Quiz II Review: Additional Summary

1. Brief review of the electromagnetic wave equation

The wave equation governs the spatial and temporal variation of a wave as it propagates. The one-dimensional wave equation can be written as:

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 . \tag{1.1}$$

For electromagnetic waves, this 1D wave equation becomes:

$$\frac{\partial^2 \vec{E}}{\partial z^2} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 , \tag{1.2}$$

where the speed of light in an optical medium with dielectric constant ε and magnetic permeability μ (μ=1 for most optical material) is $v = 1/\sqrt{\mu\varepsilon}$. In vacuum, this equals $c = 1/\sqrt{\mu_0\varepsilon_0} = 2.9979\times 10^8 [\mathrm{m/s}]$. While the electric field is most commonly used to represent the amplitude and the phase of the optical field distribution, the magnetic field can do the same because they are both governed by the same wave equation (1.2). In 3D, the wave equation is:

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 . \tag{1.3}$$

The two most useful solutions of (1.3) for a field that propagates along the wave vector $\vec{k}$ are a plane wave and a spherical wave. The length of $\vec{k}$ equals the wave number $k = 2\pi/\lambda$, and the direction vector can be noted as $\hat{k}$. Using the complex vector field notation, we can express:

$$\vec{E}_{plane} = E_0\, \mathrm{e}^{i(\vec{k}\cdot\vec{r} - \omega t + \phi)}\, \hat{k} , \tag{1.4}$$

$$\vec{E}_{point} = \frac{E_0}{r} e^{i(\vec{k}\cdot\vec{r} - \omega t + \phi)} \hat{k} . \tag{1.5}$$

In the input plane, the spherical wave can be expressed as a delta function, and a plane wave as a constant multiplied by a phase factor.

Other solutions to the wave equation also exist, with different shapes of phase-fronts, such as Gaussian beam, Airy beam, Bessel beam, etc.

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2. Approximations and conventions for the E field expression

In order to manipulate them easily, the solutions of the wave equations in (1.4) and (1.5) are modified with some approximations and conventions.

- **Why is the field complex?** All physical observables are purely real, with no imaginary component. In case of the electric field, the actual field value is represented by the real part of the complex notation as $\vec{\mathrm{E}} = \mathrm{Re}\{\vec{E}(x, y, z)\}$, to be strict. The complex form is conventionally accepted without writing the real part notation $\mathrm{Re}\{\}$ every time, and is more popular because other physical observables (such as Intensity) can be computed with minimal trigonometric calculations. Experimentally, what we can actually observe is the intensity, which is real. Using the complex notation of electric field, Intensity is simply obtained from:

$$I = |E_{tot}|^2 = (E_{tot}) \cdot (E_{tot}^{\;*}) , \tag{1.6}$$

  where $E_{tot}^{\;*}$ is the complex conjugate of the total complex electric field.

- **Why do you keep multiplying/dropping a constant factor of $i, \lambda, 2\pi, \cdots$ in front?**
  A constant term multiplied does not alter the intensity pattern at all, and is of minimal importance in optics. Different books multiply different factors by convention usually to cancel out a constant multiplier that comes out during the Fresnel/Fraunhofer diffraction. For example, the Goodman book uses the following notation for a point source:

$$E_{point,\text{Goodman}} = \frac{E_0}{i\lambda r} \mathrm{e}^{i(k \cdot r - \omega t)} , \tag{1.7}$$

  with $i\lambda$ in the denominator. Note that E_0 is already a lumped term for the field amplitude. The physically measured variable is usually the "normalized intensity", which does not depend on convention.

- **Why do you keep dropping e^{ikz}/z?**

  When you are asked to calculate for the field pattern at a screen on the x-y plane, the z value is identical for every point in the plane and therefore is a constant. Since the normalized intensity is never affected by this constant factor, omission is conventionally allowed. This is the same reason why $\mathrm{e}^{-i\omega t}$ can be omitted for a temporally coherent light source.

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3. Fresnel diffraction equation and the paraxial approximation

The most important concept in Diffraction is the Huygens principle, where every point on a wavefront is regarded as a new point source, which interferes with neighboring point sources to produce a diffracted (blurry) image. The Fresnel expression for a point source at (x_0, y_0, z_0) measured after a propagation in the z direction can be used when paraxial approximation is valid.

$$E_{point,\text{Fresnel}}(x,y,z) = \frac{E_0}{i\lambda(z-z_0)} e^{ik\left((z-z_0)+\frac{(x-x_0)^2+(y-y_0)^2}{2(z-z_0)}\right)} . \tag{1.8}$$

The Fresnel diffraction pattern of a transmission function $t(x,y)$ illuminated by $E(x,y;z=0)$, propagated by z and measured at $(x',y';z)$ can be expressed as:

$$E_{\text{Fresnel}}(x',y') = \frac{e^{ikz}}{i\lambda z} \iint E_0(x,y)t(x,y) e^{ik\left(\frac{(x'-x)^2+(y'-y)^2}{2z}\right)} dxdy . \tag{1.9}$$

- **The effect of the paraxial approximation on common functions:**

Truncation of the Taylor series to a second-order approximation.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \cong 1 - \frac{\theta^2}{2} + O(\theta^4)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \cong \theta + O(\theta^3)$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots \cong \theta + O(\theta^3)$$

$$\sec\theta = \cos^{-1}\theta = 1 + \frac{\theta^2}{2} + \frac{5\theta^4}{24} + \cdots \cong 1 + \frac{\theta^2}{2} + O(\theta^4)$$

$$\sqrt{1+\theta^2} = 1 + \frac{\theta^2}{2} - \frac{\theta^4}{8} + \cdots \cong 1 + \frac{\theta^2}{2} + O(x^4)$$

$$\sqrt{R^2+x^2} = R\sqrt{1+\left(\tfrac{x}{R}\right)^2} = R\left[1 + \frac{\left(\frac{x}{R}\right)^2}{2} - \frac{\left(\frac{x}{R}\right)^4}{8} + \cdots\right] \cong R\left[1 + \frac{x^2}{2R^2} + O\left(\left(\tfrac{x}{R}\right)^4\right)\right]$$

$$\sqrt{f^2+x^2} \cong f + \frac{x^2}{2f}$$

Propagation of light through any paraxial optical system can be rigorously derived using the Fresnel diffraction formula.

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4. Fraunhofer Approximation is a scaled Fourier Transform

Fraunhofer diffraction goes one more step from Fresnel diffraction: the quadratic phase factor $\exp[ik\frac{x_0^2+y_0^2}{2(z-z_0)}]$ in (1.8) is omitted to yield:

$$E_{point,\text{Fraunhofer}}(x,y,z)=\frac{E_0}{i\lambda(z-z_0)}\mathrm{e}^{-ik\left(\frac{xx_0+yy_0}{z-z_0}\right)}\mathrm{e}^{ik\left((z-z_0)+\frac{x^2+y^2}{2(z-z_0)}\right)}. \tag{1.10}$$

This omission is justified when $k(x_0^2+y_0^2)_{max}/2 \ll \mathrm{z}-\mathrm{z}_0$ is satisfied. The Fraunhofer diffraction pattern of a transmission function $t(x,y)$ illuminated by $E(x,y;\mathrm{z}=0)$, propagated through z and measured at $(x',y';z)$ can be expressed as:

$$\begin{aligned} E_{\text{Fraunhofer}}(x',y') &= \frac{\mathrm{e}^{ik\left(z+\frac{x'^2+y'^2}{2z}\right)}}{i\lambda z}\iint E_0(x,y)t(x,y)\mathrm{e}^{-ik\left(\frac{xx'+yy'}{z}\right)}dxdy \\ &= \frac{\mathrm{e}^{ikz}}{i\lambda z}\iint E_0(x,y)t(x,y)\mathrm{e}^{-i2\pi\left(x\frac{x'}{\lambda z}+y\frac{y'}{\lambda z}\right)}dxdy \\ &= \frac{\mathrm{e}^{ik\left(z+\frac{x'^2+y'^2}{2z}\right)}}{i\lambda z}FT\left[E_0(x,y)t(x,y)\right]_{f_x=\frac{x'}{\lambda z},f_y=\frac{y'}{\lambda z}} \end{aligned} \tag{1.11}$$

When the Fraunhofer approximation is valid, the resulting field can be simply expressed as a scaled Fourier Transform of the source field, multiplied by the transmission function at the object plane.

$$FT[f(x,y)]=\iint f(x,y)\mathrm{e}^{-i2\pi(xf_x+yf_y)}dxdy\ . \tag{1.12}$$

The Fresnel approximation is also used when transmission through an optical component inside the optical path cancels out the quadratic phase term- for example, from the FFP to the BFP of a spherical lens. Reference for lens derivation can be found in Goodman 5.2.

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5. Reference Problems

Interference, interferometer:

HW3.2, 3.3 Interference

HW4.1 Mach-Zehnder Interferometer

HW4.2 Fabry-Perot Interferometer

HW5.3 Newton's rings

HW6.5 4F system with Biprism

2012 Quiz II. Lloyd Mirror Interferometer

4F system as spatial filter:

HW4.4 FT of an MIT seal with spatial filters: this is a visual example of how a transmission element placed at the Fourier plane of a 4F system can function as a low-pass or high-pass filter, although a 4F system was not mentioned in the problem itself.

HW6.4 4F system with T-shaped filter

HW6.5 4F system with Biprism

Grating

HW5.4 Triangular Grating

HW6.1 Grating Spectrometer