Introduction to Numerical Analysis for Engineers

- Fundamentals of Digital Computing
  - Digital Computer Models
  - Convergence, accuracy and stability
  - Number representation
  - Arithmetic operations
  - Recursion algorithms
- Error Analysis
  - Error propagation – numerical stability
  - Error estimation
  - Error cancellation
  - Condition numbers

Floating Number Representation

$$r = mb^e$$

m	Mantissa
b	Base
e	Exponent

Examples

Decimal $\quad 0.00527 = 0.527_{10} \times 10^{-2_{10}}$

Binary $\quad 10.1_2 = 0.101_2 \times 2^{2_{10}} = 0.101_2 \times 2^{10_2}$

Convention

Decimal $\quad 0.1 \leq m < 1.0$

Binary $\quad 0.1_2 = 0.5_{10} \leq m < 1.0$

General $\quad b^{-1} \leq m < b^0$

Max mantissa $\quad 0.11\ldots1 = 0.999999$

Min mantissa $\quad 0.10\ldots0 = 0.5$

Max exponent $\quad 2^7 - 1 = 127 \quad 2^{127} \simeq 1,7 \times 10^{38}$

Min exponent $\quad -2^7 = -128 \quad 2^{-128} \simeq 2.9 \times 10^{-39}$

Error Analysis

Number Representation

Absolute Error

$$\bar{\epsilon} = |\bar{m} - m| \le \frac{1}{2} b^{-t}$$

Relative Error

$$\bar{\alpha} = \frac{|\bar{m} - m| b^e}{|m| b^e} \le \frac{\frac{1}{2} b^{-t}}{b^{-1}} \le \frac{1}{2} b^{1-t}$$

Addition and Subtraction

$$r_1 \pm r_2 = m_1 b^{e_1} \pm m_2 b^{e_2}$$

Shift mantissa of largest number

$$e_1 > e_2$$

Result has exponent of largest number

$$r_1 \pm r_2 = \left(m_1 \pm m_2 b^{e_2 - e_1} \right) b^{e_1} = m b^{e1}$$

Absolute Error

$$\bar{\epsilon} \le \bar{\epsilon_1} + \bar{\epsilon_2}$$

Relative Error

$$\bar{\alpha} = \frac{|\bar{m} - m|}{|m|}$$

Unbounded

Multiplication and Division

$$r_1 \times r_2 = m_1 m_2 b^{e1+e2}$$

$$m = m1m2 < 1$$

$$0.1_2 \times 0.1_2 = 0.01_2$$

Relative Error

$$\bar{\alpha} \le \bar{\alpha_1} + \bar{\alpha_2}$$

Bounded

Error Propagation

Spherical Bessel Functions

Differential Equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}(x^2 - n(n+1))y = 0$$

Solutions

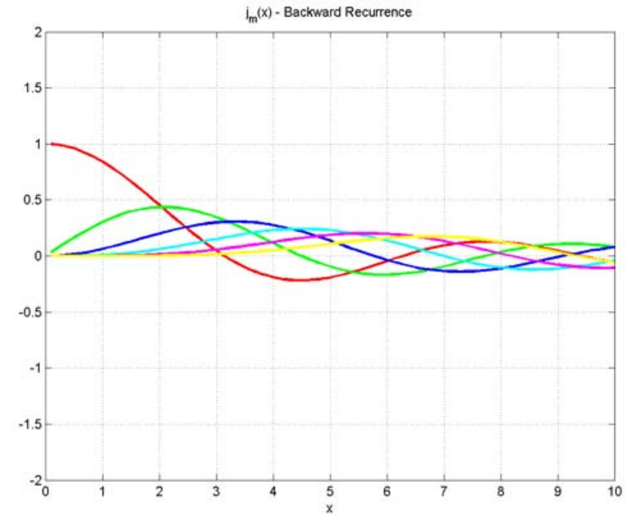
$$j_n(x) y_n(x)$$

n	$j_n(x)$	$y_n(x)$
0	$\frac{\sin x}{x}$	$-\frac{\cos x}{x}$
1	$\frac{\sin x}{x^2} - \frac{\cos x}{x}$	$-\frac{\cos x}{x^2} - \frac{\sin x}{x}$

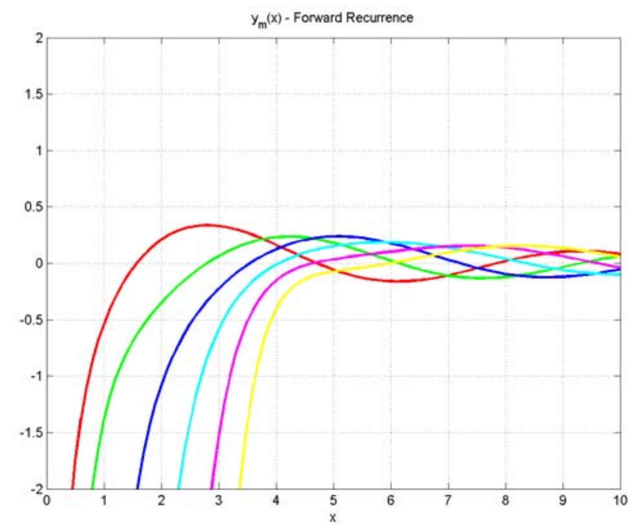
$$j_n(x) \rightarrow 0 \begin{cases} n \rightarrow \infty \\ x \rightarrow 0 \end{cases}$$

$$y_n(x) \rightarrow -\infty \begin{cases} n \rightarrow \infty \\ x \rightarrow 0 \end{cases}$$

$j_n(x)$



$y_n(x)$



Error Propagation

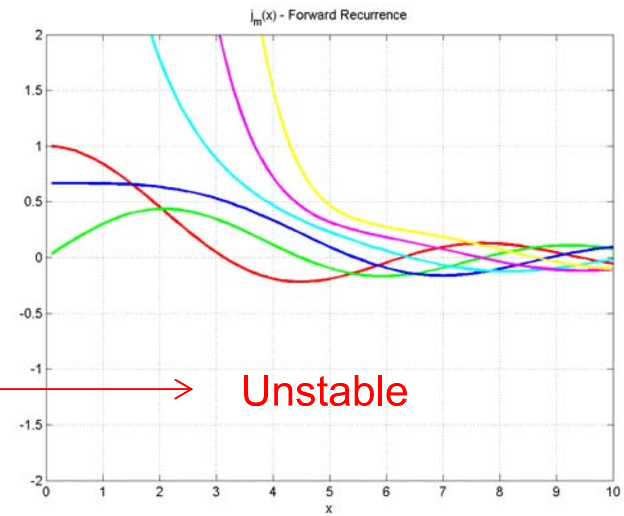
Spherical Bessel Functions

Forward Recurrence

$$j_{n+1}(x) = \frac{2n+1}{x} j_n(x) - j_{n-1}(x)$$

Forward Recurrence

$$\frac{2n+1}{x} j_n(x) \simeq j_{n-1}(x)$$



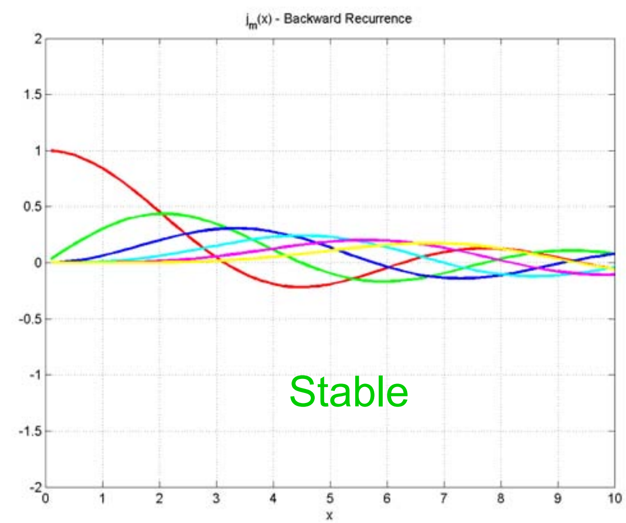
Backward Recurrence

$$j_{n-1}(x) = \frac{2n+1}{x} j_n(x) - j_{n+1}(x)$$

Miller's algorithm

$$j_N(x) = 1\ ,\ \ j_{N+1}(x) = 0\ ,\ \ j_0(x) = \frac{\sin x}{x}$$

$$N \sim x + 20$$



Error Propagation

Euler's Method

Differential Equation

$$\frac{dy}{dx} = f(x,y)\ ,\ y_0 = p$$

Example

$$f(x,y) = x\ (y = x^2/2 + p)$$

Discretization

$$x_n = nh$$

Finite Difference (forward)

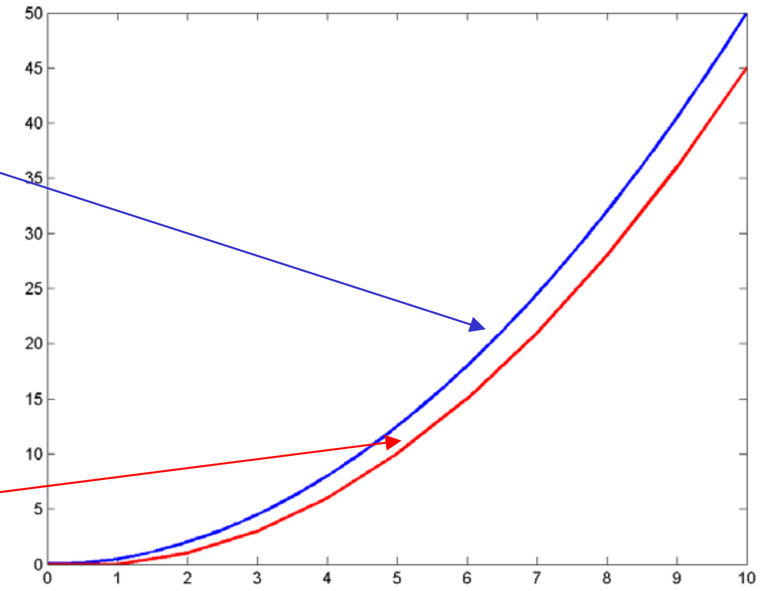
$$\frac{dy}{dx}\Big|_{x=x_n} \simeq \frac{y_{n+1} - y_n}{h}$$

Recurrence

$$y_{n+1} = y_n + hf(nh,y)$$

Central Finite Difference

$$\frac{dy}{dx}\Big|_{x=x_n} \simeq \frac{y_{n+1} - y_{n-1}}{2h}$$



euler.m

Error Propagation

$$y = f(x_1, x_2, \ldots x_n)$$

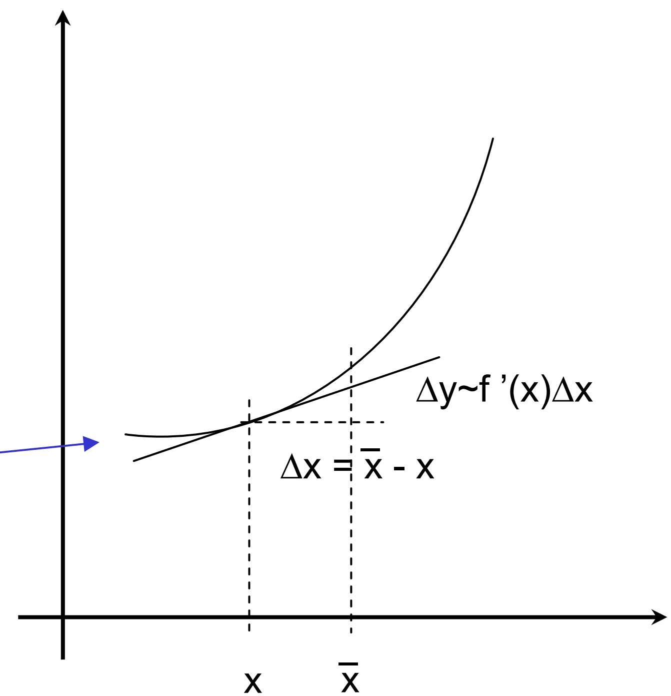
Absolute Errors

$$\epsilon_1, \epsilon_2, \ldots \epsilon_n$$

$$\epsilon_y$$

Function of one variable

$$y = f(x) \quad \bar{y} = f(\bar{x})$$



General Error Propagation Formula

$$\Delta y \simeq \sum_{i=1}^{n} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \Delta x_i$$

$$\epsilon_y \leq \sum_{i=1}^{n} \left| \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \right| |\epsilon_i|$$

Error Propagation Example

Multiplication

$$
\begin{aligned}
y &= x_1 x_2 \\
\Rightarrow \quad \log y &= \log x_1 + \log x_2 \\
\Rightarrow \quad \frac{1}{y}\frac{\partial y}{\partial x_i} &= \frac{1}{x_i} \\
\Rightarrow \quad \frac{\partial y}{\partial x_i} &= \frac{y}{x_i}
\end{aligned}
$$

=>

Error Propagation Formula

$$\left|\frac{\Delta y}{y}\right| \leq \sum_{i=1}^{2} \left|\frac{\Delta x_i}{x_i}\right|$$

$$\alpha_y \leq \sum_{i=1}^{2} \alpha_i$$

$$y = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$$

$$\boxed{\alpha_y \leq \sum_{i=1}^{n} |m_i| \alpha_i}$$

Relative Errors Add for Multiplication

Error Propagation
Expectation of Errors

Addition

$$y = x_1 + x_2 + \cdots + x_n$$

Truncation

$$\Delta x_i = \bar{x}_i - x_i \leq 0$$

Error Expectation

$$E(\Delta x_i) = -b^{-t}/2$$

$$E(\Delta y) = -nb^{-t}/2$$

Rounding

$$E(\Delta x_i) = 0$$

$$E(\Delta y) = -nE(\Delta x_i) = 0$$

Standard Error

$$E(\Delta_s y) \simeq \sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2 \epsilon_i^2}$$

$$y = x_1 + x_2 + \cdots + x_n$$

$$E(\Delta_s y) \simeq \sqrt{\sum_{i=1}^{n} \epsilon_i^2} = \sqrt{n}\epsilon$$

Standard Error better measure of expected errors

Error Propagation
Error Cancellation

Function of one variable

$$y = f(x) = \sqrt{x^2+1} - x + 200; \quad x = 100 \pm 4$$

$$\frac{\partial z_1}{\partial x} = \frac{x}{\sqrt{x^2+1}}, \quad \frac{\partial z_2}{\partial x} = -1$$

$$z_1 = \sqrt{x^2+1} \quad \epsilon_1 = 4$$
$$z_2 = 200 - x \quad \epsilon_2 = 4$$
$$y = z_1 + z_2$$

$$\Delta y \simeq \frac{dz_1}{dx}\Delta x + \frac{dz_2}{dx}\Delta x$$

$$= \left(\frac{x}{\sqrt{x^2+1}} - 1\right)\Delta x \simeq_{x \gg 1} \frac{-1}{x^2}\Delta x$$

Error cancellation

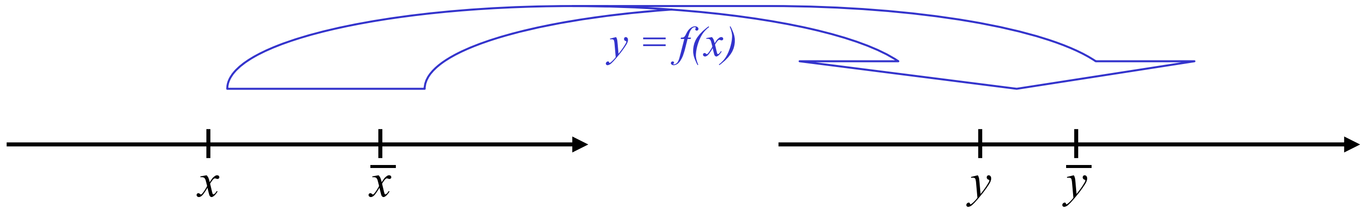
Max. error $E(\Delta y) = 8$

Stand. error $E(\Delta_s y) = 4\sqrt{2} = 5.6$

$$x = 100 \pm 4 \Rightarrow |\Delta y| \leq 4\, 10^{-4} \leq 0.5\, 10^{-3}$$

$$y = 200.005 \pm 0.5\, 10^{-3}$$

Error Propagation
Condition Number



$\bar{x} = x(1 + \alpha)$ $\qquad$ $\bar{y} = y(1 + \beta)$

Problem Condition Number

$$
\begin{aligned}
K_P &= \frac{|\beta|}{|\alpha|} \\
&= \left|\frac{f(\bar{x}) - f(x)}{f(x)}\right| / \left|\frac{\bar{x} - x}{x}\right| \\
&= \left|\frac{f(\bar{x}) - f(x)}{\bar{x} - x}\right| \times \left|\frac{x}{f(x)}\right| \\
&\simeq \left|x\frac{f'(x)}{f(x)}\right|
\end{aligned}
$$

$$K_P \gg 1$$

Problem ill-conditioned

Error cancellation example

$$y = f(x) \;=\; \sqrt{x^2 + 1} - x + 200; \quad x = 100 \pm 4$$

$$K_P \;=\; \left|100\,\frac{-10^{-4}}{200.005}\right| = 0.5\;10^{-4}$$

Well-conditioned problem

Error Propagation
Condition Number

Problem Condition Number

$$y = \sqrt{x^2+1} - x$$

$$x = 100 \Rightarrow y = 0.5 \cdot 10^{-2}$$

$$y' = \frac{x}{\sqrt{x^2+1}} - 1 \simeq -\frac{1}{x^2} = -10^{-4}$$

$$K_P = \left| 100 \frac{-1 \cdot 10^{-4}}{0.5 \cdot 10^{-4}} \right| = 2.0$$

4 Significant Digits

$$\bar{y} = \sqrt{0.1 \cdot 10^5 + 1} - 0.1 \cdot 10^3$$

$$= \sqrt{(0.1000 + 0.00001) \cdot 10^5} - 0.1 \cdot 10^3 = 0$$

$$|\beta| = \left| \frac{\bar{y} - y}{y} \right| = \frac{0.5 \cdot 10^{-2}}{0.5 \cdot 10^{-2}} = 1$$

$$|\alpha| = \left| \frac{\bar{x} - x}{x} \right| \le \frac{1}{2} 10^{1-t} \simeq \frac{1}{2} 10^{-3}$$

Algorithm Condition Number

$$K_A = \frac{|\beta|}{|\alpha|} \simeq 2000$$

K_A is algorith condition number, which may be much larger than the K_P due to limited number representation.

Solution

- Higher precision
- Rewrite algorithm

Well-conditioned Algorithm

$$y = \frac{1}{\sqrt{x^2+1} + x}$$

$$\bar{y} = \frac{1}{0.1 \cdot 10^3 + 0.1 \cdot 10^3} = 0.5 \cdot 10^{-}2$$

$$|\beta| \simeq 0 \Rightarrow K_A \simeq 0 \ll 1$$