= 3 + 0.382\cos(0.6\pi n - 0.6\pi)$$

Multiplication in frequency domain

$$x[n] = 3 + 3\cos(0.6\pi n)$$



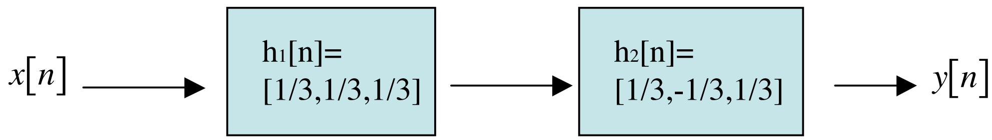
closer $\hat{\omega}$
to a zero,
the smaller the
output.

$$|\mathcal{H}(\hat{\omega})| = \tfrac{1}{3}(1 + 2\cos\hat{\omega}) \qquad |\mathcal{H}(0)| = 1 \quad |\mathcal{H}(0.6\pi)| = 0.127$$

$$\angle\mathcal{H}(\hat{\omega}) = -\hat{\omega} \qquad \angle\mathcal{H}(0) = 0 \quad \angle\mathcal{H}(0.6\pi) = -0.6\pi$$

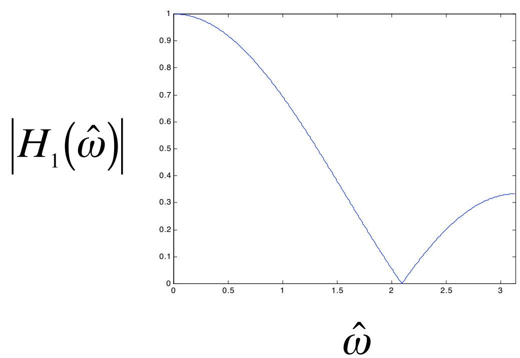
$$y[n] = 3(1) + 3(0.127)\cos(0.6\pi n - 0.6\pi)$$

$$= 3 + 0.382\cos(0.6\pi n - 0.6\pi)$$

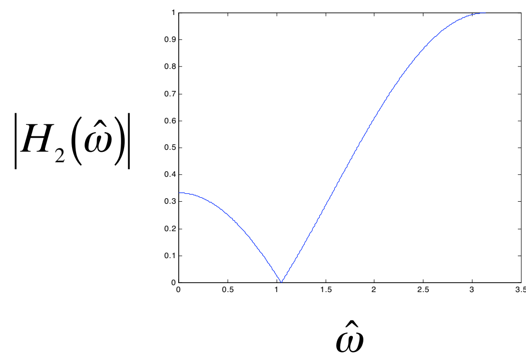


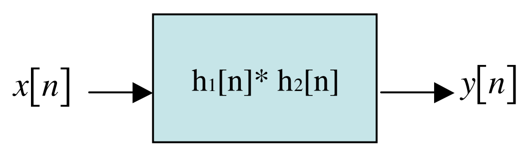
$h_1[n]=[1/3,1/3,1/3]$ $h_2[n]=[1/3,-1/3,1/3]$

lowpass

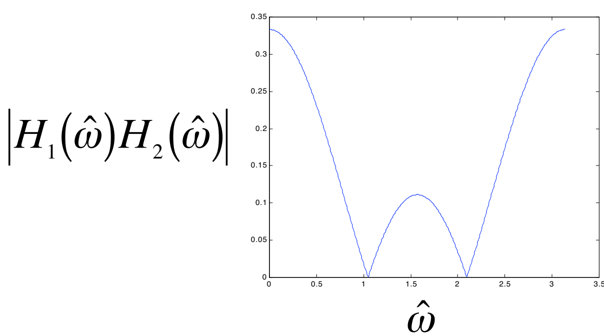


highpass





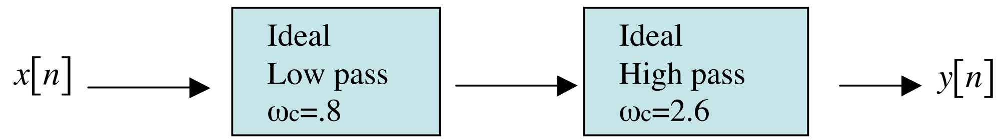
bandstop

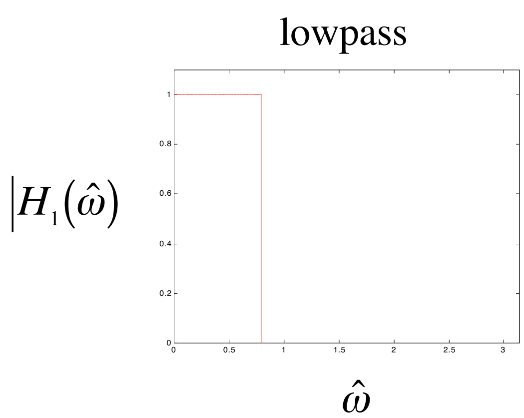


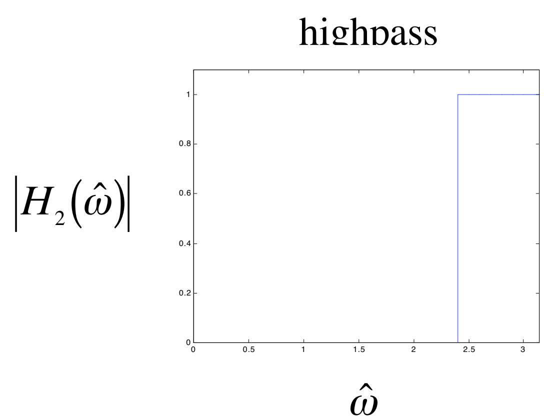
$h_1[n]*h_2[n]=[1/9,0,1/9,0,1/9]$

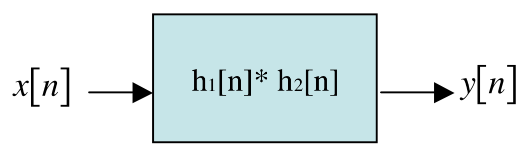
$$\mathcal{H}(\hat{\omega})=\mathcal{H}_1(\hat{\omega})\cdot\mathcal{H}_2(\hat{\omega})$$

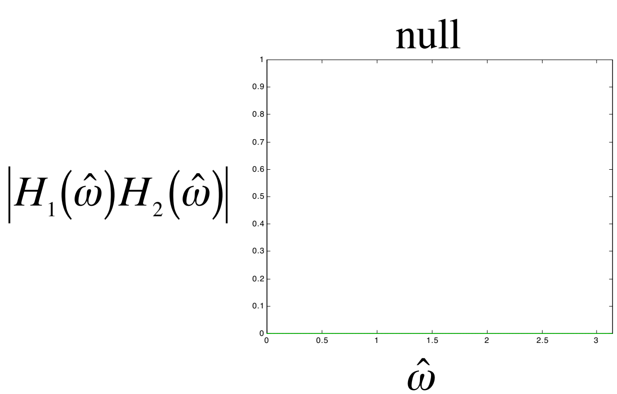
sample domain convolution
= freq domain multiplication
(and visa versa)



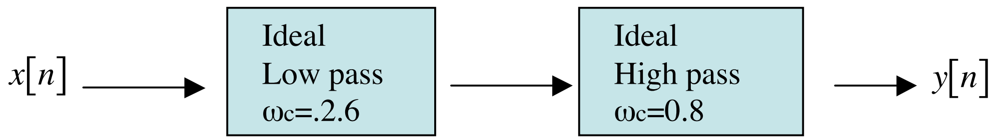


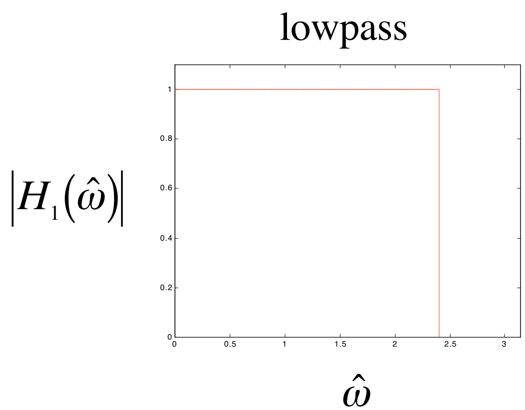


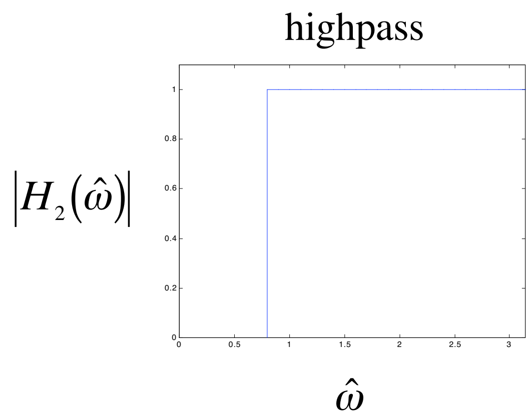


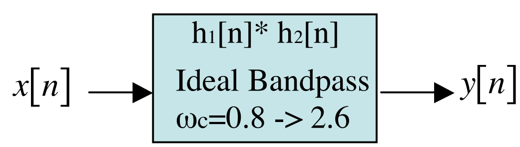


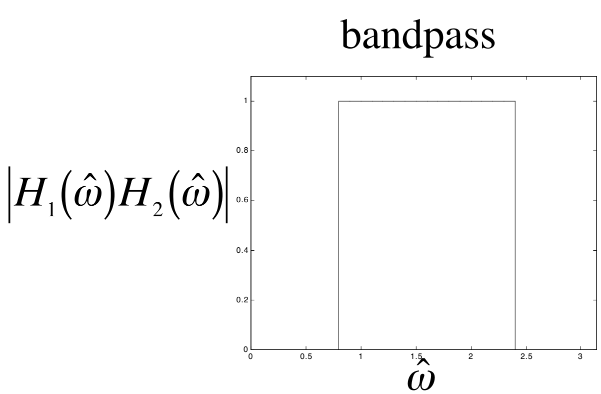
passbands don’t overlap



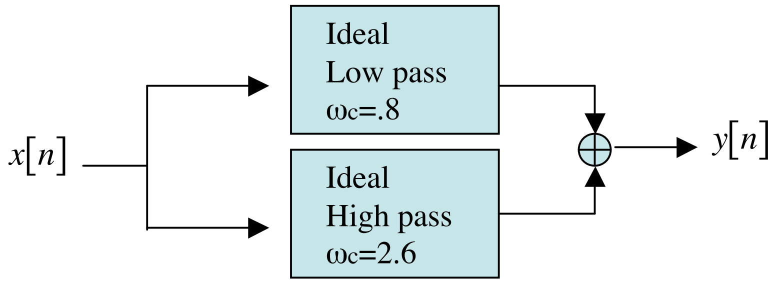




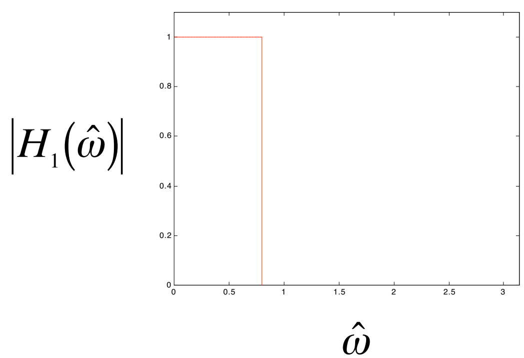




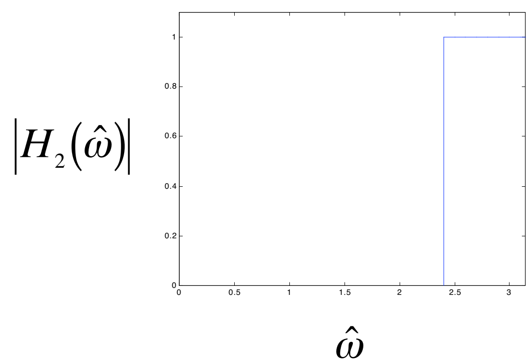
passbands overlap

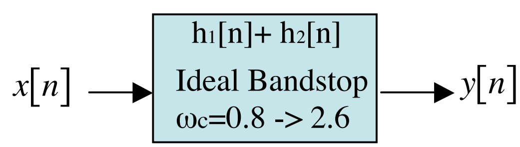


lowpass

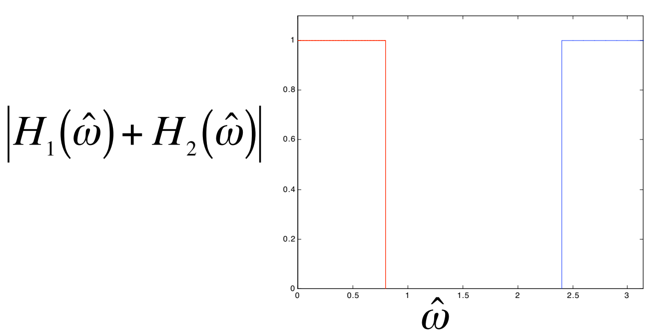


highpass

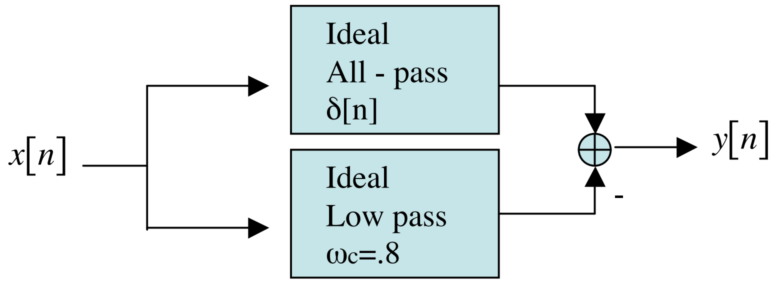




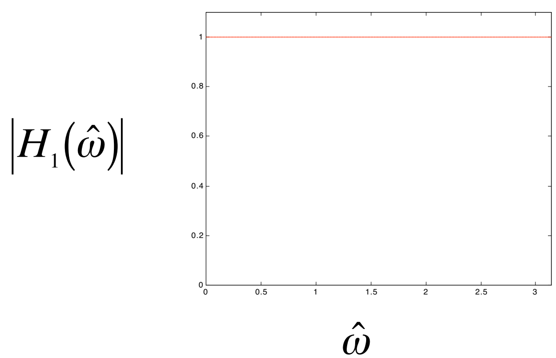
bandstop



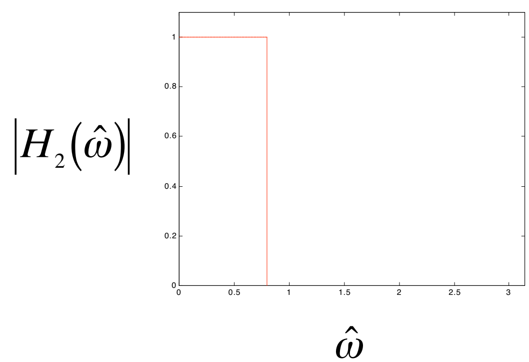
passbands don't overlap

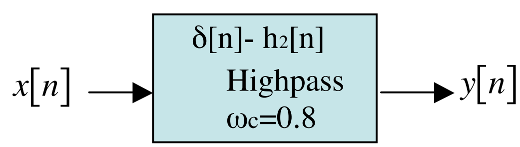


all-pass

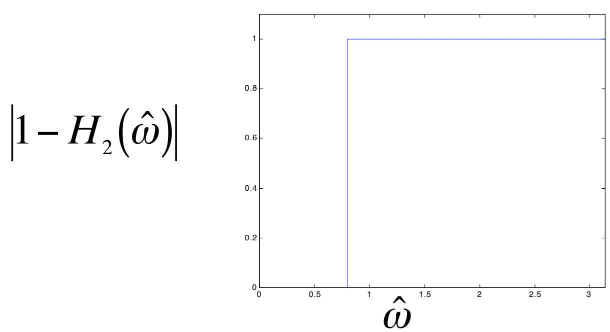


lowpass





highpass

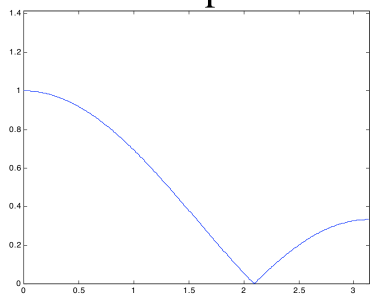


$h_1[n]=[1/3,1/3,1/3]$

$h_2[n]=\delta[n]-h_1[n]=[2/3,-1/3,-1/3]$

lowpass

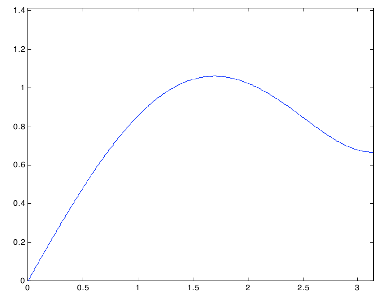
$$\left|H_1(\hat{\omega})\right|$$



$\hat{\omega}$

highpass

$$\left|H_2(\hat{\omega})\right| = \left|1 - H_1(\hat{\omega})\right|$$



$\hat{\omega}$

$h_3[n] = h_1[n] * h_2[n]$

$= [2/9,\ 1/9,\ 0,\ -2/9,\ -1/9]$

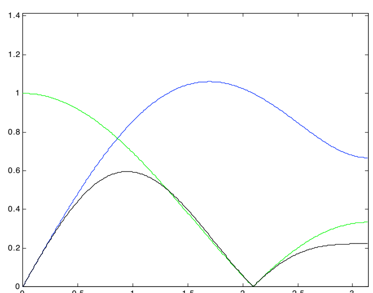
$h_5[n]$

$h_4[n] = h_1[n] + [1/3\ \ -1/3\ \ 1/3]$

$= [2/3\ \ 0\ \ 2/3]$

bandpass

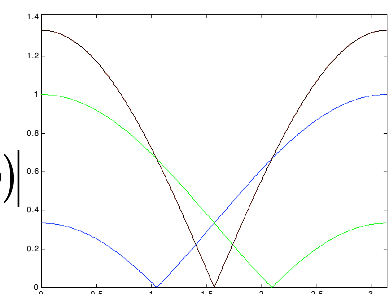
$$\left|H_3(\hat{\omega})\right| = \left|H_1(\hat{\omega})H_2(\hat{\omega})\right|$$



$\hat{\omega}$

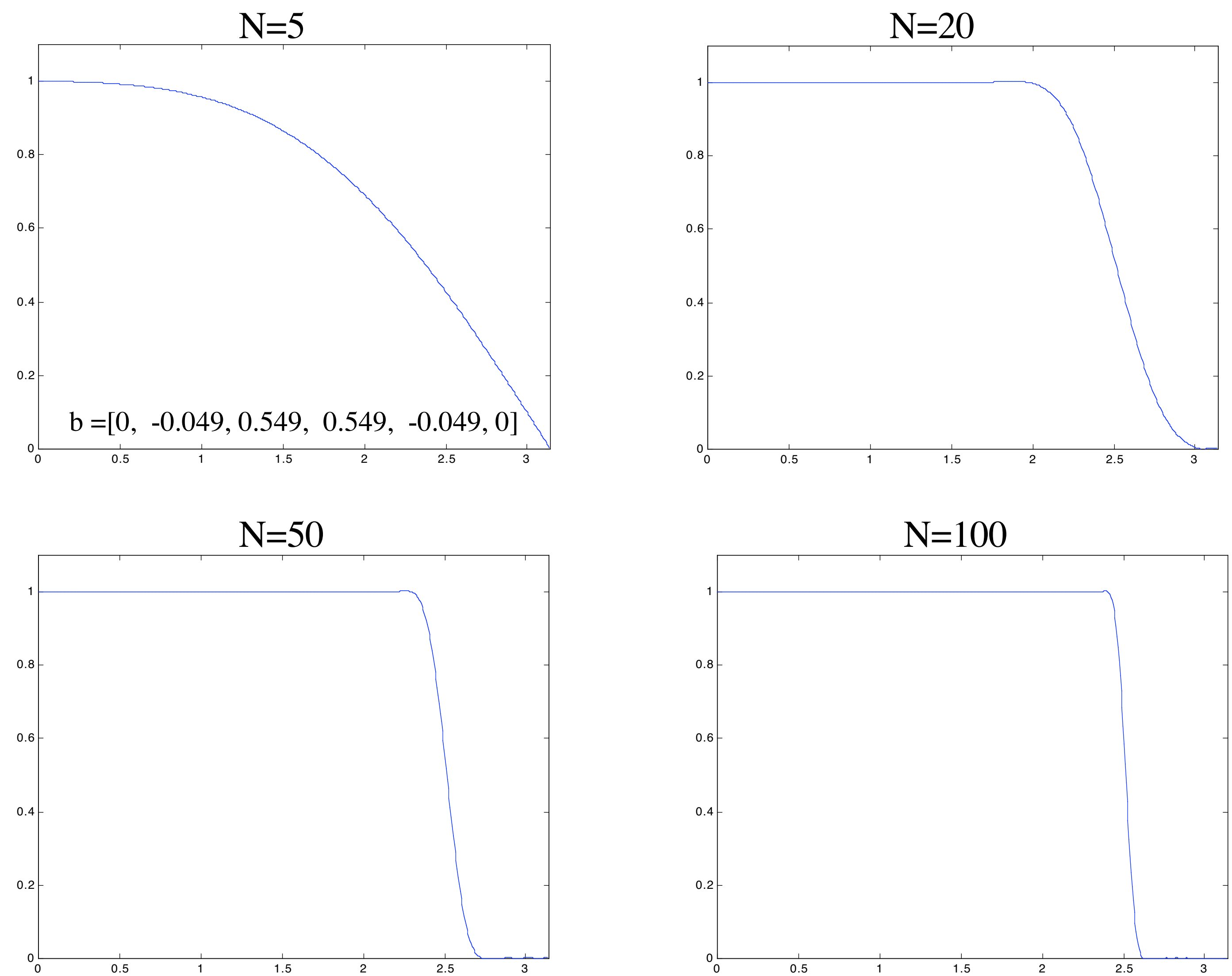
bandstop

$$\left|H_4(\hat{\omega})\right| = \left|H_4(\hat{\omega})\right| + \left|H_5(\hat{\omega})\right|$$



$\hat{\omega}$

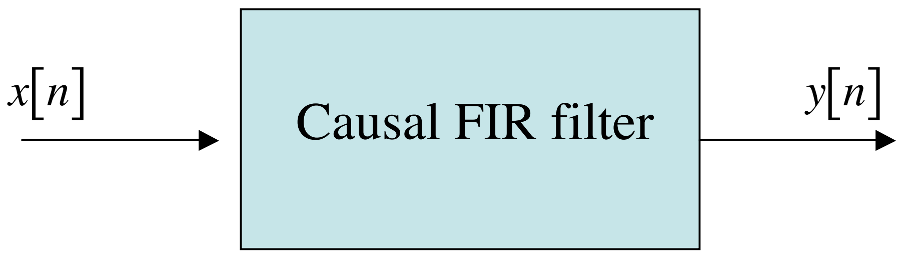
FIR low-pass filter response vs. number of taps N



More coefficients ("taps"), can achieve more ideal response;
Longer to compute, more memory, greater phase shift

Note: These aren't L-point averagers,
fir1(), Hammingwindow, Wn=0.8

Causal FIR Filter



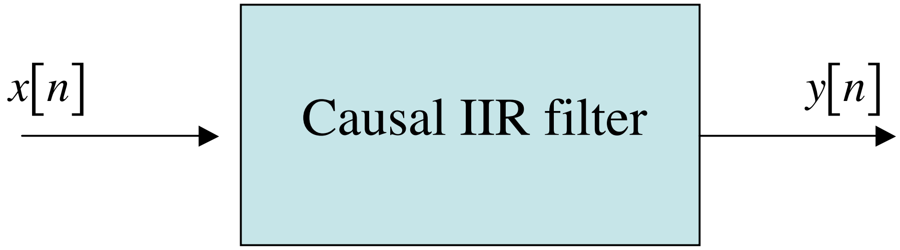
In an FIR filter, the output y at each sample n is a weighted sum of the present input, x[n], and past inputs, x[n-1], x[n-2],…, x[n-M].

$$y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Causal IIR filter

Infinite Impulse Response



For an IIR filter, the output y at each sample n is a weighted sum of the present input, x[n], past inputs, x[n-1], x[n-2],…, x[n-M], and past outputs, y[n-1], y[n-2],…, y[n-N]..

$$y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$
$$+ a_1 y[n-1] + a_2 y[n-2] + \ldots + a_M y[n-N]$$

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \qquad N \geq M \quad \text{proper system}$$

IIR Impulse Response

$$x[n] = 1\delta[n]$$

$$y[n] = \tfrac{1}{2}\, y[n-1] + 2x[n]$$

To calculate output at step n, need previous outputs.
At step n=0, assume initial rest conditions. x[n]=0, y[n]=0 for n<0

$$\begin{aligned} y[0] &= \tfrac{1}{2}\, y[0-1] + 2x[0] \\ &= \tfrac{1}{2}\cdot 0 + 2\cdot 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} y[1] &= \tfrac{1}{2}\, y[1-1] + \tfrac{1}{4}\, x[1] \\ &= \tfrac{1}{2}\, y[0] + \tfrac{1}{4}\, x[1] \\ &= \tfrac{1}{2}\cdot 2 + \tfrac{1}{4}\cdot 0 \\ &= 1 \end{aligned}$$

for n>0

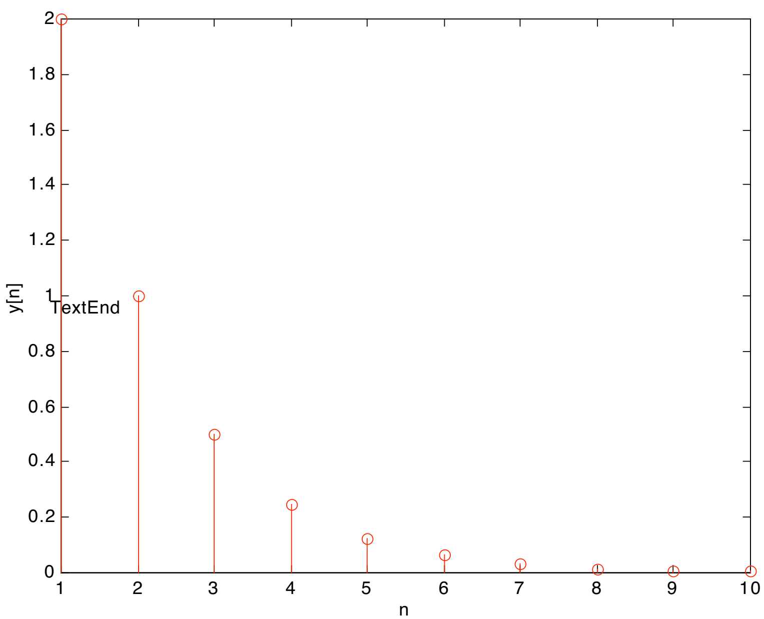
$$\begin{aligned} y[n] &= \tfrac{1}{2}\, y[n-1] + 2\cancelto{0}{x[n]} \\ &= \tfrac{1}{2}\, y[n-1] \end{aligned}$$

$$n = \left[\,{}^{-}1 \quad {}^{-}2 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots\right]$$

$$x[n] = \left[\,0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \ldots\right]$$

$$y[n] = \tfrac{1}{2}\, y[n-1] + 2x[n]$$

$$y[n] = \left[\,0 \quad 0 \quad 2 \quad 1 \quad \tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{8} \quad \ldots 2 \cdot \tfrac{1}{2}^{\,n}\right] = h[n]$$ impulse response



decays exponentially,
goes on forever
“infinite impulse response”

IIR impulse response

$$y\left[n\right] = \sum_{l=0}^{N} a_k x\left[n-l\right] + \sum_{k=0}^{M} b_k x\left[n-k\right]$$

IIR filter

$$x\left[n\right] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$

Delta function

↓

$$y\left[n\right]\Big|_{x=\delta[n]} = h[n] \neq \sum_{k=0}^{M} b_k \delta\left[n-k\right]$$

impulse response

Can't read filter coefficients off of impulse response

$$y\left[n\right] = \sum_{k=-\infty}^{\infty} h[n] x\left[n-k\right]$$

Convolution sum:
LTI systems: FIR/IIR

IIR convolution

$$x[n] = x[k]\delta[n-k]$$

We can decompose x[n]
into scaled delayed impulses.
Here assume finite length x[n].

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution sum:
LTI: FIR / IIR

$$y[n] = x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \ldots$$

Each impulse has a scaled impulse response.
The final output is a sum of those scaled and delayed
impulse responses. Now the impulse responses are
Infinite in length.

IIR convolution

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-2]$$

$$h[n] = \begin{bmatrix} 0 & 0 & 2 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \ldots 2 \cdot \frac{1}{2}^{n} \end{bmatrix}$$

n=-2 -1 0 1 2 3 4 …

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

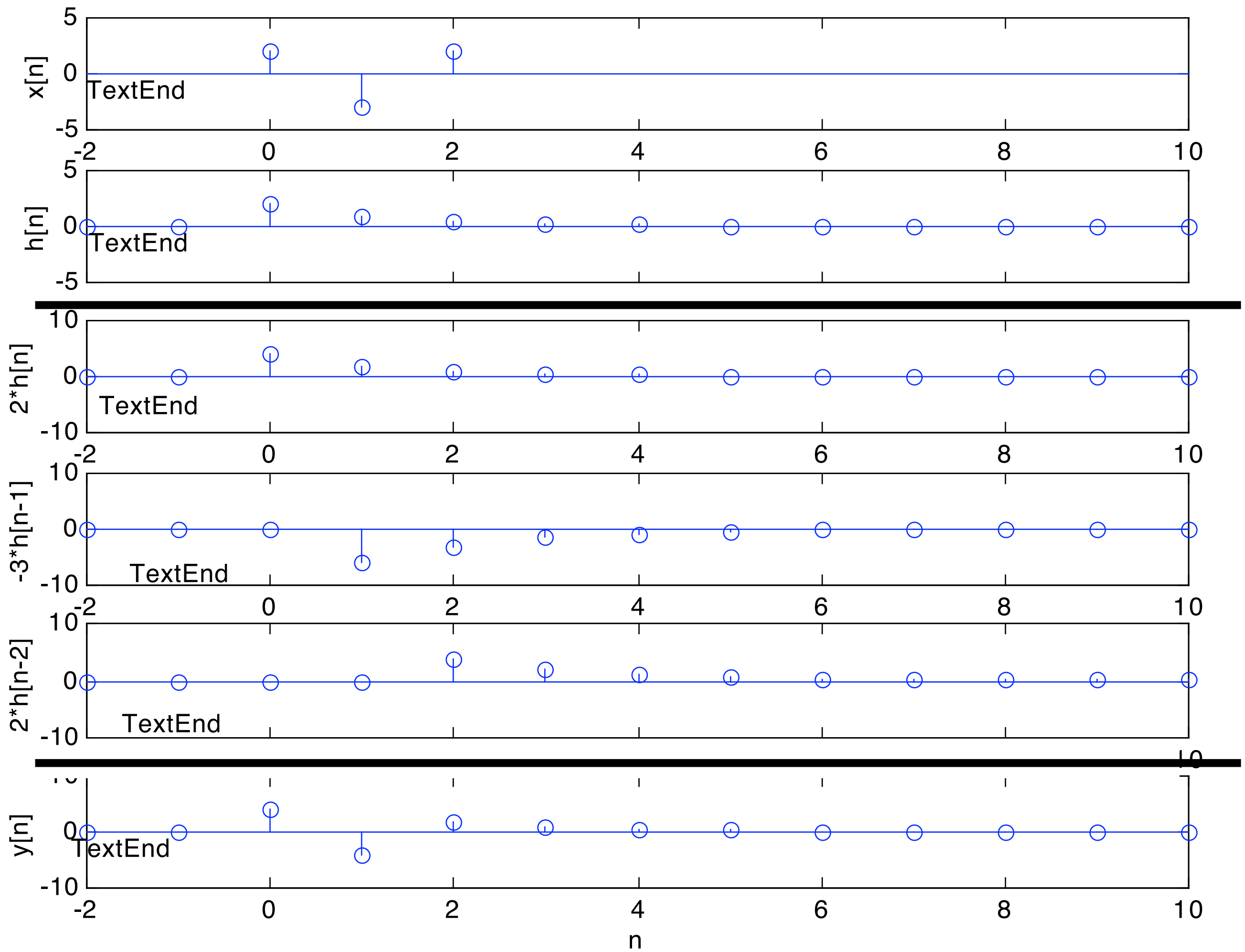
Each impulse has a scaled impulse response. The final output is a sum of those scaled and delayed impulse responses.

$$y[n] = 2 \cdot h[n] - 3 \cdot h[n-1] + 2 \cdot h[n-2]$$

n>2

$$= \left[\begin{array}{ccccccccc} 0 & 0 & 4 & 2 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \ldots 2 \cdot \frac{1}{2}^{n-1} \\ & 0 & 0 & -6 & -3 & \frac{-3}{2} & \frac{-3}{4} & \frac{-3}{8} & \ldots - 3 \cdot \frac{1}{2}^{n-2} \\ & & & 0 & 4 & 2 & 1 & \frac{1}{2} & \ldots 2 \cdot \frac{1}{2}^{n-3} \\ \hline 0 & 0 & 4 & -4 & 2 & 1 & \frac{1}{2} & \frac{1}{4} & \ldots 2\left[\frac{1}{2}\right]^{n-2} \end{array} \right] \ldots \left[\frac{1}{2}\right]^{n-3}\left[2 \frac{1}{2}^{2} - 3\frac{1}{2} + 2\right]$$

IIR convolution



$$y[n] = \left[0 \quad 0 \quad 4 \quad -4 \quad 2 \quad 1 \quad \tfrac{1}{2} \quad \tfrac{1}{4} \quad \ldots 2 \cdot \left[\tfrac{1}{2}\right]^{n-2}\right]$$

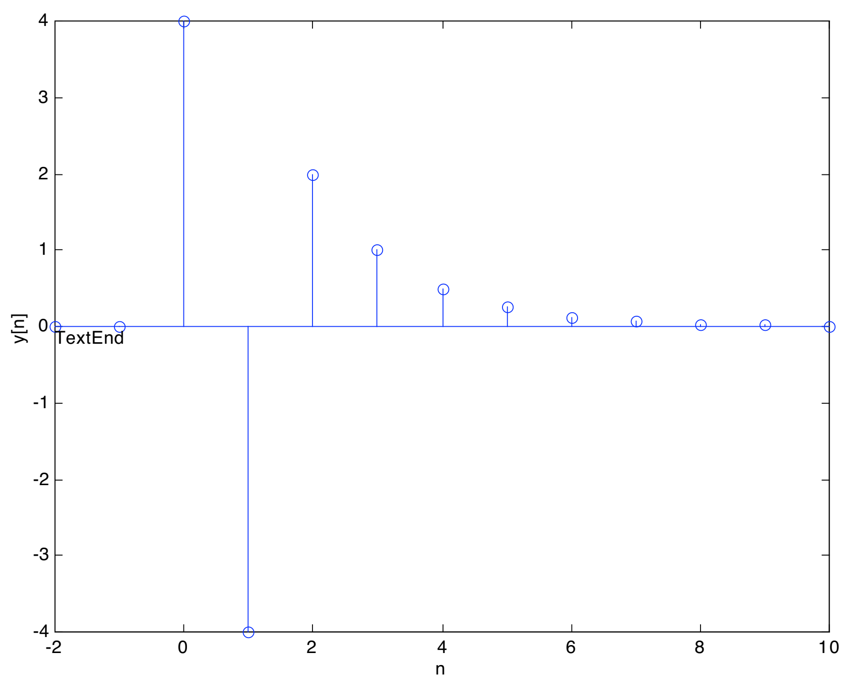
IIR convolution

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$

$$h[n] = \left[\begin{matrix} 0 & 0 & 2 & 1 & \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{8} & \ldots 2\cdot\tfrac{1}{2}^{n} \end{matrix}\right]$$

$$y[n] = h[n] * x[n]$$

$$y[n] = \left[\begin{matrix} 0 & 0 & 4 & -4 & 2 & 1 & \tfrac{1}{2} & \tfrac{1}{4} & \ldots 2\cdot\left[\tfrac{1}{2}\right]^{n-2} \end{matrix}\right]$$



Frequency response

$$y[n]=\sum_{k=0}^{\infty}h[k]x[n-k]$$

convolution

$$x[n]=Ae^{j\phi}e^{j\hat{\omega}n}$$

Complex exponential input

$$\downarrow$$

$$y[n]=\sum_{k=0}^{\infty}h[k]Ae^{j\phi}e^{j\hat{\omega}(n-k)}$$

$$=\left(\sum_{k=0}^{\infty}h[k]e^{-j\hat{\omega}k}\right)Ae^{j\phi}e^{j\hat{\omega}n}$$

$$=\mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

frequency response

let

$$\mathcal{H}(\hat{\omega})=\sum_{k=0}^{\infty}h[k]e^{j\hat{\omega}k}$$

This sum must converge for $\mathcal{H}(\hat{\omega})$ to exist.

$\mathcal{H}(\hat{\omega})$ is the Discrete Time Fourier Transform of the impulse function h[n]

Frequency response

FIR

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$h[k] = \sum_{k=0}^{M} b_k \delta[n-k]$$

$$\downarrow$$

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} h[k] e^{j\hat{\omega}k}$$

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{j\hat{\omega}k}$$

Easy to go from difference equation to frequency response because h[n] finite length and h[n] = [b_0, b_1,…].

IIR

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

$$h[k] \neq \sum_{k=0}^{\infty} b_k \delta[n-k]$$

$$\downarrow$$

X Argh!

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$

Tough to go from diff.eqn. to freq. response because h[n] infinite length, h[n]=f(a_l,b_k) is complicated, and $\mathcal{H}(\hat{\omega})$ may be unbounded.

temporal space - n

complex frequency space- z

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

$$h[k] \neq \sum_{k=0}^{\infty} b_k \delta[n-k]$$

z-transform →

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{\prod_{i=0}^{M} (z - z_{zi})}{\prod_{i=0}^{N} (z - z_{pi})}$$

Fourier transform ↓ @#!

Argh! road block

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$

frequency space - ω

← Hurray!

$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

The frequency response is H(z) evaluated on unit circle

Benefits of z-plane and z-transforms:

1. Get around road block by using z-plane and z-transforms.
Compute system function from diff.eq. coefficients, then evaluate on the unit circle to find the frequency response.

2. z-plane (pole/zeros) will tell us if system stable and frequency response exists.

3. By using z-transforms, solution to diff.eq goes from solving convolution in n-space to solving algebraic equations in z-domain (easier).

And lots more…!

Frequency response

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

convolution

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

Complex exponential input

$$\downarrow$$

$$y[n] = \sum_{k=0}^{\infty} h[k]Ae^{j\phi}e^{j\hat{\omega}(n-k)}$$

$$= \left(\sum_{k=0}^{\infty} h[k]e^{-j\hat{\omega}k}\right)Ae^{j\phi}e^{j\hat{\omega}n}$$

$$= \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

frequency response

let

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k]e^{j\hat{\omega}k}$$

This sum must converge for $\mathcal{H}(\hat{\omega})$ to exist.

$\mathcal{H}(\hat{\omega})$ is the Discrete Time Fourier Transform of the impulse function h[n]

System function

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

convolution

$$x[n] = z^n$$

power sequence
of complex numbers

$\downarrow$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{(n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)z^n$$

let
$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$

$$= H(z)z^n$$

$$= |H(z)|e^{j\angle H(z)}z^n$$

scaled and shifted
power sequence

LTI

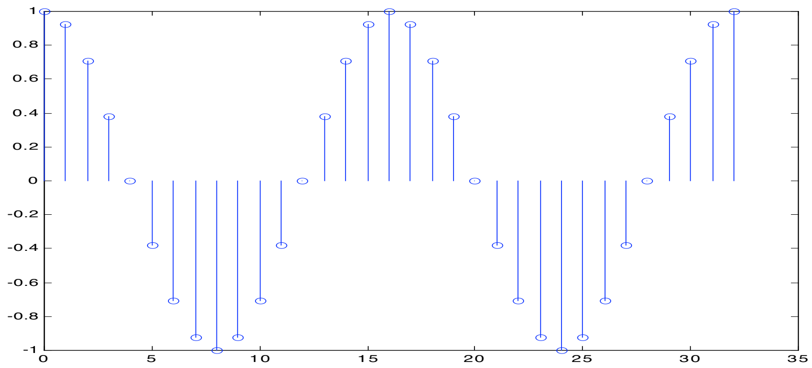
$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

system function

characteristic functions of LTI systems

$$z^n$$

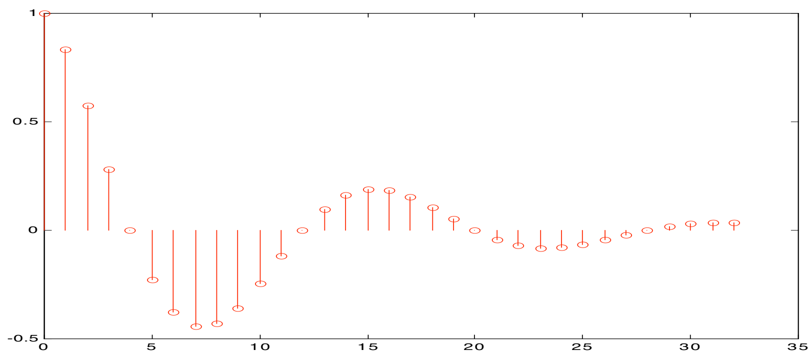
$$\mathrm{Re}\left(z^n\right) \qquad \frac{z^n + (z^*)^n}{2}$$



$$z = e^{j\frac{2\pi}{16}}$$

sinusoid

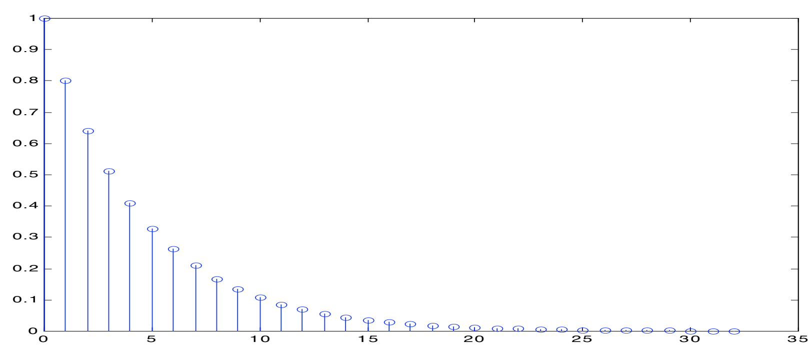
$$\mathrm{Re}\left(z^n\right) \qquad \frac{z^n + (z^*)^n}{2}$$



$$z = 0.9e^{j\frac{2\pi}{16}}$$

damped sinusoid

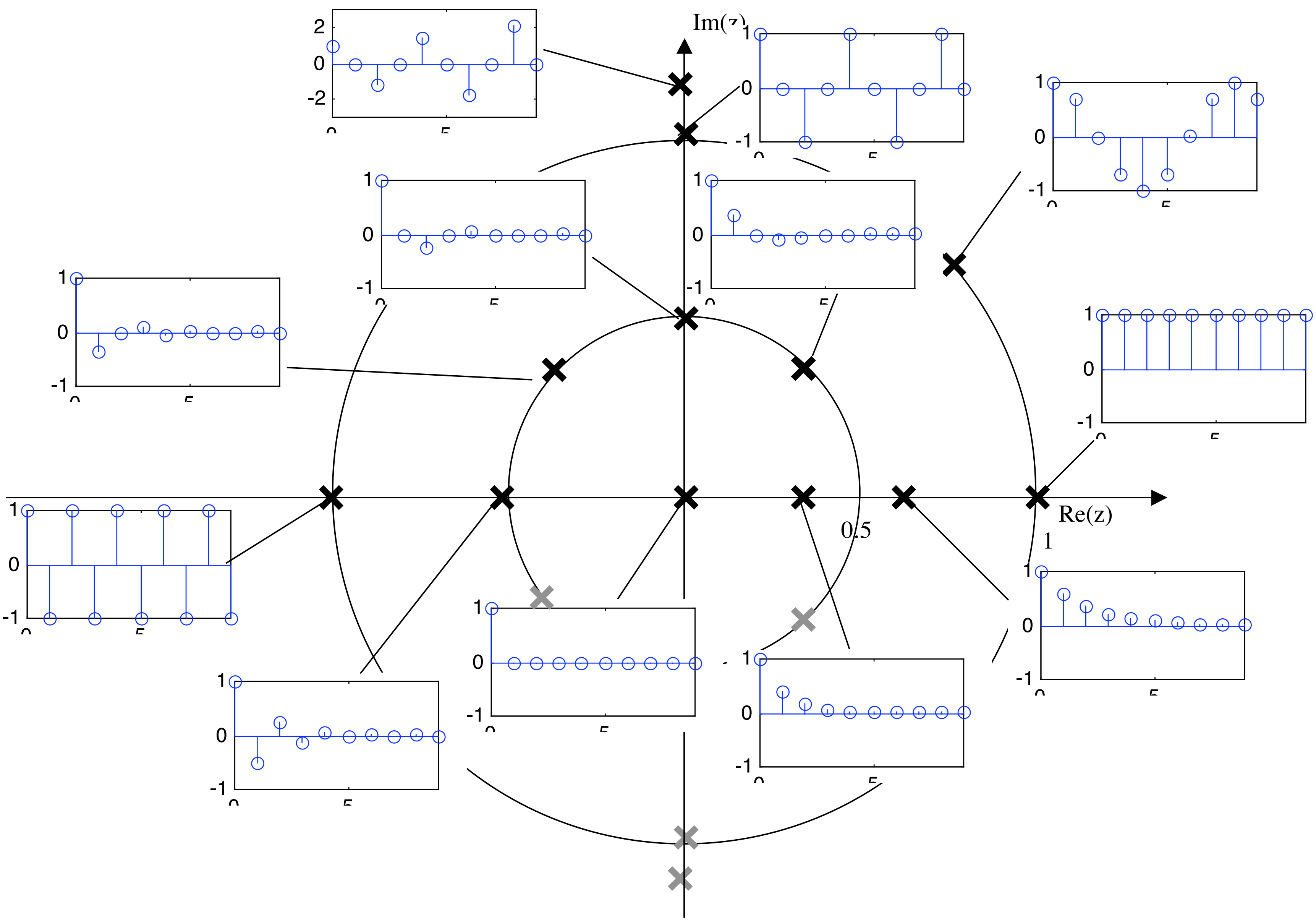
$$z^n$$



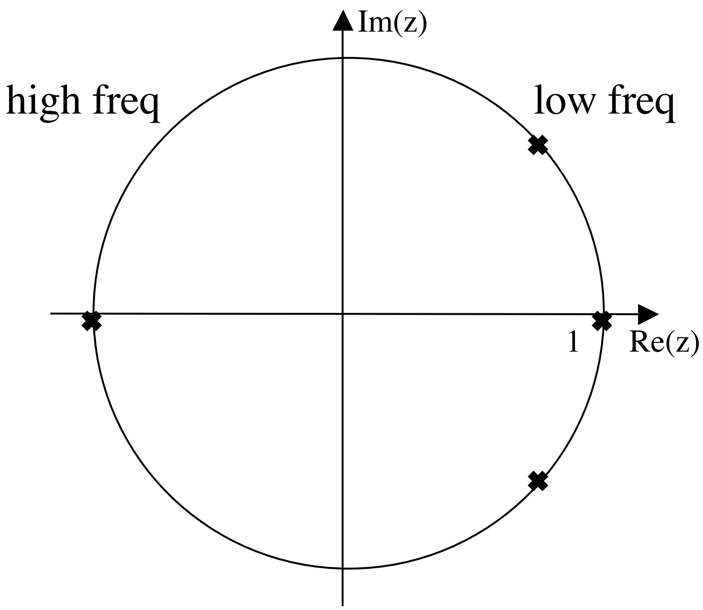
$$z = 0.8$$

exponentials

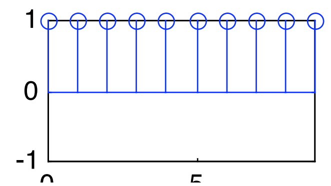
z-plane and sample responses z^n



z-plane $\qquad$ z^n

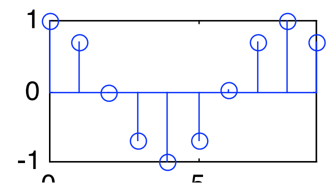


$z = 1$



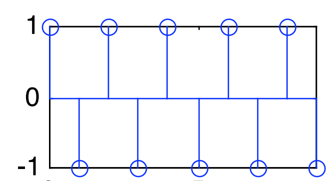
DC, ω=0

$z = e^{\pm j\hat{\omega}}$



unit circle
sinusoids

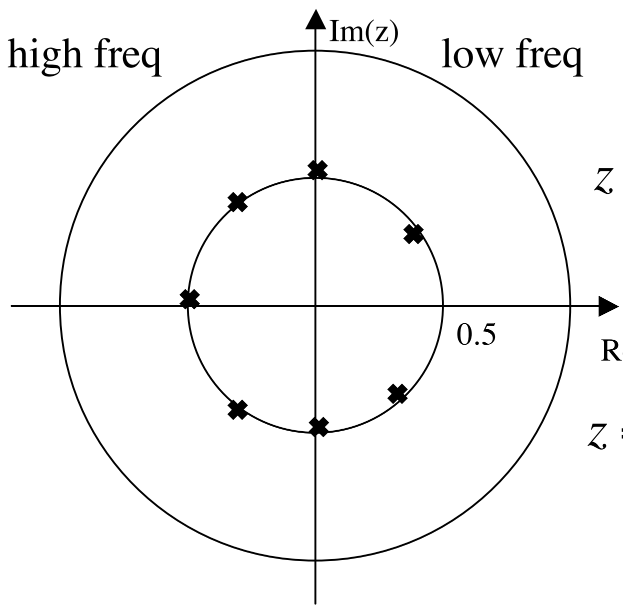
$z = -1$



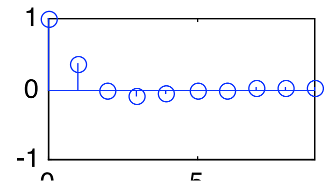
Nyquist sampled
sinusoid

z-plane

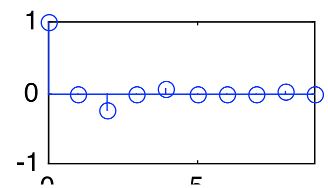
z^n



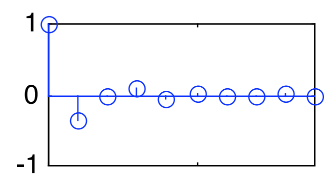
$z = 0.5e^{\pm j\frac{\pi}{4}}$



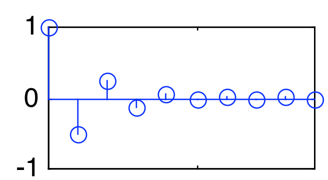
$z = 0.5e^{\pm j\frac{\pi}{2}}$



$z = 0.5e^{\pm j\frac{3\pi}{4}}$



$z = -0.5$



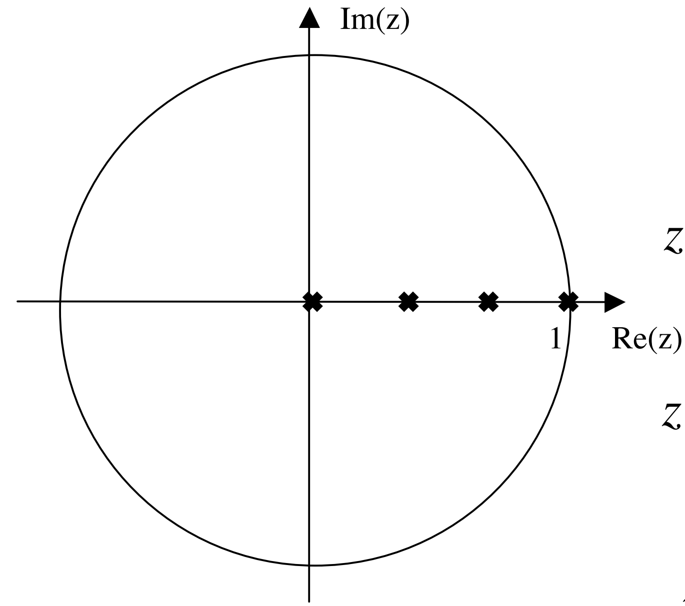
|z|<1

Damped sinusoids

Nyquist sampled damped sinusoid

z-plane

z^n

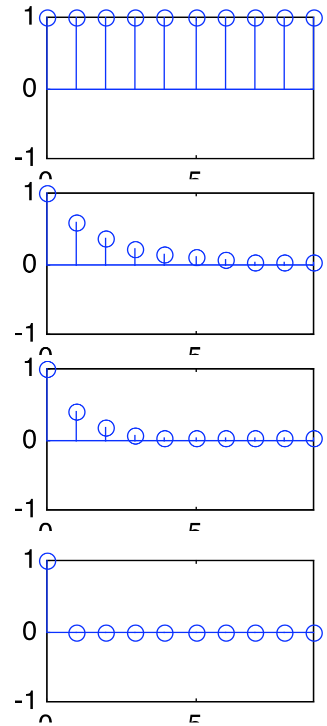


$z = 1$

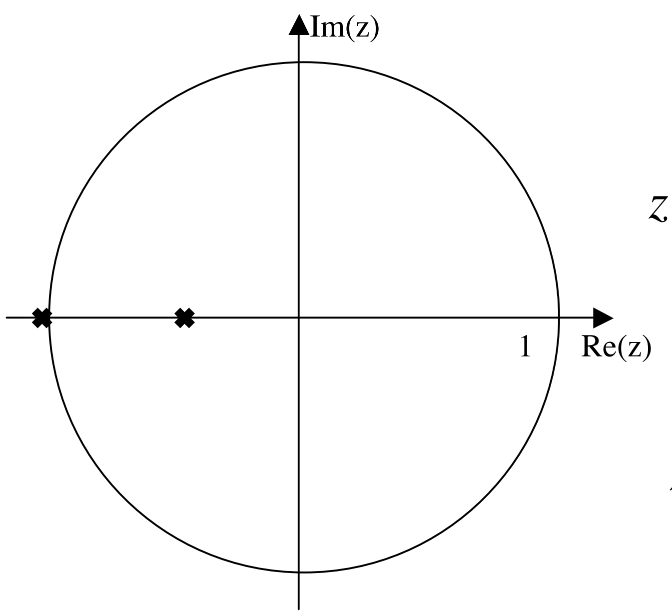
$z = 0.6$

$z = 0.4$

$z = 0$

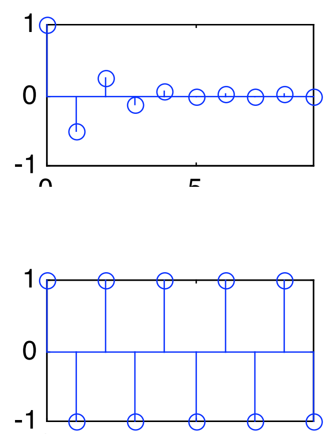


+Re(z) axis
exponential
decay



$z = -0.5$

$z = -1$



-Re(z) axis

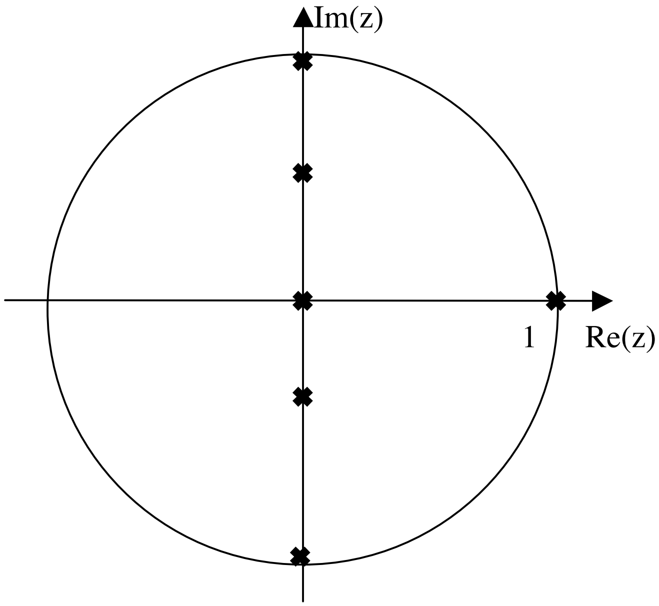
$\omega_s/2$

2 samples/cycle

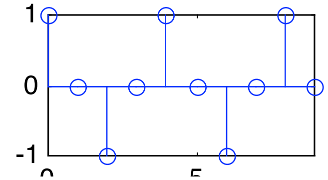
Nyquist sampled
signals

z-plane

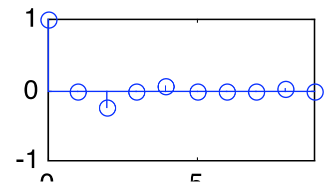
z^n



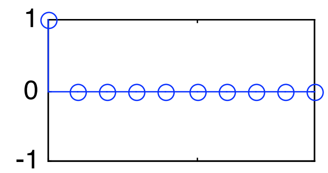
$z = \pm 1j$



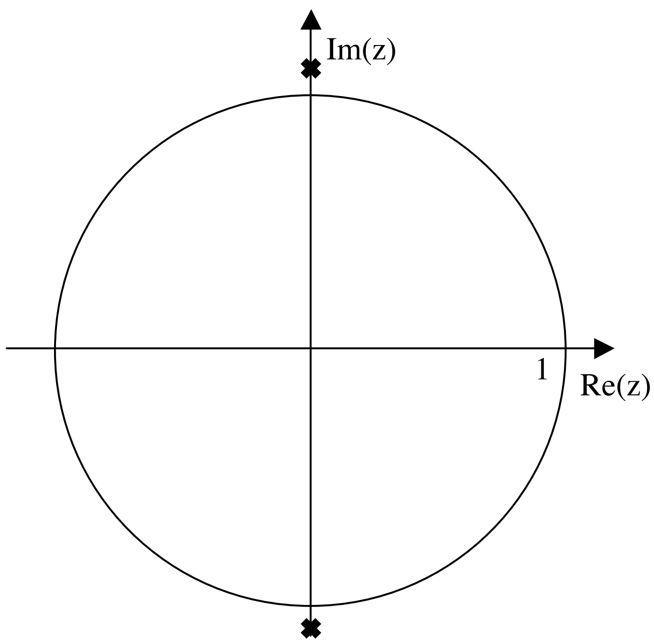
$z = \pm j0.5$



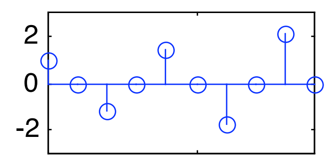
$z = 0$



Im(z) axis
every other
sample 0



$z = \pm j1.1$



|z| > 1 unstable

|z| < 1 stable

System function

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

convolution

$$x[n] = z^n$$

power sequence
of complex numbers

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{(n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)z^n$$

let
$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$

$$= H(z)z^n$$

$$= |H(z)|e^{j\angle H(z)}z^n$$

scaled and shifted
power sequence

LTI

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

system function

z-transform

$$x[n] \overset{z}{\Leftrightarrow} X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$ definition

Ex.

impulse

$$x[n] = \delta[n] \qquad \overset{z}{\Leftrightarrow} \qquad X(z) = \sum_{k=-\infty}^{\infty} \delta[k] z^{-k}$$

$$= z^{-0}$$

Ex.

$$= 1$$

$$y[n] = h[n] * x[n]$$
$$= h[n] * \delta[n]$$
$$\Updownarrow \ z$$
$$Y(z) = H(z) \cdot 1$$
$$= H(z)$$
$$\Updownarrow \ z$$
$$y[n] = h[n]$$

H(z) is the z-transform of the impulse response h[n]

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

LTI

Ex.

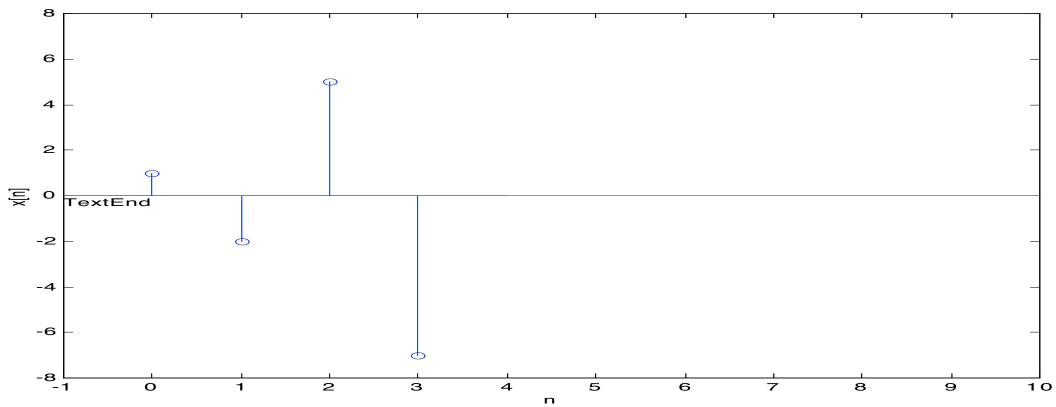
n_0 sample delayed impulse

n_0^{th} power of z^{-1}

$$x[n] = \delta[n - n_0] \quad \Leftrightarrow \quad X(z) = \sum_{k=-\infty}^{\infty} \delta[k - n_0] z^{-k}$$

$$= z^{-n_0} = \left(z^{-1}\right)^{n_0}$$

Ex. finite response



$$x[n] = 1\delta[n] - 2\delta[n-1] + 5\delta[n-2] - 7\delta[n-3]$$

$$\Updownarrow z$$

$$X(z) = 1 - 2z^{-1} + 5z^{-2} - 7z^{-3}$$

Ex. $y[n] = \tfrac{1}{3}x[n] + \tfrac{1}{3}x[n-1] + \tfrac{1}{3}x[n-2]$ FIR

$$h[n] = \tfrac{1}{3}\delta[n] + \tfrac{1}{3}\delta[n-1] + \tfrac{1}{3}\delta[n-2]$$

$$H(z) = \sum_{k=0}^{2} h[k]z^{-k}$$

$$= h[0]z^{0} + h[1]z^{-1} + h[2]z^{-2}$$

$$= \tfrac{1}{3} + \tfrac{1}{3}z^{-1} + \tfrac{1}{3}z^{-2}$$

$$= \tfrac{1}{3}\left(z^{-1}\right)^{0} + \tfrac{1}{3}\left(z^{-1}\right)^{1} + \tfrac{1}{3}\left(z^{-1}\right)^{2}$$ polynomial in z^{-1}

$$h[n] = \sum_{k=0}^{2} b_k\delta[n-k] \Leftrightarrow H(z) = \sum_{k=0}^{2} b_k z^{-k}$$ FIR only!

sequence $\Leftrightarrow$ polynomial

n-domain (sample space) z-domain (complex freq space)

Ex. $$H(z) = \tfrac{1}{3} + \tfrac{1}{3}z^{-1} + \tfrac{1}{3}z^{-2} = \frac{z^2 + z + 1}{3z^2}$$

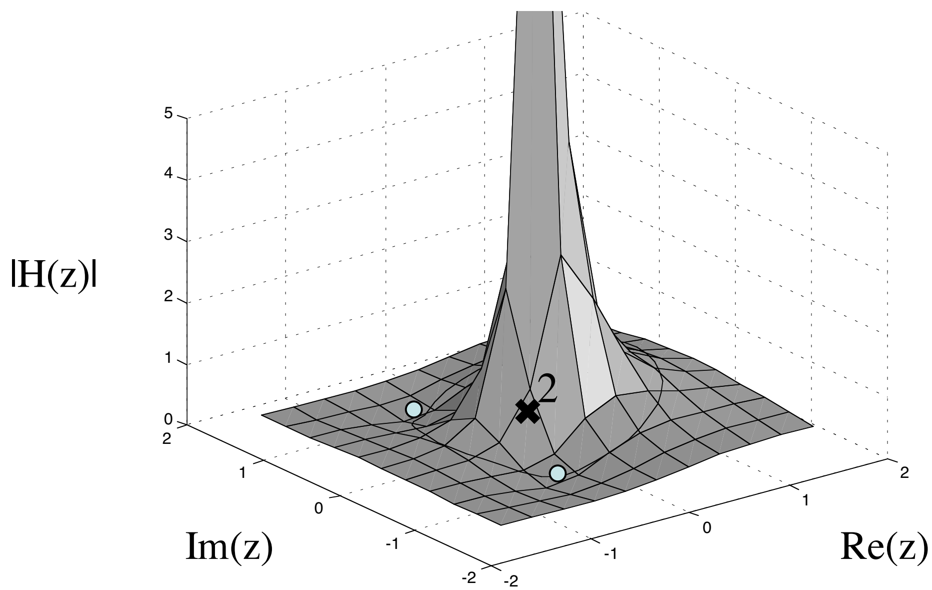
$$y[n] = H(z)z^n$$

num=0 $H(z) = 0$ $y[n] = 0$ $z^2 + z + 1 = 0$ zeros

$z = \tfrac{1}{2}\left(-1 \pm j\sqrt{3}\right) = e^{\pm j2\pi/3}$ roots of numerator

denom=0 $H(z) = \infty$ $y[n] = \infty$ $z^2 = 0$ poles

$z = 0,0$ roots of denominator

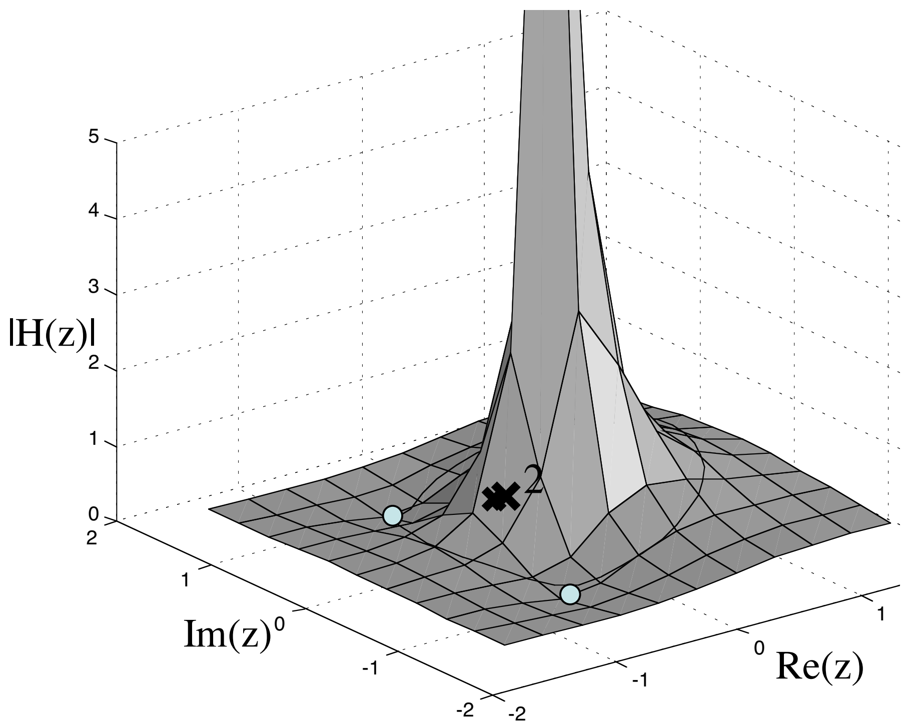


FIR filters only have zeros on unit circle, and poles are either at 0 or ∞.

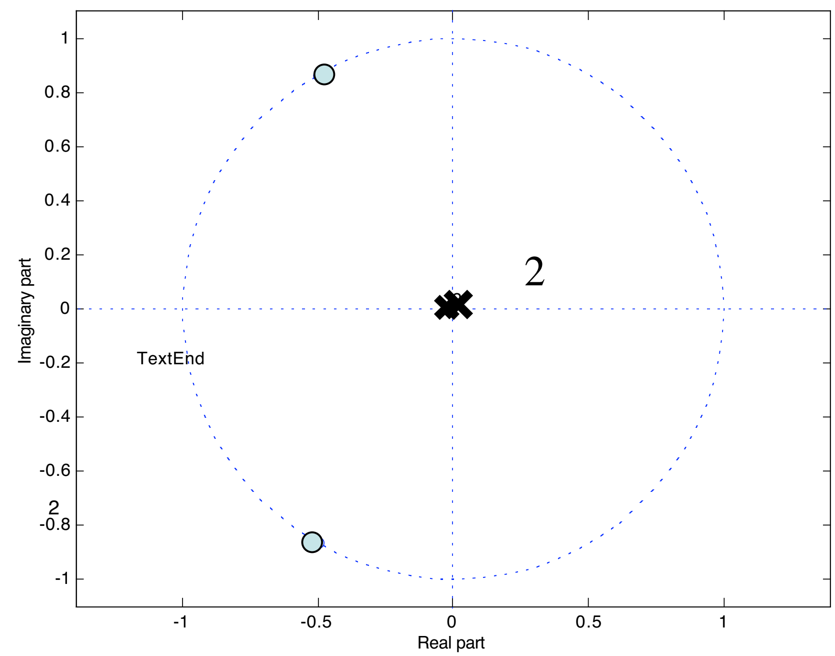
#poles=#zeros

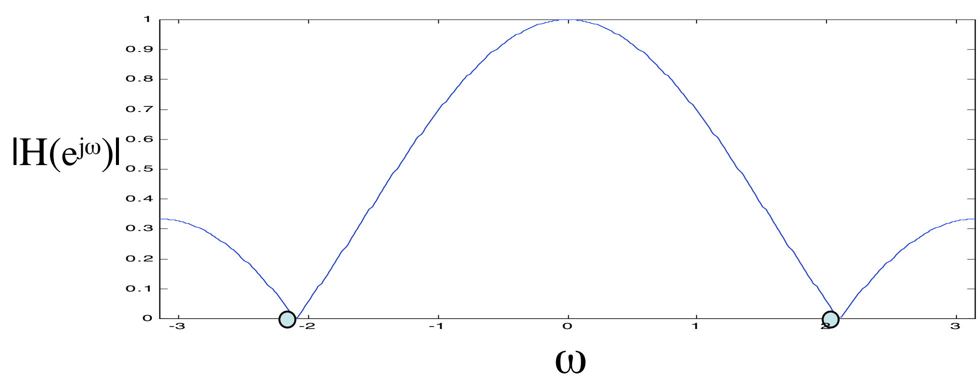
"extra" zero/poles are at z=∞.

system response |H(z)|



pz plot





frequency response

$$\mathcal{H}(\omega) = H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}}$$

The frequency response is
H(z) evaluated on unit circle

Ex. IIR

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

$$h[n] = \sum_{l=1}^{N} a_l h[n-l] + \sum_{k=0}^{M} b_k \delta[n-k]$$

$$\downarrow$$

$$H(z) = \sum_{l=1}^{N} a_l H(z) z^{-l} + \sum_{k=0}^{M} b_k z^{-k}$$

$$H(z)\left(1 - \sum_{l=1}^{N} a_l z^{-l}\right) = \sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

freqz(b,a)

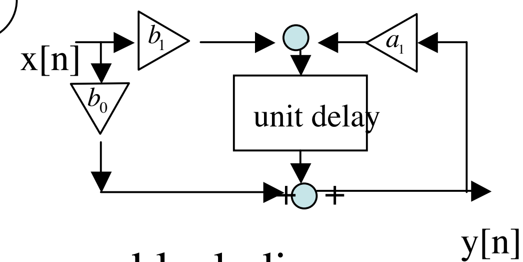
sequence $\Leftrightarrow$ polynomial

n-domain (sample space) z-domain (complex freq space)

Equivalent ways to represent the system

① $$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \quad \Leftrightarrow$$

difference equation

② 

block diagram

$x[n]=\delta[n]$ ⇕

⇕ inspection

③ $$h[n] = y[n]\Big|_{x[n]=\delta[n]} \overset{Z}{\Leftrightarrow} H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{\prod_{i=0}^{M} (z - z_{zi})}{\prod_{i=0}^{N} (z - z_{pi})}$$

impulse response sequence

④ system function polynomial

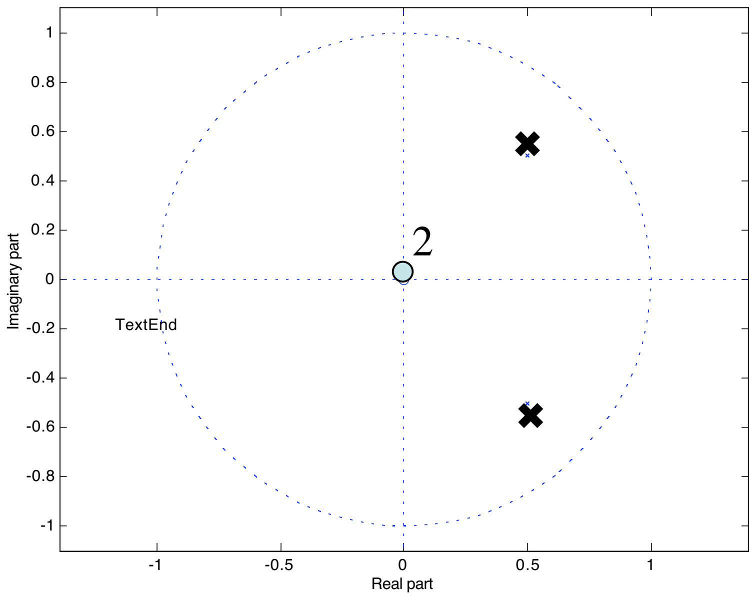
pole-zero locations ⑤

$z = e^{j\omega}$ ⇕

⑥ $$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

frequency response

All poles must be inside unit circle for $\mathcal{H}(\omega)$ to converge and the system to be stable.
(FIR filter always stable)



IIR filter

difference equation

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$H(z) = \frac{1}{1 - z^{-1} + 0.5z^{-2}} = \frac{z^2}{z^2 - z + 0.5}$$

$$= \frac{\prod_{i=0}^{M}(z - z_{zi})}{\prod_{i=0}^{N}(z - z_{pi})}$$

$$= \frac{z \cdot z}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

frequency response

$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

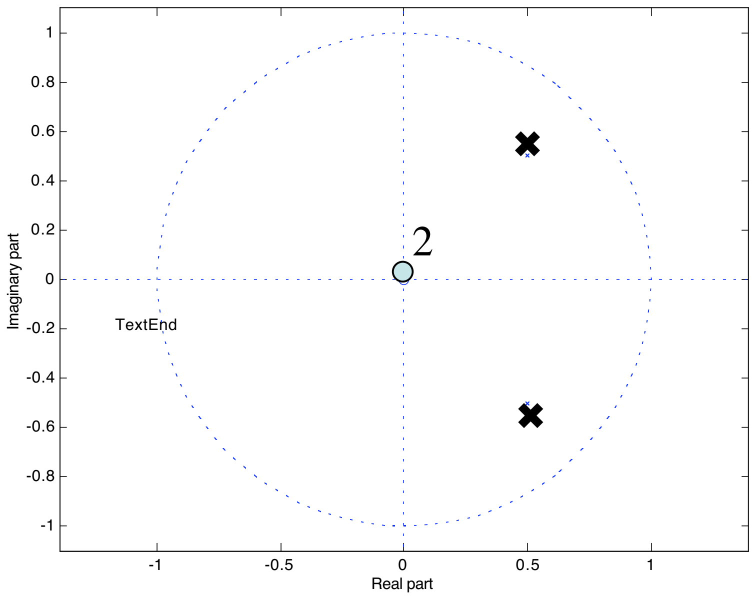
zeros: 0, 0

poles: 0.5±j0.5

zeros: 0, 0

poles: 0.5±j0.5

IIR filter



$$H(z) = \frac{\prod_{i=0}^{M}(z - z_{zi})}{\prod_{i=0}^{N}(z - z_{pi})}$$

$$= \frac{z \cdot z}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

$$= \frac{z^2}{z^2 - z + 0.5} = \frac{1}{1 - z^{-1} + 0.5z^{-2}}$$

$$\frac{1}{1 - z^{-1} + 0.5z^{-2}} = \frac{1}{1 - z^{-1} + 0.5z^{-2}} \cdot \frac{z^2}{z^2}$$

$$= \frac{z^2}{z^2 - z + 0.5}$$

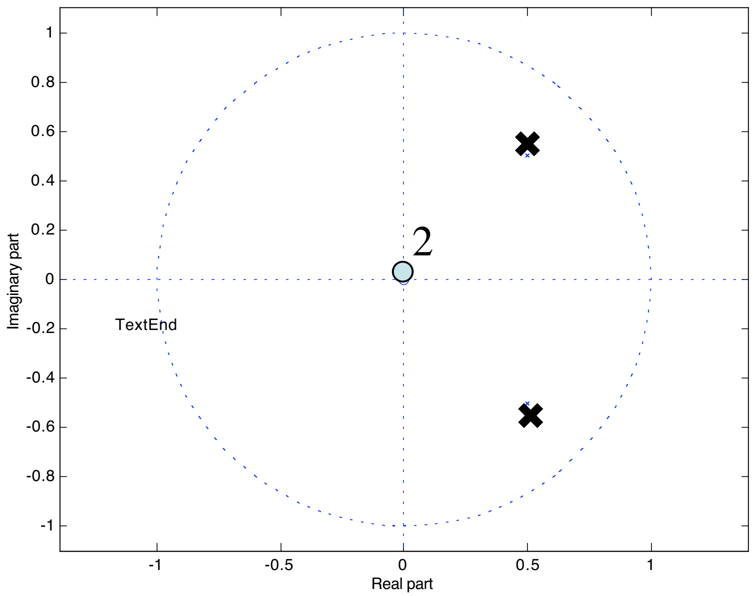
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

frequency response

$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

difference equation

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$



zeros: 0, 0 IIR filter

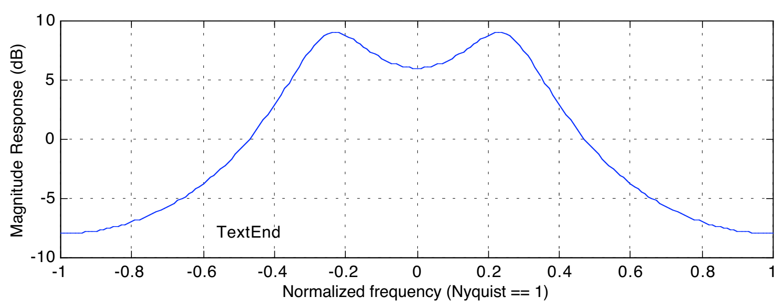
poles: 0.5±j0.5

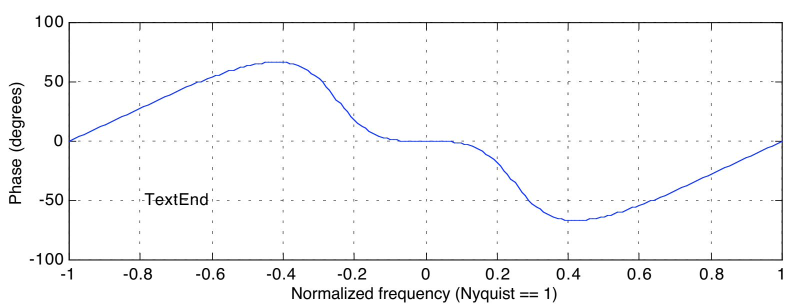
$$H(z) = \frac{z^2}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

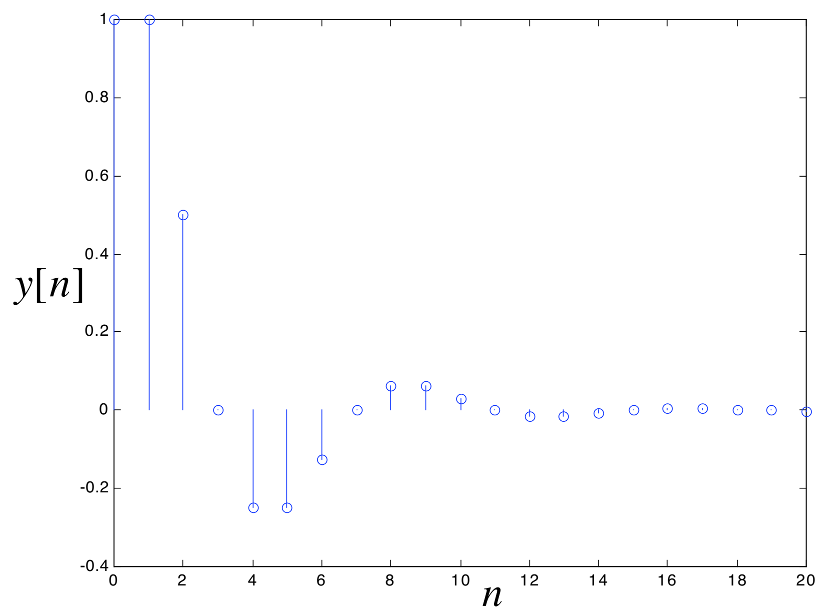
$$= \frac{z^2}{z^2 - z + 0.5} = \frac{1}{1 - z^{-1} + 0.5z^{-2}}$$

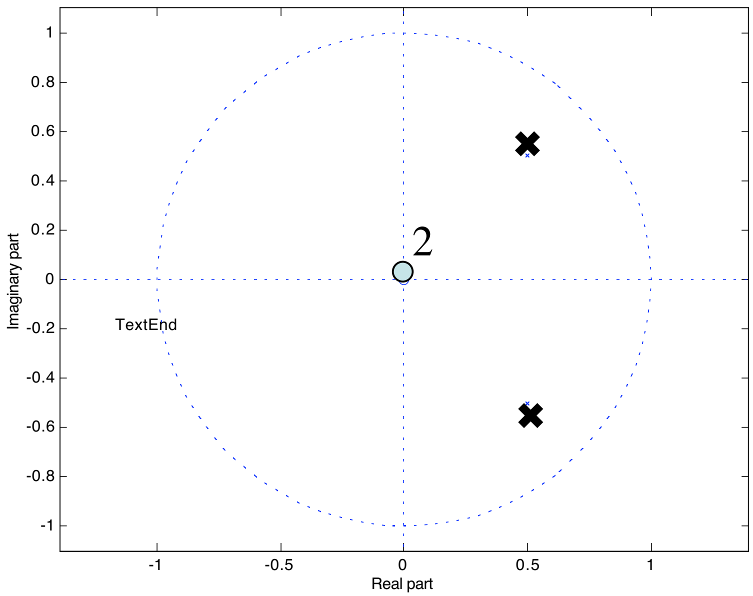
$$H(\hat{\omega}) = H(z)\Big|_{z = e^{j\hat{\omega}}}$$

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$









zeros: 0, 0 IIR filter

poles: 0.5±j0.5

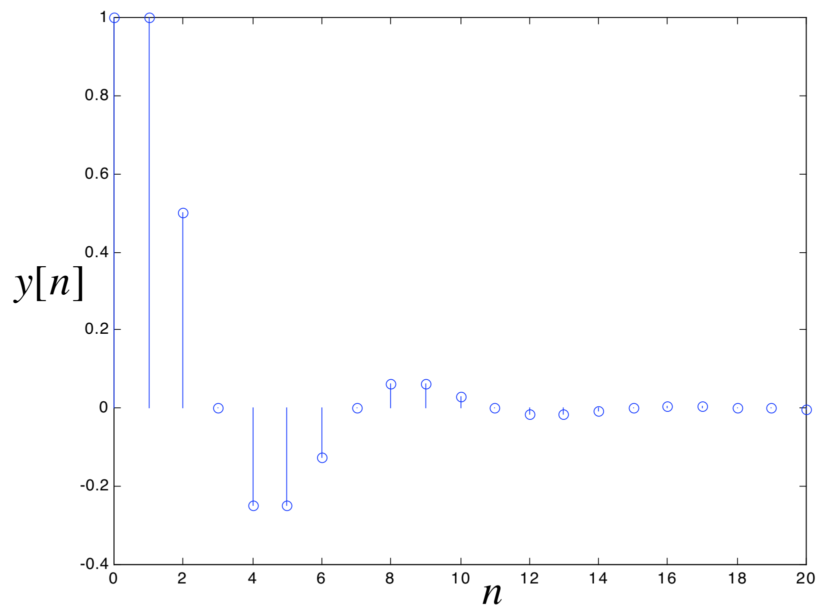
$$H(z) = \frac{z^2}{(z-0.5-j0.5)(z-0.5+j0.5)}$$

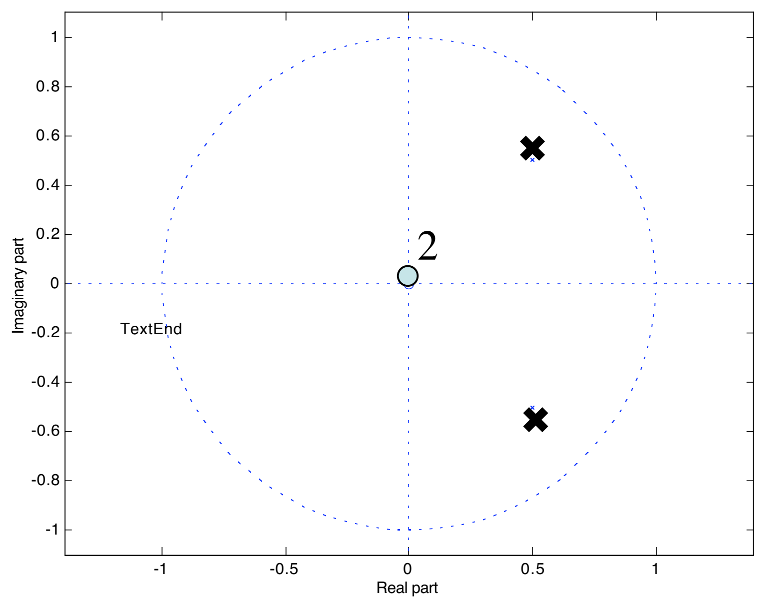
$$= \frac{z^2}{z^2 - z + 0.5} = \frac{1}{1 - z^{-1} + 0.5z^{-2}}$$

$$H(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$

To go from p-z map to impulse response: Look at locations of poles. Closer poles are to zeros, the weaker their response.





zeros: 0, 0 IIR filter

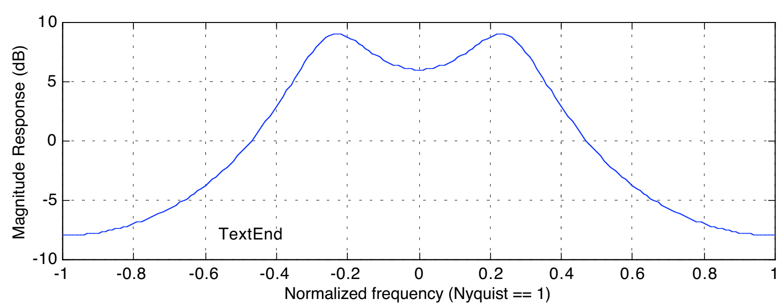
poles: 0.5±j0.5

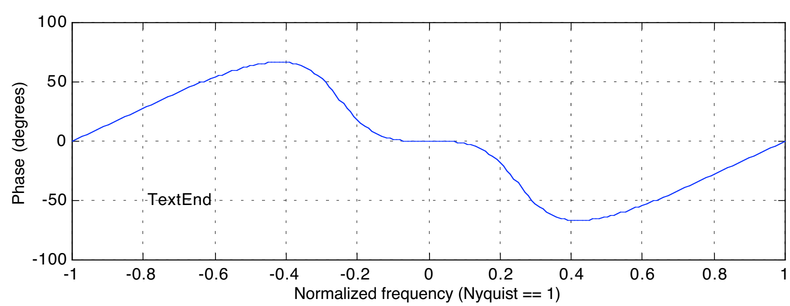
$$H(z)=\frac{z^2}{(z-0.5-j0.5)(z-0.5+j0.5)}$$

$$=\frac{z^2}{z^2-z+0.5}=\frac{1}{1-z^{-1}+0.5z^{-2}}$$

$$H(\hat{\omega})=H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$y[n]=x[n]+y[n-1]-0.5y[n-2]$$





From zp map to freq response:
Go around unit circle from DC to Nyquist. Response goes down near zeros, goes up near poles. Closer poles are to zeros, the weaker their effect.

z-transform

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \Leftrightarrow X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

sequence ⇔ polynomial

n-domain (sample space) z-domain (complex freq space)

$$h[n] \Leftrightarrow H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

impulse response sequence

system function polynomial

H(z) is the z-transform of the impulse response h[n].

LTI

z-transform

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \Leftrightarrow X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

sequence $\Leftrightarrow$ polynomial

n-domain (sample space) z-domain (complex freq space)

Why're we interested?

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$ convolution sum

$$Y(z) = H(z)X(z)$$ polynomial multiplication

z-transform

$$x[n] \Leftrightarrow X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

impulse

$$x[n] = \delta[n] \qquad \Leftrightarrow \qquad X(z) = \sum_{k=-\infty}^{\infty} \delta[k] z^{-k}$$

$$= z^{-0}$$

$$= 1$$

Ex.

$$y[n] = h[n] * x[n]$$
$$= h[n] * \delta[n]$$
$$\Updownarrow \ z$$
$$Y(z) = H(z) \cdot 1$$
$$= H(z)$$
$$\Updownarrow \ z$$
$$y[n] = h[n]$$

H(z) is the z-transform of the impulse response h[n]

LTI

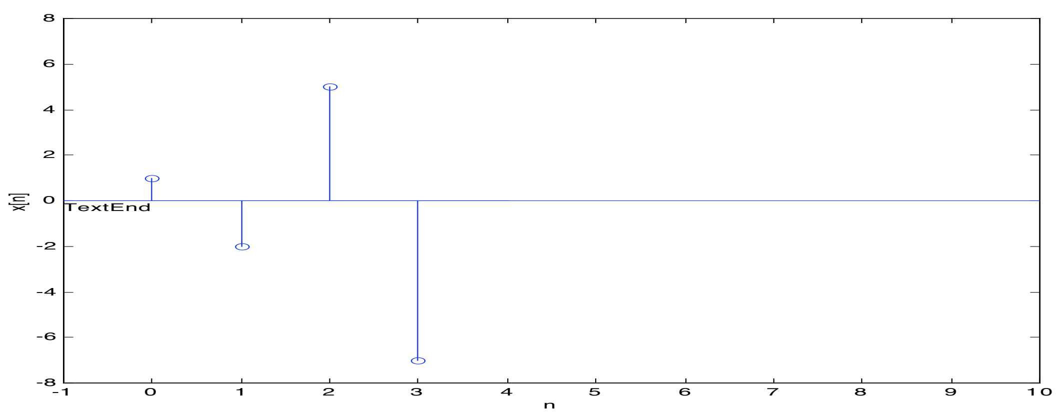
n_0 sample delayed impulse

n_0^{th} power of z^{-1}

$$x[n] = \delta[n - n_0] \quad \Leftrightarrow \quad X(z) = \sum_{k=-\infty}^{\infty} \delta[k - n_0] z^{-k}$$

$$= z^{-n_0} = \left(z^{-1}\right)^{n_0}$$

Ex. finite response



$$x[n] = 1\delta[n] - 2\delta[n-1] + 5\delta[n-2] - 7\delta[n-3]$$

$$\Updownarrow z$$

$$X(z) = 1 - 2z^{-1} + 5z^{-2} - 7z^{-3}$$

Infinite signals

$$x[n]u[n] \Leftrightarrow X(z) = \sum_{k=0}^{\infty} x[k]z^{-k}$$ right-sided signals

$$x[n] = a^n u[n] \quad \Leftrightarrow \quad X(z) = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(az^{-1}\right)^k$$

$$= 1 + az^{-1} + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 \ldots$$

geometric series

Infinite signals

$$x[n] = a^n u[n] \quad \Leftrightarrow \quad X(z) = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= 1 + az^{-1} + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 \ldots$$

geometric series

$$\begin{aligned} X(z) &= 1 + az^{-1} + \left(az^{-1}\right)^2 + \left(az^{-1}\right)^3 \ldots \\ -\, az^{-1}X(z) &= \quad - az^{-1} - \left(az^{-1}\right)^2 - \left(az^{-1}\right)^3 - \left(az^{-1}\right)^4 \ldots \end{aligned}$$

$$\left(1 - az^{-1}\right)X(z) = 1$$

$$X(z) = \frac{1}{1 - az^{-1}} \qquad \left|az^{-1}\right| < 1 \text{ or } |z| > |a| \qquad \text{region of convergence}$$

Infinite series: IIR filter

$$x[n] = a^n u[n] \Leftrightarrow X(z) = \sum_{k=0}^{\infty} a^k z^{-k}$$

x[n]=0 n<0
right sided

$$X(z) = \frac{1}{1 - az^{-1}}$$

$$|z| > |a|$$ region of convergence

Finite series: FIR filter

$$x[n] = a^n \left(u[n-M] - u[n-N]\right) \Leftrightarrow X(z) = \sum_{k=M}^{N-1} a^k z^{-k}$$

$$X(z) = \frac{\left(az^{-1}\right)^M - \left(az^{-1}\right)^N}{1 - az^{-1}}$$

all z region of convergence

$$\lim_{z \to a} X(z) = N - M$$

Ex. $x[n] = 1u[n]$ $X(z) = ?$

$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

$$= \sum_{k=0}^{\infty} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(z^{-1}\right)^{k}$$

let $a = 1$

$$= \frac{1}{1 - z^{-1}}$$

$|z| > |1|$

We know:

$$\sum_{k=0}^{\infty} \left(az^{-1}\right)^{k} = \frac{1}{1 - az^{-1}}$$

$|z| > |a|$

roc

Ex. $x[n] = \cos(\hat{\omega} n) u[n] \qquad X(z) = ?$

$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k} = \sum_{k=0}^{\infty} \cos(\hat{\omega} k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{j\hat{\omega} k} + e^{-j\hat{\omega} k}}{2} \right) z^{-k}$$

$$= \tfrac{1}{2} \sum_{k=0}^{\infty} e^{j\hat{\omega} k} z^{-k} + \tfrac{1}{2} \sum_{k=0}^{\infty} e^{-j\hat{\omega} k} z^{-k}$$

$$= \tfrac{1}{2} \sum_{k=0}^{\infty} \left(e^{j\hat{\omega}} z^{-1} \right)^k + \tfrac{1}{2} \sum_{k=0}^{\infty} \left(e^{-j\hat{\omega}} z^{-1} \right)^k$$

let $a = e^{j\hat{\omega}}$ let $a = e^{-j\hat{\omega}}$

$$= \tfrac{1}{2} \frac{1}{1 - e^{j\hat{\omega}} z^{-1}} + \tfrac{1}{2} \frac{1}{1 - e^{-j\hat{\omega}} z^{-1}}$$

We know:

$$\sum_{k=0}^{\infty} \left(a z^{-1} \right)^k = \frac{1}{1 - a z^{-1}}$$

$$|z| > |a|$$

roc

intersection of roc’s

$$|z| > \left| e^{j\hat{\omega}} \right| \cap \left| e^{-j\hat{\omega}} \right|$$

$$|z| > |1|$$

Ex. $x[n] = \cos(\hat{\omega}n)u[n]$ $X(z) = ?$ Cont.

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k} = \sum_{k=0}^{\infty} \cos(\hat{\omega}k)z^{-k}$$

$$= \tfrac{1}{2}\frac{1}{1-e^{j\hat{\omega}}z^{-1}} + \tfrac{1}{2}\frac{1}{1-e^{-j\hat{\omega}}z^{-1}}$$

$$= \tfrac{1}{2}\frac{z}{z-e^{j\hat{\omega}}} + \tfrac{1}{2}\frac{z}{z-e^{-j\hat{\omega}}}$$

$$= \tfrac{1}{2}z\frac{2z-e^{-j\hat{\omega}}-e^{j\hat{\omega}}}{z^2-\left(e^{-j\hat{\omega}}+e^{j\hat{\omega}}\right)z+1}$$

$$= z\frac{z-\cos(\hat{\omega})}{z^2-2\cos(\hat{\omega})z+1} \qquad |z| > |1|$$

Table of z-transforms

signal	z-transform	ROC
$\delta[n-k]$	z^{-k}	$all \quad z$
$u[n]$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$n^2 u[n]$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\gamma^n \cos(\alpha n) u[n]$	$\frac{z(z-\gamma\cos\alpha)}{z^2-(2\gamma\cos\alpha)z+\gamma^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
$\gamma^n \sin(\alpha n) u[n]$	$\frac{z\gamma\sin\alpha}{z^2-(2\gamma\cos\alpha)z+\gamma^2}$	$\lvert z\rvert > \lvert\gamma\rvert$

z-transform properties

$$a_1 x_1[n] + a_2 x_2[n] \Leftrightarrow a_1 X_1(z) + a_2 X_2(z)$$

linearity/superposition

$$x[n-m] \Leftrightarrow z^{-m} X(z)$$

sample shift

$$h[n] * x[n] \Leftrightarrow H(z)X(z)$$

sample domain convolution
z domain multiplication

$$nx[n] \Leftrightarrow -z\frac{dX(z)}{dz}$$

multiply by a ramp
z domain differentiation

$$a^n x[n] \Leftrightarrow X\left(\tfrac{z}{a}\right)$$

multiply by an exponential
z domain scaling

Ex. $x[n] = na^n u[n]$ $X(z) = ?$

We already know

$$a^n u[n] \Leftrightarrow \frac{1}{1 - az^{-1}} \quad \text{and} \quad ny[n] \Leftrightarrow -z\frac{dY(z)}{dz}$$

$$\frac{d}{dz}\left(\frac{1}{1 - az^{-1}}\right) = \frac{d}{dz}\left(\frac{z}{z - a}\right)$$

$$= \frac{-a}{(z - a)^2} = \frac{-az^{-2}}{\left(1 - az^{-1}\right)^2}$$

$$na^n u[n] \Leftrightarrow -z\frac{az^{-2}}{\left(1 - az^{-1}\right)^2}$$

$$\Leftrightarrow -\frac{az^{-1}}{\left(1 - az^{-1}\right)^2}$$

Next:

Solve difference equation using z-transforms.

Convolution problem in the temporal domain becomes an algebra problem in the z-domain.

Need to know how to convert from z-domain back to temporal domain (inverse z-transform).

$$y[n] = \tfrac{1}{3}x[n] + \tfrac{1}{3}x[n-1] + \tfrac{1}{3}x[n-2]$$
$$h[n] = \tfrac{1}{3}\delta[n] + \tfrac{1}{3}\delta[n-1] + \tfrac{1}{3}\delta[n-2]$$
$$x[n] = \cos(\hat{\omega}n)u[n]$$

$$y[n] = h[n] * x[n] \Leftrightarrow Y(z) = H(z)X(z)$$

$$H(z) = \tfrac{1}{3} + \tfrac{1}{3}z^{-1} + \tfrac{1}{3}z^{-2} = \frac{z^2 + z + 1}{3z^2}$$
$$X(z) = z\frac{z - \cos(\hat{\omega})}{z^2 - 2\cos(\hat{\omega})z + 1}$$

$$Y(z) = \frac{z^2 + z + 1}{3z^2}\left(z\frac{z - \cos(\hat{\omega})}{z^2 - 2\cos(\hat{\omega})z + 1}\right)$$
$$y[n] = \mathcal{Z}^{-1}\left[\frac{z^2 + z + 1}{3z^2}\left(z\frac{z - \cos(\hat{\omega})}{z^2 - 2\cos(\hat{\omega})z + 1}\right)\right]$$