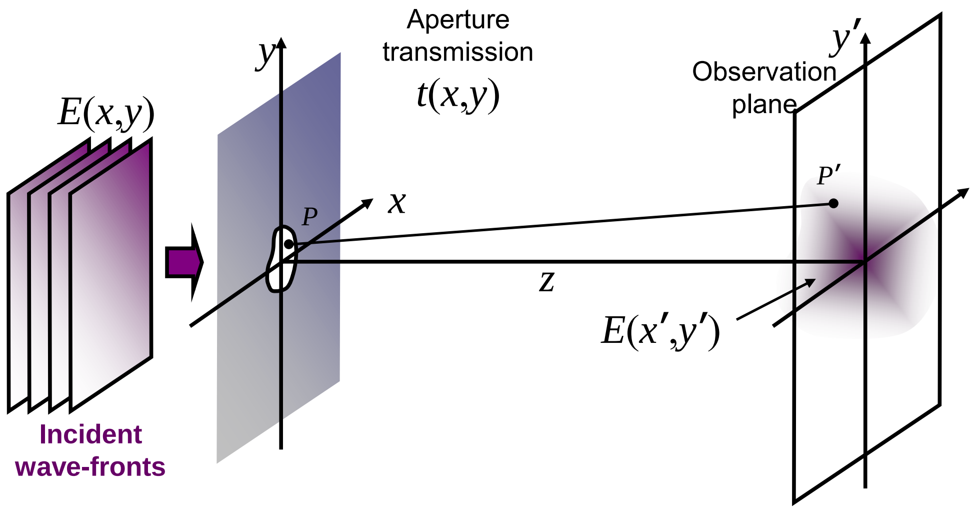
Outline:

A. General Diffraction Geometry
B. Diffraction Using a Lens
C. Diffraction Gratings

A. General Diffraction Geometry



- Goal: What is the electric field $E(x', y')$ measured at a distance z from the plane of the aperture?

The field is affected by 3 contributions:

a) The illumination source $\boxed{E(x, y)}$
b) The transmission function through an aperture $\boxed{t(x, y)}$
c) (Dipole or Huygens source) radiation at location (x, y), propagating to screen (x', y'): $\boxed{h(x'-x, y'-y, z)}$

Note: The step c) is known as Huygens principle: every point along a wave-front emits a spherical wave that interferes with all others. Several scientists, including Kirchhoff, and Bethe-Boukamp (1946) attempted to quantify this idea based on Maxwell Equations, but the strength and orientation of the source in metallic holes at optical wavelength is now a hot topic under debate, since Ebbesen's experiments in 1998.

Here we take the simplest case of sphere waves:

$$h(x' - x, y' - y, z) = \frac{\exp(ikr)}{r} \tag{1}$$

where $$r = \sqrt{(x' - x)^2 + (y' - y)^2 + z^2} \tag{2}$$

The resulting field is then a **convolution** of all the three factors:

$$E(x', y') = \iint h(x' - x, y' - y, z)t(x, y)E(x, y)dxdy \tag{3}$$

- Fraunhoffer diffraction: Far field (z>>x', y', x, y)

$$r = \sqrt{(x' - x)^2 + (y' - y)^2 + z^2}$$

$$r \approx z\left(1 + \frac{(x'-x)^2+(y'-y)^2}{2z^2}\right) \tag{4}$$

$$\exp(ikr) \approx \exp\left(ikz + ik\frac{(x' - x)^2 + (y' - y)^2}{2z}\right)$$

$$\approx \exp(ikz)exp\left(-ik\frac{xx'+yy'}{z}\right)\exp\left(ik\frac{x'^2+y'^2+x^2+y^2}{2z}\right) \tag{5}$$

(*up to here, it is* ***Fresnel condition*** *for diffraction*)

Now if we further assume the quadratic terms are negligible:
(***Fraunhofer condition, difficult to achieve!***)

$$\frac{k\left(x'^2+y'^2\right)}{2z} \ll 1, \quad \frac{k\left(x^2+y^2\right)}{2z} \ll 1 \tag{6}$$

Then we have a set of *plane waves (rays)* launched at x, y:

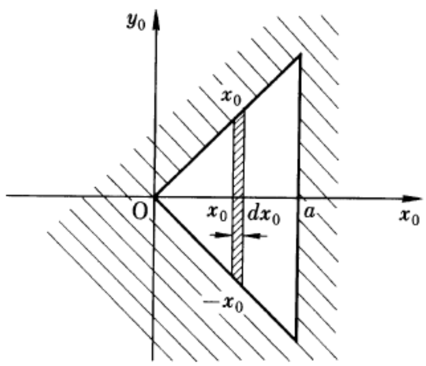
$$exp\left(-ik\frac{xx'+yy'}{z}\right) \approx \exp(-ik(\theta_{x'}x + \theta_{y'}y)) \tag{7}$$

$$\theta_{x'} \approx \frac{x'}{z}, \theta_{y'} \approx \frac{y'}{z}, \text{or } k_x \approx k\frac{x'}{z}, k_y \approx k\frac{y'}{z}$$

$$\boxed{E(x', y') \approx \frac{1}{z}\iint \exp(-ik(\theta_{x'}x + \theta_{y'}y))t(x, y)E(x, y)dxdy} \tag{8}$$

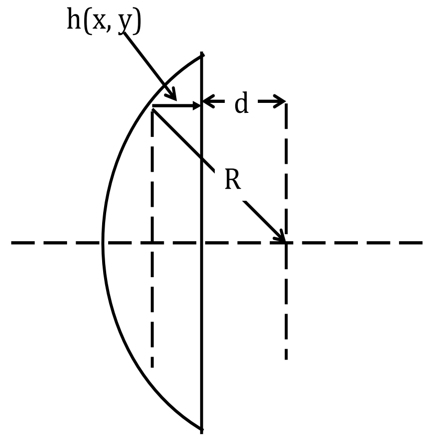
What we measure at the far field is a **Fourier transform** of t(x,y)E(x,y)!

Practice problem: Find the Fraunhofer diffraction pattern of a triangular aperture as shown in the following figure**.** The edges of the triangle are expressed at x=a, y=x, and y=-x, respectively. The screen is placed at $z=z_0$.



Hint: In this case, the aperture along the y direction depends on the position x. So we may integrate first along the y direction, and then along x-direction in the next.

B. Diffraction using a lens



A lens introduces a phase delay proportional to its thickness h, at a given point (x, y):

$$t(x,y) = \exp[ik(n-1)h(x,y)] \quad (9)$$

$$h(x,y) = \sqrt{[R^2 - (x^2+y^2)]} - d \quad (10)$$

In the thin lens limit, we find:

$$h(x,y) \approx R[1-(x^2+y^2)/2R^2] - d \quad (11)$$

$$t(x,y) \approx \exp\{ik(n-1)[R-d-(x^2+y^2)/2R]\} \quad (12)$$

$$t(x,y) \approx \exp[ik(n-1)(R-d)] \times exp[-ik(n-1)(x^2+y^2)/2R] \quad (13)$$

$$t(x,y) \approx \exp[ik(n-1)(R-d)] \times exp\left[-ik\frac{(x^2+y^2)}{2f}\right] \quad (14)$$

$$1/f \equiv (n-1)/R$$
$$\Delta \equiv (n-1)(R-d) \to 0 \text{ (thin lens)}$$

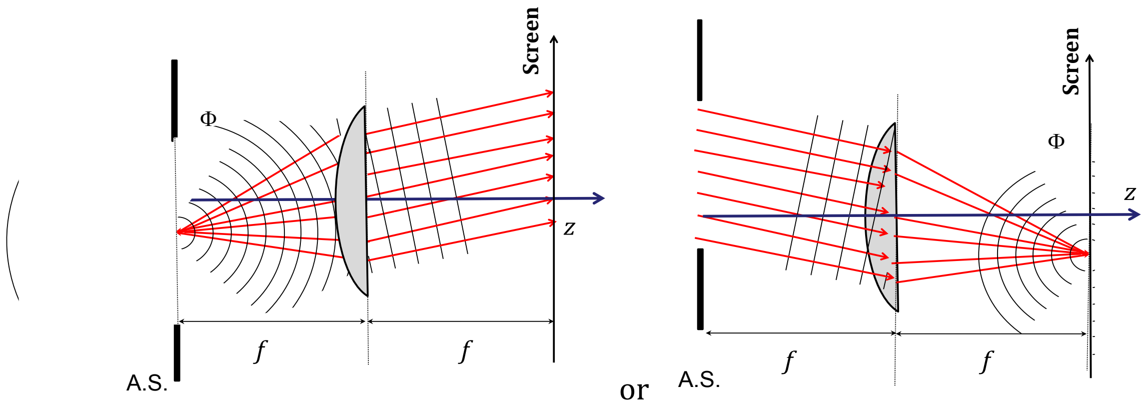
$$\exp(ikr)\, t(x,y) \approx \exp(ik(z+\Delta)) \exp\left(-ik\frac{xx'+yy'}{z}\right)$$
$$exp\left[ik(x^2+y^2)\left(\frac{1}{2z}-\frac{1}{2f}\right)\right]\; exp\left[ik\left(\frac{x'^2+y'^2}{2z}\right)\right] \quad (15)$$

The quadratic term of x^2+y^2 will vanish, provided that:

$$\boxed{\frac{1}{2z}-\frac{1}{2f} = 0, or\ z = f}$$

You can apply the same argument to the quadratic phase term of $(x'^2+y'^2)$ by placing the screen at z=*f* away from the lens.

We'll see the Fourier Transform of $E(x.y)$ (**Fraunhofer diffraction**) by placing the aperture and a screen at the focal planes of a lens, even it is not far away!



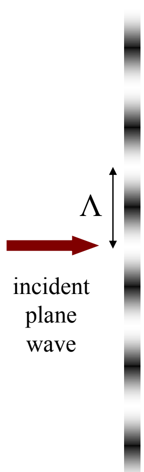
Note: This is consistent with our previous analysis based on ray optics. For a set of diverging ray vectors (x, θ) that emerge from the aperture at the front focal plane, the lens converts the rays to a set of parallel beams. At the back focal plane, we measure a set of new ray vectors (x', θ'):

$$\begin{pmatrix} x' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 0 \end{bmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} \tag{16}$$

In another word, geometric optics addresses the *propagation* (or optimal optical path length) from aperture stop to the screen appropriately. The effect of diffraction is reduced to Huygens principle, or how the "secondary" field emerges right behind the aperture.

C. Diffraction Gratings

A diffraction grating is a slab with a periodic modulation of any sort on one of its surfaces. The modulation can be in the form of transmission, reflection, or the phase delay of a beam.



For example, let's begin with a sinusoidal modulation of the transmission function:

$$t(x,y) = t_0 + t_1 cos\left(\frac{2\pi x}{\Lambda}\right) \tag{17}$$

the Fraunhofer diffracted field is:

$$E(x',y') \approx \frac{\exp(ikz)}{z} \iint \exp(-ik_x x - ik_y y))t(x,y)E(x,y)dxdy \tag{18}$$

What if the periodic modulation of the transmission is not sinusoidal, e.g. a square modulation?

Since it's periodic, we can use a Fourier series for it (or consider a comb function in convolution with the square windows):

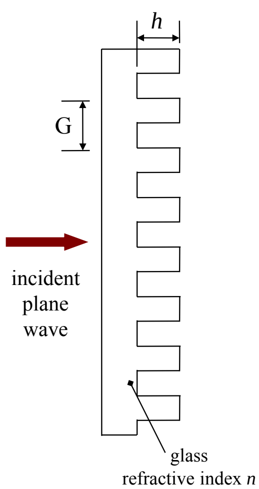
$$t(x) = \sum_{m=0}^{\infty} t_m cos\left(\frac{2\pi m x}{\Lambda}\right) \tag{19}$$

An order of a diffraction grating occurs if:

$$\boxed{k_x = m\frac{2\pi}{\Lambda}}, \text{ or } \Lambda\sin(\theta_m) = m\lambda, \tag{20}$$

where m is an integer. The above equation assumed normal incidence.

- **Phase grating:** In the case for phase grating, we may modulate the beams by varying the phase delay of incident beam through a set of patterns on transparent window.



For example, let's look at a binary grating with modulation of the phase function:

$$t(x) = \exp(ik(n-1)h(x)) \tag{21}$$

$$h(x) = h \times rect\left(\frac{x}{W}\right) \otimes comb\left(\frac{x}{G}\right) << \lambda \tag{22}$$

the 1D Fraunhofer diffracted field is: (assuming E(x)=1)

$$E(x') \approx \int \exp(-ik_x x)\, t(x) dx \tag{23}$$

$$E(x') \approx \int \quad \exp(-ik_x x)\,[1 + ik(n-1)h(x)]dx \tag{24}$$

$$E(k_x) \approx \delta(k_x) + ik(n-1)\int \quad \exp(-ik_x x)\left[h \times rect\left(\frac{x}{W}\right) \otimes comb\left(\frac{x}{G}\right)\right] dx \tag{25}$$

$$E(x') \approx \delta(k_x) + ik(n-1)h\,[W sinc(Wk_x)] comb(\frac{G}{2\pi}k_x) \tag{26}$$