MAS.160 / MAS.510 / MAS.511 Signals, Systems and Information for Media Technology
Fall 2007

MAS160/510 Supplemental Notes
Basis functions and transforms

While it's typical to concentrate on representing signals as weighted sums of complex exponential functions, here we're going to examine the more general case of linear combinations of *any* basis functions.

Let's assume we have a set of N basis functions (where N may or may not be finite) $\phi_0(t), \phi_1(t), ...\phi_{N-1}(t)$. We will represent a function $x(t)$:

$$x(t) \approx \hat{x}(t) = \sum_{i=0}^{N-1} a_i \phi_i(t). \tag{1}$$

Note that for the moment we use the approximation symbol $\approx$ rather than the equal symbol because we haven't spelled out the conditions under which the sum is exact rather than an approximation.

For the moment, assume the basis functions ϕ_i are real-valued. They are said to be *orthogonal* over some interval $(t_1 \leq t \leq t_2)$ if for all j

$$\int_{t_1}^{t_2} \phi_i(t)\phi_j(t) = \begin{cases} 0, & i \neq j \\ \lambda_j, & i = j \end{cases}. \tag{2}$$

For complex-valued basis sets this complicates slightly to

$$\int_{t_1}^{t_2} \phi_i(t)\phi_j^*(t) = \begin{cases} 0, & i \neq j \\ \lambda_j, & i = j \end{cases}. \tag{3}$$

If $\lambda_j = 1$ for all j, then the basis functions are said to be *orthonormal.*

We can use the definition of orthogonality in (3) above to determine the transform coefficients. To compute a_k for a particular basis function ϕ_k we multiply both sides of (1) by ϕ_k^* and integrate:

$$\int_{t_1}^{t_2} \phi_k^* \hat{x}(t) dt = \int_{t_1}^{t_2} \phi_k^* \left[\sum_{i=0}^{N-1} a_i \phi_i(t) \right] dt. \tag{4}$$

Now we reverse the order of summation and integration:

$$\int_{t_1}^{t_2} \phi_k^* \hat{x}(t) dt = \sum_{i=0}^{N-1} a_i \int_{t_1}^{t_2} \phi_k^* \phi_i(t) dt \tag{5}$$

And by (3) we know that the right side is equal to $a_k \lambda_k$, since all other terms of the summation are zero. So the coefficient for the kth basis function is simply

$$a_k = \frac{1}{\lambda_k} \int_{t_1}^{t_2} \phi_k^* \hat{x}(t) dt. \tag{6}$$

In the case where the basis functions are complex exponentials (which are orthonormal), this becomes

$$a_k = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^{-jk\omega_0 t} \hat{x}(t) dt. \tag{7}$$

A common measure of the degree to which the sum of weighted basis functions approximates the desired function $x(t)$ is the *mean squared error*, or MSE:

$$MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x(t) - \sum_{i=0}^{N-1} a_i \phi_i(t) \right]^2 dt. \tag{8}$$

Now if for all functions $x(t)$ in some class the MSE goes to zero in the limit:

$$\lim_{N \to \infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x(t) - \sum_{i=0}^{N-1} a_i \phi_i(t) \right]^2 dt = 0 \tag{9}$$

we say that the series approximation converges in the mean to $x(t)$, and that the set of $\{\phi_i(t)\}$ is *complete* for all the functions $x(t)$ over that interval.

Walsh Functions

An interesting set of orthonormal basis functions is the *Walsh functions*. These are defined only on the range $(0 \le t \le 1)$.

There are a number of conventions for ordering the Walsh basis set. All give the same set of functions, but they generate them in differing orders. Here we will be using *sequency* order, where the index k gives the number of zero-crossings in the interval.

The first sixteen Walsh functions are shown in sequency order in Figure 1. Note that for even k the ends match in sign, while for odd k they are opposite. We can generate these by a *recursion formula*. First we define the zeroth-order basis function:

$$\phi_0(t) = \left\{ \ 1, \quad 0 \le t \le 1 \quad . \right. \tag{10}$$

Now, if there isn't a zero-crossing at $1/2$ we can add one simply by inverting the second half of the interval. So for odd k,

$$\phi_k(t) = \left\{ \begin{array}{ll} \phi_{k-1}(t), & 0 \le t < \frac{1}{2} \\ -\phi_{k-1}(t), & \frac{1}{2} < t \le 1 \end{array} \right. . \tag{11}$$

2

For even k, we can repeat a squeezed version of $\phi_{k/2}$. But if $k/2$ is odd, we need to invert the second copy so the ends match up, or else there would be an extra, unwanted crossing in the middle:

$$\phi_k(t) = \begin{cases} \phi_{k/2}(2t), & 0 \le t < \frac{1}{2} \\ (-1)^{k/2}\phi_{k/2}(2t-1), & \frac{1}{2} < t \le 1 \end{cases}. \tag{12}$$

Example: Represent the function

$$x(t) = \sin(\pi t) \tag{13}$$

with Walsh functions over the interval $(0 \le t \le 1)$. Find the first four coefficients.

To find the coefficient for each basis function $\phi_k(t)$ we solve (6):

$$a_k = \int_0^1 \phi_k(t) sin(\pi t) dt. \tag{14}$$

For $k = 0$, this is simply

$$a_0 = \int_0^1 (1)\sin(\pi t)dt = -\frac{1}{\pi}\cos(\pi t)\Big|_0^1 = 0.637. \tag{15}$$

Similarly,

$$a_1 = \int_0^{1/2} (1)\sin(\pi t)dt + \int_{1/2}^1 (-1)\sin(\pi t)dt = 0 \tag{16}$$

$$a_2 = \int_0^{1/4} (1)\sin(\pi t)dt + \int_{1/4}^{3/4} (-1)\sin(\pi t)dt + \int_{3/4}^1 (1)\sin(\pi t)dt = -0.264 \tag{17}$$

$$a_3 = \int_0^{1/4} (1)\sin(\pi t)dt + \int_{1/4}^{1/2} (-1)\sin(\pi t)dt + \int_{1/2}^{3/4} (1)\sin(\pi t)dt + \int_{3/4}^1 (-1)\sin(\pi t)dt = 0. \tag{18}$$
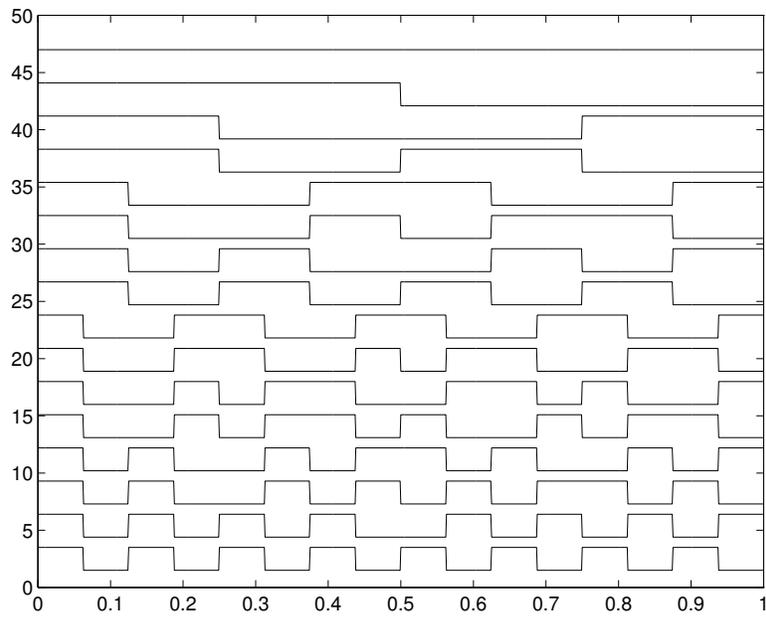
Figure 1: The first 16 Walsh basis functions on the interval from 0 to 1.