Supplement to Lecture 3–b
Image formation by a composite lens

Consider the system of two lenses shown in Figure 1. A unit-length object is located 5 units to the left of lens L1 (focal length 10 units). L1 is located 5 units to the left of lens L2 (focal length 10 units as well.) We seek to determine the location of the image, and the magnification m_l. We will carry out the derivation using three different methods: repeated application of the lens law, matrix algebra, and exploiting the concept of principal planes (which we have to first locate.)

Applying the lens law repeatedly

This method is described by Figures 2–5. We begin by considering the first lens (L1) in isolation, *i.e.* without L2. L1 will form an *intermediate* image of the object. As usual, we guess that the image is formed at a distance s_1' to the right of L1, and we also guess positive magnification $m_{1,l}$, *i.e.* erect (upright) image.

The imaging system with L1 in isolation is shown in Figure 2. Applying the lens law to this system we obtain

$$\frac{1}{5} + \frac{1}{s_1'} = \frac{1}{10} \Rightarrow s' = -10; \qquad \text{and}$$

$$m_{1,l} = -\frac{s_1'}{5} = -\frac{-10}{5} = +2.$$

Since we found a negative image distance s_1' we conclude that our initial guess about the intermediate image location was incorrect: the intermediate image is located to the left of L1, so it is a *virtual* image. Since we found a positive magnification, we conclude that the intermediate image is *erect*. The correct configuration of object and intermediate image for L1 in isolation is shown in Figure 3.

We now turn to consider L2 in isolation, *i.e.* without L1. The intermediate image formed by L1 acts as intermediate object for L2. Since we found the intermediate image to form 10 units to the left of L1, it is 15 units to the left of L2. The configuration of L2 in isolation with the intermediate object and an initial guess for real and erect image with (positive) magnification $m_{2,l}$ is shown in Figure 4. Applying the lens law to this system we obtain

$$\frac{1}{15} + \frac{1}{s'} = \frac{1}{10} \Rightarrow s' = +30; \qquad \text{and}$$

$$m_{2,l} = -\frac{s'}{15} = -\frac{+30}{15} = -2.$$

Since we found a positive image distance s' we conclude that our initial guess about the intermediate image location was correct: the final image is located to the right of L1, so it is a *real* image. Since we found a negative magnification, we conclude that the final image is *inverted.* The overall magnification is

$$m_l = m_{1,l}m_{2,l} = (+2) \times (-2) = -4.$$

The correct configuration of object and final image is shown in Figure 5.

Using the matrix formulation

The system consists of a cascade of five elements, represented respectively by matrices as follows:

$\mathbf{M}_5$	free–space propagation from object to L1
$\mathbf{M}_4$	lens L1
$\mathbf{M}_3$	free–space propagation from L1 to L2
$\mathbf{M}_2$	lens L2
$\mathbf{M}_1$	free–space propagation from L2 to the image

The matrices are given, according to our paraxial approximation formulae, as follows:

$$\mathbf{M}_5 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}; \qquad \mathbf{M}_4 = \begin{pmatrix} 1 & -1/10 \\ 0 & 1 \end{pmatrix};$$

$$\mathbf{M}_3 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}; \qquad \mathbf{M}_2 = \begin{pmatrix} 1 & -1/10 \\ 0 & 1 \end{pmatrix};$$

$$\text{and} \qquad \mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ s' & 1 \end{pmatrix}.$$

The image distance s' is the unknown that we seek to determine.

The overall system matrix is the product of the five matrices, ordered such that the element corresponding to the last element in the cascade appears first in the product; thus,

$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\mathbf{M}_3\mathbf{M}_4\mathbf{M}_5.$$

Consider a ray departing from the object, as shown in Figure 6. Let us denote the lateral coordinate and angle of departure of that ray as x, α, respectively; and let us denote the lateral coordinate and angle of arrival of that ray at the image plane as x', α', respectively. Since both the object and image are in air (index of refraction $n = 1$), the pairs of lateral and angular ray coordinates are related as

$$\begin{pmatrix} x' \\ \alpha' \end{pmatrix} = \mathbf{M} \begin{pmatrix} x \\ \alpha \end{pmatrix} \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ \alpha \end{pmatrix}$$

The matrix multiplication can be carried out in straightforward fashion (but rather tedious; it helps to note that $\mathbf{M}_2\mathbf{M}_3 = \mathbf{M}_4\mathbf{M}_5$, however.) The result is

$$\begin{pmatrix} x' \\ \alpha' \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{4} & -\dfrac{3}{20} \\ -\dfrac{s'}{4} + \dfrac{15}{2} & -\dfrac{3s'}{20} + \dfrac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ \alpha \end{pmatrix}.$$

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As in the case of the simple thin lens, the imaging condition is

$$\frac{\partial x'}{\partial \alpha} = 0 \Rightarrow M_{21} = 0,$$

from which we find $s' = +30$. To find the lateral magnification, we observe that at the image plane the second row of the matrix equation is rewritten as

$$x' = \left(M_{21}\alpha + M_{22}x \right) \Big|_{M_{21}=0} \Rightarrow x' = M_{22} \Big|_{M_{21}=0} x.$$

So the lateral magnification is

$$m_l = M_{22} \Big|_{M_{21}=0} = \left(-\frac{3s'}{20} + \frac{1}{2} \right) \Big|_{M_{21}=0 \Rightarrow s'=30} \Rightarrow m_l = -4.$$

Using the principal planes

First we need to locate the principal planes (that's some extra work, so the method of using the principal planes is efficient if we already know where they are.)

Locating the principal planes using repeated application of the lens law

We begin with the 2^{nd}PP. We consider a parallel ray bundle (object at infinity) and choose one ray from the bundle at height, say, x, as shown in Figure 7. Our objective is to find where the optical system will focus this ray (intersect it with the optical axis; if the intersection location is independent of x, then all the rays from infinity are in focus at that point.) If we extend the focusing ray backwards till it intersects the incoming ray, then the plane of intersection is the 2^{nd}PP.

Once again, we first consider L1 in isolation. L1 by itself would focus the incoming rays from infinity to a distance equal to one focal length to the right of the lens, *i.e.* 10 units. We refer to this as "intrremediate focus." Then we consider L2 in isolation. Since 10 units to the right of L1 is 5 units to the right of L2, the intermediate focus of L1 acts as a virtual object for L2, as shown in Figure 9. The physical meaning of the virtual object is that the ray bundle coming towards L2 is already convergent, so the point of intersection of the rays is to the right of L2 (or, if you wish, the rays meet at a sink rather than a source.)

To find where the image of the virtual object to L2 in isolation is located, we apply the lens law with a virtual object (negative object distance):

$$\frac{1}{-5} + \frac{1}{s'} = \frac{1}{10} \Rightarrow s' = \frac{10}{3}.$$

This is the Back Focal Point (BFP), *i.e.* the location where the combination of L1&L2 focuses an incoming ray bundle from infinity. The distance between the last element in the system (L2 in this case) and the BFP is the Back Focal Length (BFL). So BFL $= 10/3$ distance units in this system.

To locate the principal plane, we also need to find the angle of intersection with the optical axis of a ray that arrived at infinity at a height x. This is done in two steps.

First, we notice that when L1 is in isolation, the angle α' of the focusing ray with the optical axis (Figure 8) is

$$\alpha' = -\frac{x}{10}.$$

Next we calculate the angular magnification of the optical system of L2 in isolation, with the intermediate virtual object as shown in Figure 9. We find

$$m_a = -\frac{s}{s'} = -\frac{-5}{10/3} = \frac{3}{2}.$$

The positive angular magnification indicates that an incoming convergent ray bundle will become even more strongly convergent after passing through L2. From that result, we find

$$\frac{\alpha'}{\alpha} = \frac{3}{2} \Rightarrow \alpha' = -\frac{3x}{20}.$$

From Figure 7 we then have

$$-\frac{x}{\text{EFL}} = \alpha' \Rightarrow \text{EFL} = \frac{20}{3}.$$

Since the BFL is located $10/3$ to the right of L2, the 2^{nd}PP is located $10/3$ to the left of L2.

To find the 1^{st}PP, in principle we would have to flip the optical system from left to right and repeat the above procedure. However, here we notice the obvious symmetry, so we conclude that the 1^{st}PP is $10/3$ units to the right of L1.

Locating the principal planes using matrix algebra

Consider again Figure 7. The vector description of the parallel ray bundle arriving to the left of L2 from a point object at infinity is

$$\begin{pmatrix} \alpha_{\text{in}} \\ x_{\text{in}} \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix},$$

where x is the lateral coordinate (height, measured positively from the optical axis) of an arbitrary ray. To find the 2nd principal plane, we must first locate the back focal plane, $i.e.$ where the parallel ray bundle comes to a focus, or equivalently the location where the lateral coordinate of all incoming rays becomes $x_{\text{out}} = 0$. Let us denote as ξ the distance to the right of L2 of the back focal plane (that is, ξ is the same Back Focal Length (BFL) that we mentioned in the previous method.) To find ξ, we can apply the familiar matrix cascade as follows:

$$\begin{pmatrix} \alpha_{\text{out}} \\ x_{\text{out}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/10 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/10 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{\text{in}} \\ x_{\text{in}} \end{pmatrix}$$

After multiplying the square matrices and substituting $\alpha_{\text{in}} = 0$ (for an incoming parallel ray bundle) and $x_{\text{out}} = 0$ (focusing condition for the incoming parallel ray bundle) we find

$$\begin{pmatrix} \alpha' \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} & -\dfrac{3}{20} \\ \dfrac{\xi}{2} + 5 & -\dfrac{3\xi}{20} + \dfrac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}$$

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(Note: α_{out} is the same angle α' shown in Figures 7 and 9.) From the second row of the above matrix equation we obtain

$$-\frac{3\xi}{20} + \frac{1}{2} = 0 \Rightarrow \xi = \frac{10}{3} = \text{BFL}.$$

Now we can locate the 2nd principal plane in one of two possible ways: First, we can use the fact that the "12" element of the composite matrix we computed above equals $-1/\text{EFL}$; therefore, $\text{EFL} = 20/3$. The 2nd principal plane is located one EFL behind (*i.e.*, to the left of) the BFP (please make sure not to be confused by the acronyms!!). Since the BFP is $10/3$ distance units to the right of L2, $20/3$ units behind the BFP would place the 2nd principal plane $10/3$ distance units to the left of L2. (That's gratifyingly the same 2nd principal plane location we calculated with the repeated application of lens law method.)

Alternatively, we can use the fact that the angle of approach of the focusing ray towards the optical axis is

$$\alpha_{\text{out}} = -\frac{3x}{20}.$$

(This follows from the first row of the above matrix equation.) Using Figure 7 again, we have

$$-\frac{x}{\text{EFL}} = \alpha' = -\frac{3x}{20} \Rightarrow \text{EFL} = \frac{3}{20}.$$

To find the 1st principal plane we can flip the optical system from left to right and repeat the same procedure. The result would be identical, since this optical system is symmetric, as we've already pointed out.

Using the principal planes to find the image location and magnification

Now that we know the locations of the principal planes, we can apply the lens law to the composite system, except now we use as object distance the distance of the object from the 1^{st}PP, as image distance we use the (unknown) distance of the image to the 2^{nd}PP, and instead of focal length we use the EFL. The two distances are denoted as s_o and s_o', respectively, in Figure 10. Applying the lens law, we obtain

$$\frac{1}{s_o} + \frac{1}{s_o'} = \frac{1}{\text{EFL}} \Rightarrow \frac{1}{5 + \dfrac{10}{3}} + \frac{1}{s' + \dfrac{10}{3}} = \frac{3}{20} \Rightarrow s' = 30.$$

We can also obtain the overall lateral magnification of the system as

$$m_l = -\frac{s_o'}{s_o} = -\frac{s' + \dfrac{10}{3}}{5 + \dfrac{10}{3}} = -4.$$

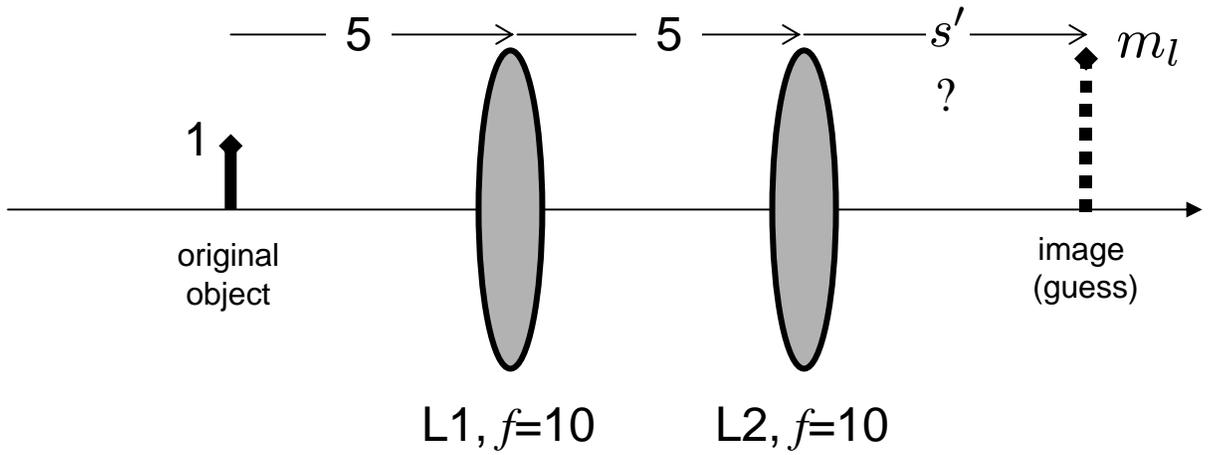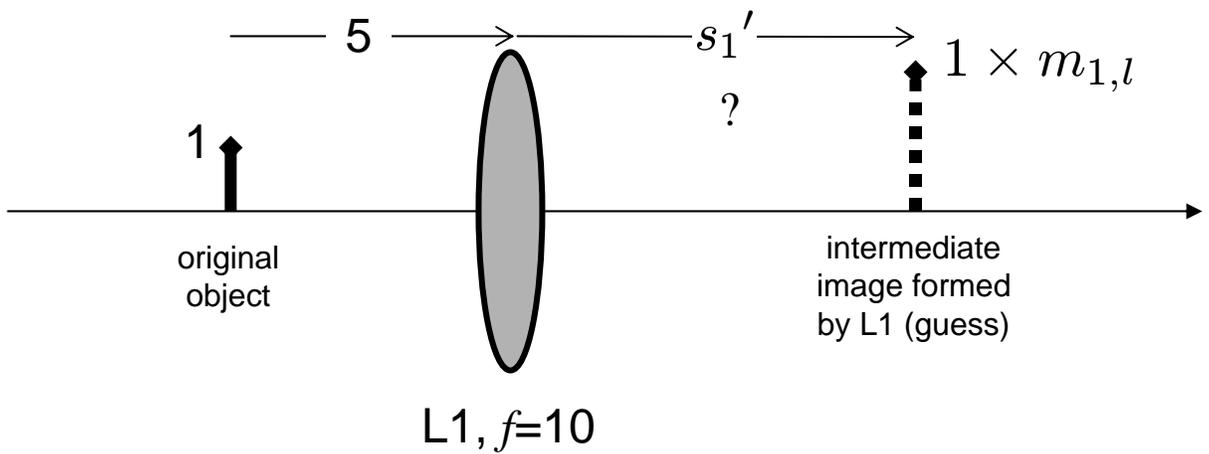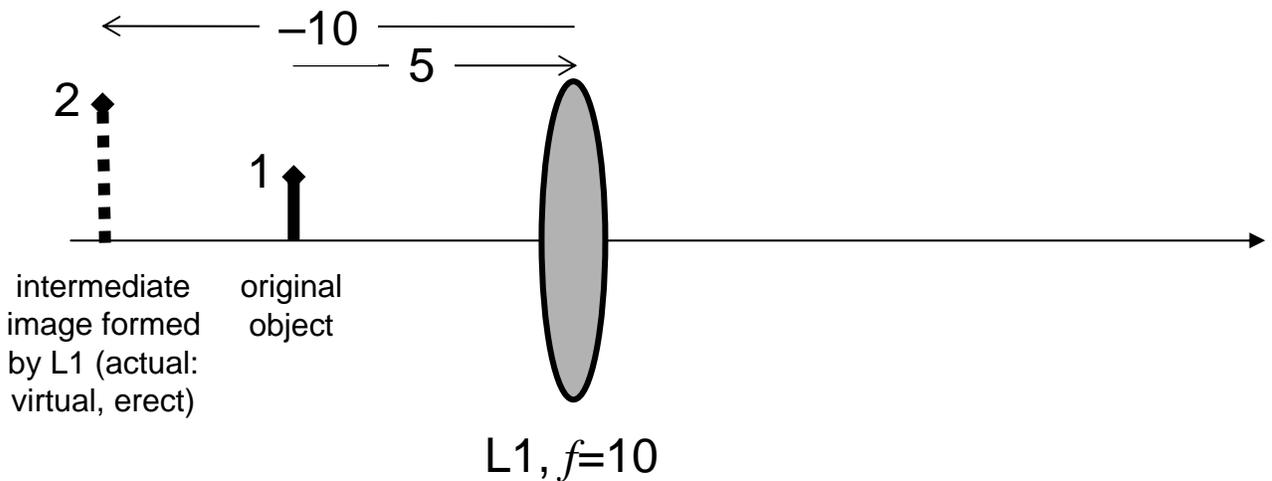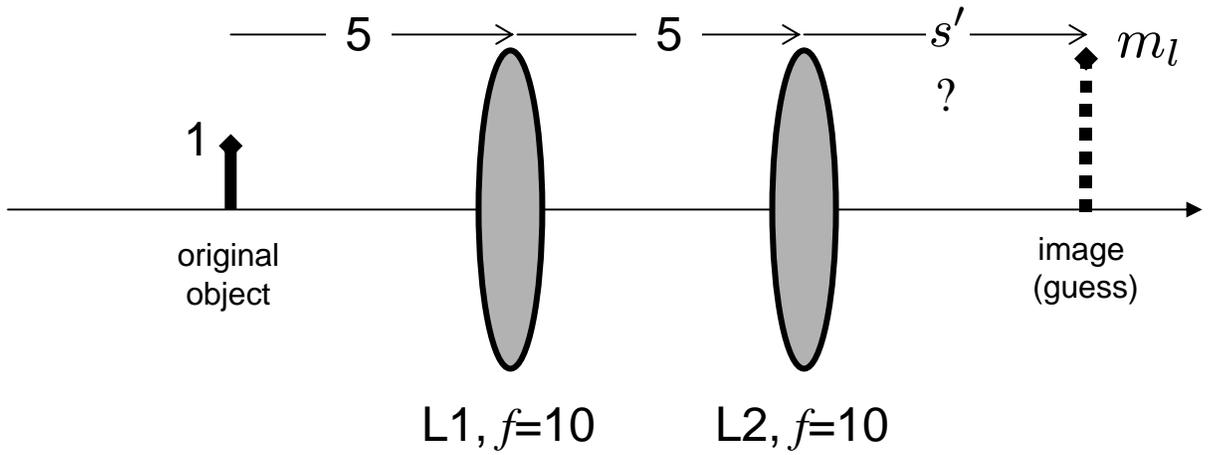Not surprisingly, we arrived at the same results as with the previous methods.
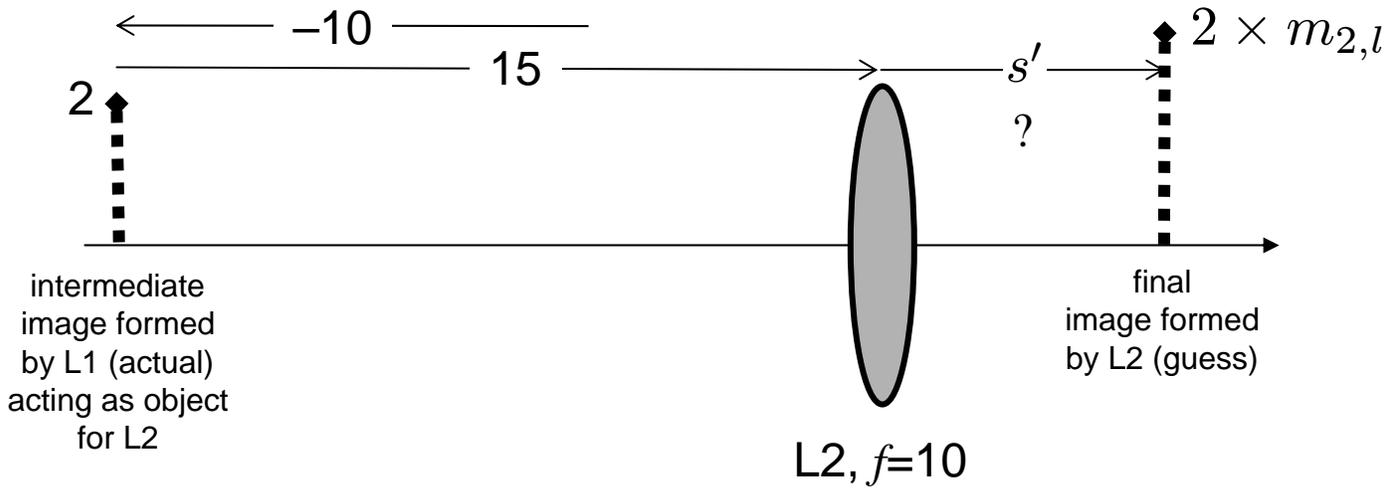
Figure 1

Figure 2

Figure 3

Figure 1 (repeated)

Figure 4
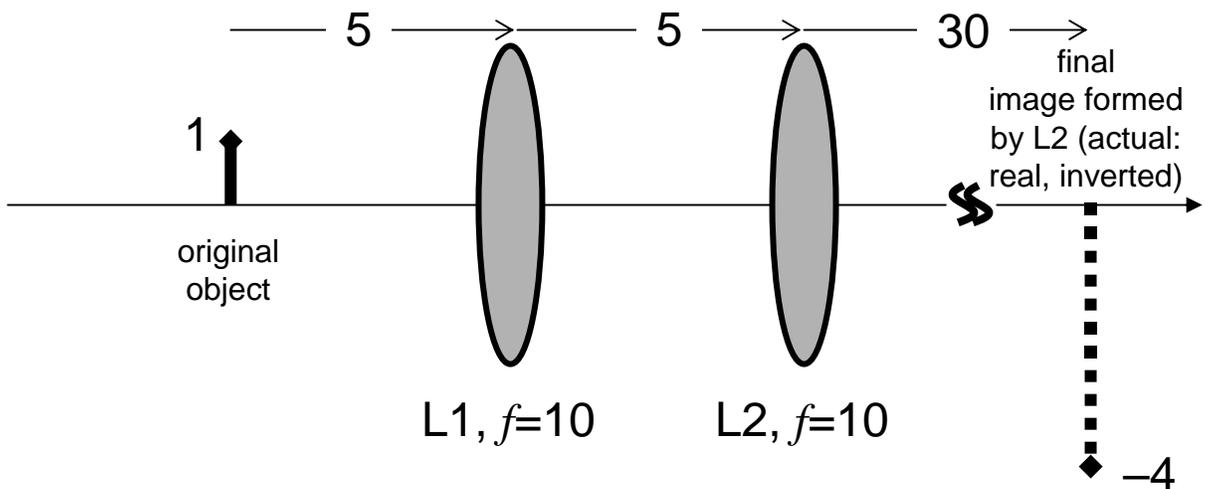

Figure 5
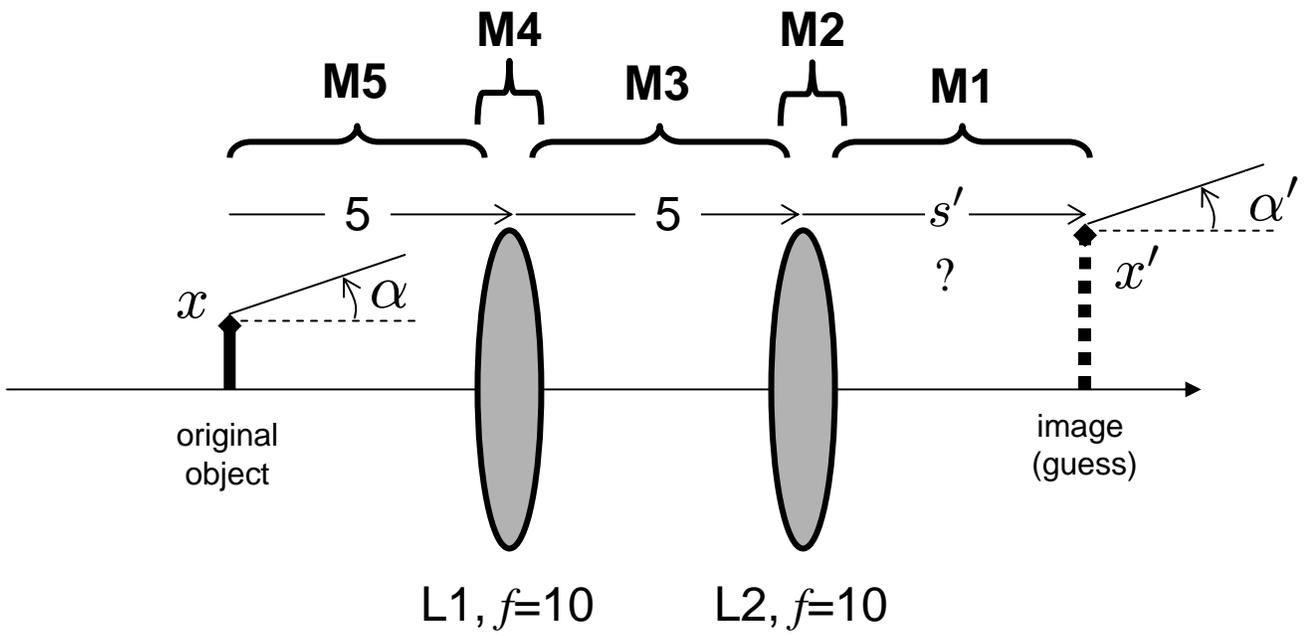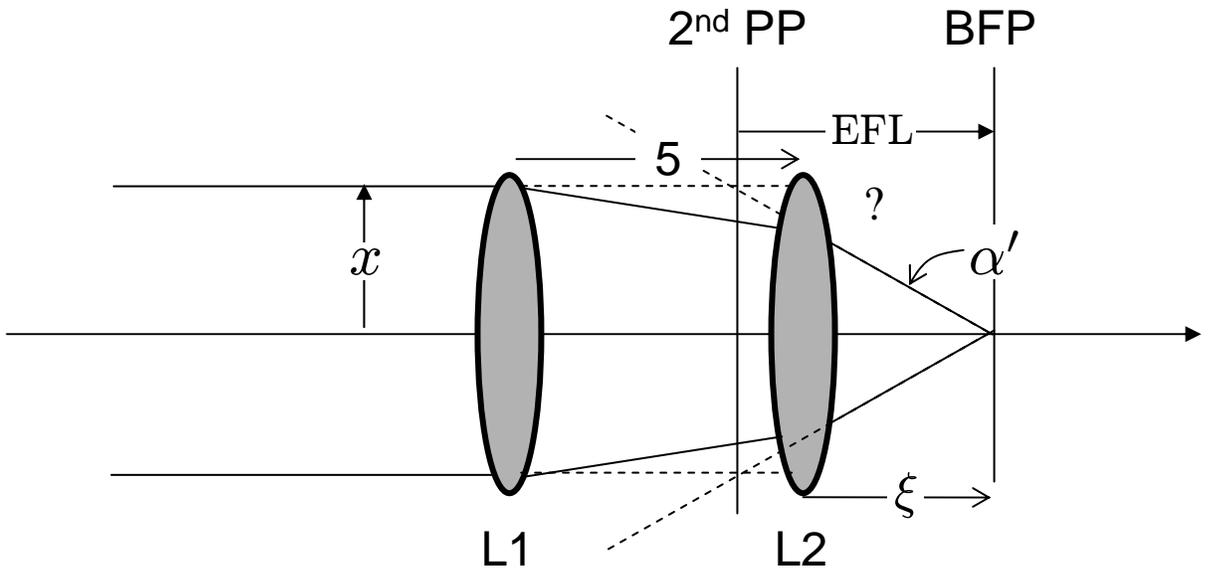
Figure 6

Figure 7

10

x

α

image of object
at infinity formed
by L1
(intermediate object)

L1

Figure 8

$s = -5$

x

α' α

L2

s'

?

image of object
at infinity formed
by L1 acting as
virtual object
For L2

Figure 9

(real) image of the
virtual intermediate
object

1st PP

2nd PP

10/3

10/3

5

5

s'
?

x'

x

original
object

image
(guess)

s_o

?
s_o'

L1

L2
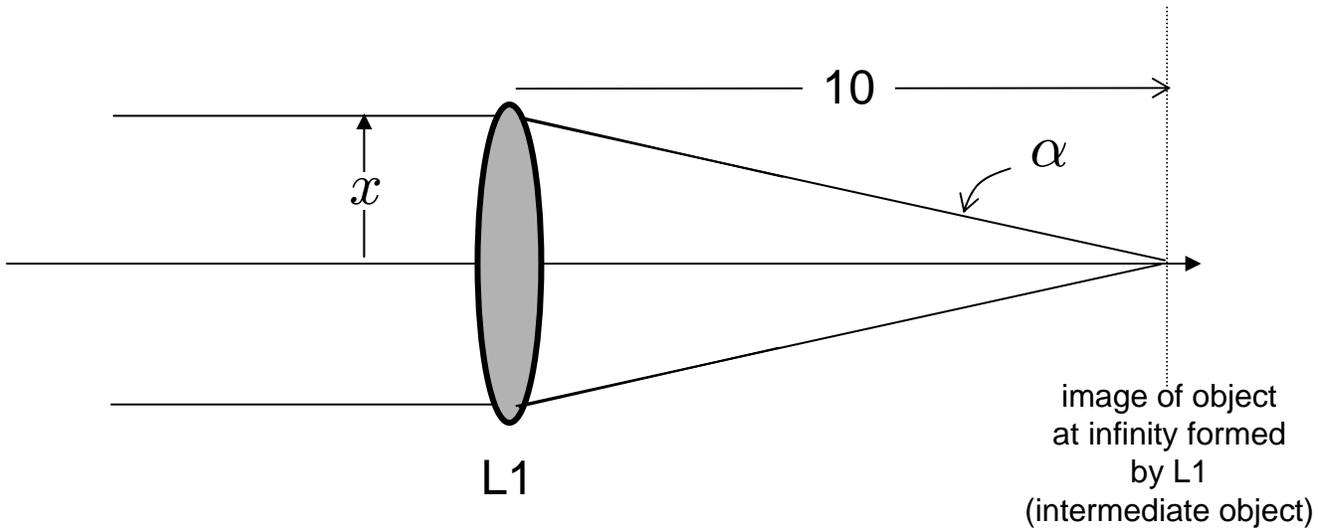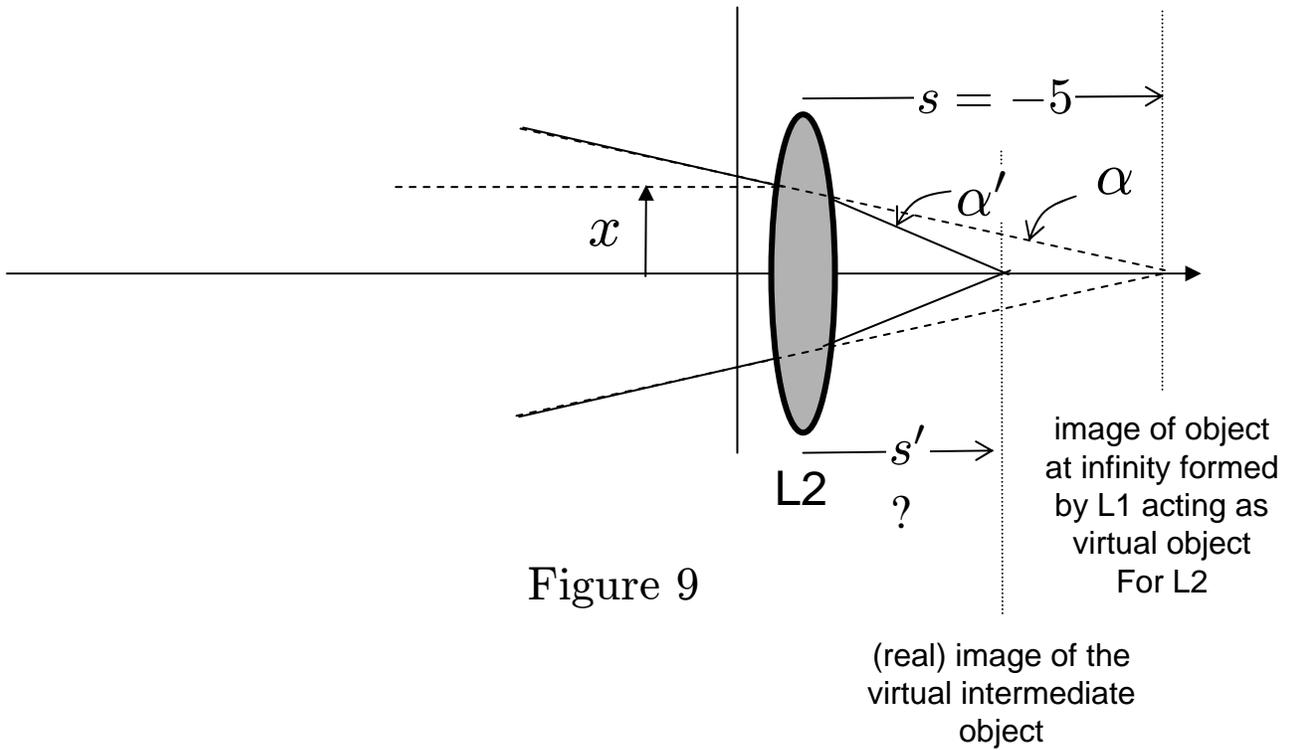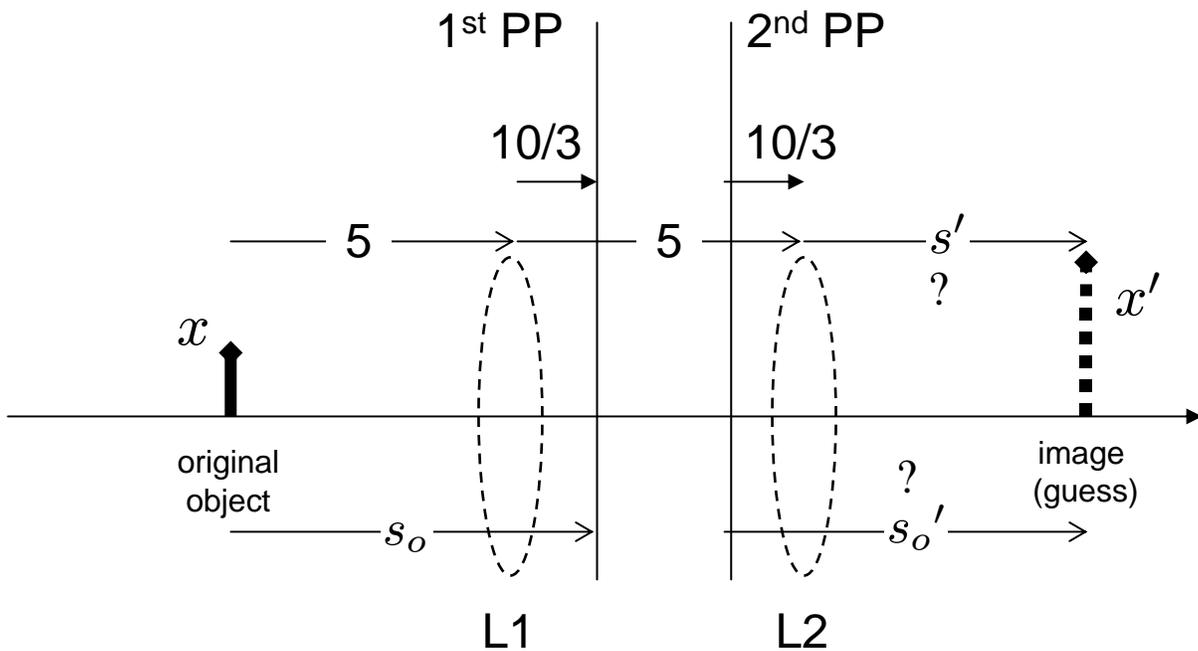
Figure 10

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Spring 2009