2. (15%) The index of refraction in a GRadient INdex (GRIN) medium is given by

$$n(r) = \begin{cases} \sqrt{2 - r^2}, & \text{if } 0 < r < 1; \\ 1, & r \geq 1, \end{cases}$$

where $r = \sqrt{x^2 + z^2}$ is the cylindrical polar coordinate.

2.a) Write down the set of Hamiltonian ray–tracing differential equations for the ray trajectories $\mathrm{d}x/\mathrm{d}s$, $\mathrm{d}z/\mathrm{d}s$ and moments $\mathrm{d}p_x/\mathrm{d}s$, $\mathrm{d}p_z/\mathrm{d}s$, where s is the indexing variable along the rays. Do not attempt to solve the 4×4 set of Hamiltonian equations.

2.b) Prove that, within the disk $r < 1$,

$$\left(\frac{\mathrm{d}p_x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}p_z}{\mathrm{d}s}\right)^2 = \frac{2}{p_x^2 + p_z^2} - 1.$$

2.c) Is the Screen Hamiltonian preserved in this system?

PLEASE TURN OVER

Problem 2:

Given the following GRIN medium,

$$n(r) = \begin{cases} \sqrt{2-r^2} & 0 < r < 1 \\ 1 & r \geq 1 \end{cases} \qquad r = \sqrt{x^2+z^2}$$

a) The Hamiltonian equations are, $(r < 1)$

$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x} = -\frac{1}{2}\frac{2p_x}{\sqrt{p_x^2+p_z^2}} = -\frac{p_x}{n}$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial p_z} = -\frac{1}{2}\frac{2p_z}{\sqrt{p_x^2+p_z^2}} = -\frac{p_z}{n}$$

$$\frac{dp_x}{ds} = -\frac{\partial H}{\partial x} = -\frac{1}{2}\frac{-2x}{n} = \frac{x}{n}$$

$$\frac{dp_z}{ds} = -\frac{\partial H}{\partial z} = -\frac{1}{2}\frac{-2z}{n} = \frac{z}{n}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{ds} = -\dfrac{p_x}{n} = -\dfrac{p_x}{\sqrt{2-x^2-z^2}} \\ \dfrac{dz}{ds} = -\dfrac{p_z}{n} = -\dfrac{p_z}{\sqrt{2-x^2-z^2}} \\ \dfrac{dp_x}{ds} = \dfrac{x}{n} = \dfrac{x}{\sqrt{2-x^2-z^2}} \\ \dfrac{dp_z}{ds} = \dfrac{z}{n} = \dfrac{z}{\sqrt{2-x^2-z^2}} \end{cases}$$

where,

$$H = n(\underline{q}) - [p_x^2+p_z^2]^{1/2} = 0$$

$$= [2-x^2-z^2]^{1/2} - [p_x^2+p_z^2]^{1/2} = 0$$

$$n = \sqrt{p_x^2+p_z^2}$$

b) $$\left(\frac{dp_x}{ds}\right)^2 + \left(\frac{dp_z}{ds}\right)^2 = \left(\frac{x}{n}\right)^2 + \left(\frac{z}{n}\right)^2 = \frac{x^2+z^2}{n^2} = \frac{x^2+z^2}{p_x^2+p_z^2}$$

$$= \frac{r^2}{p_x^2+p_z^2} = \frac{2-n^2}{p_x^2+p_z^2} = \frac{2}{p_x^2+p_z^2} - 1$$

$n^2 = p_x^2 + p_z^2$

2.1

c) Since $\frac{\partial n}{\partial z} \neq 0$, the Screen Hamiltonian is <u>not</u> conserved. This may be verified by direct substitution

$$h = -\sqrt{n^2 - p_x^2} = -\sqrt{2 - x^2 - z^2 - p_x^2}$$

and we see that,

$$\frac{\partial h}{\partial z} = \frac{z}{\sqrt{2 - x^2 - z^2 - p_x^2}} \neq 0$$

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