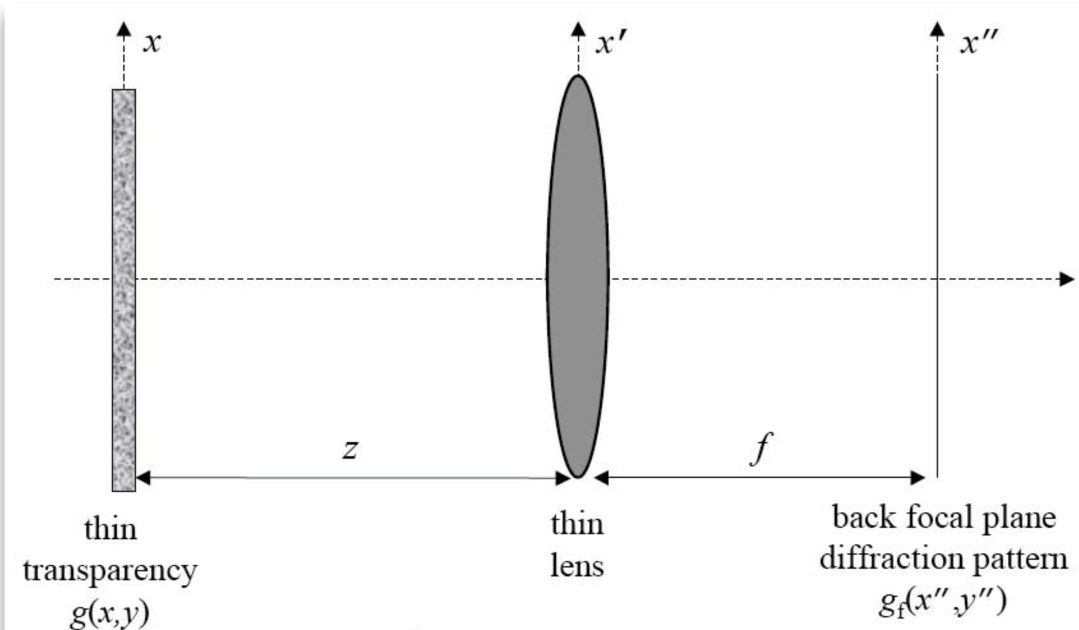
Fourier transforming property of lenses



Field before lens $g_{\text{lens}-}(x') = \int g(x)\exp\left\{i\pi\frac{(x'-x)^2}{\lambda z}\right\}\mathrm{d}x$

Field after lens $g_{\text{lens}+}(x') = g_{\text{lens}-}(x')\exp\left\{-i\pi\frac{x'^2}{\lambda f}\right\}$

Field at back f.p. $g_{\mathrm{f}}(x'') = \int g_{\text{lens}+}(x')\exp\left\{i\pi\frac{(x''-x')^2}{\lambda f}\right\}\mathrm{d}x'$

1D calculation

$$g_{\mathrm{f}}(x'') = \exp\left\{i\pi\frac{x''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}\int g(x)\exp\left\{-i2\pi\frac{xx''}{\lambda f}\right\}\mathrm{d}x$$

2D version

$$g_{\mathrm{f}}(x'',y'') = \exp\left\{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}\iint g(x,y)\exp\left\{-i2\pi\frac{xx''+yy''}{\lambda f}\right\}\mathrm{d}x\,\mathrm{d}y$$

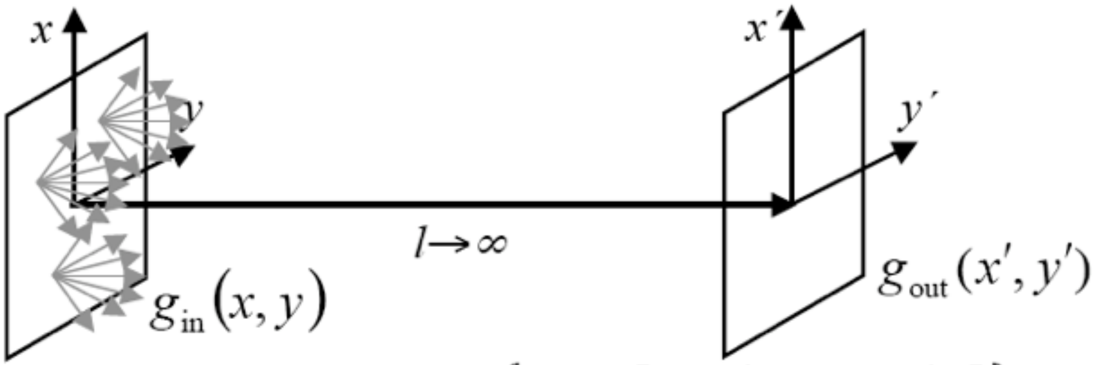
$$g_{\mathrm{f}}(x'',y'') = \exp\left\{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}\iint g(x,y)\exp\left\{-i2\pi\frac{xx''+yy''}{\lambda f}\right\}\mathrm{d}x\,\mathrm{d}y$$

$$\therefore\; g_{\mathrm{f}}(x'',y'') = \underbrace{\exp\left\{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}}_{\text{spherical wave-front}}\;\underbrace{G\left(\frac{x''}{\lambda f},\frac{y''}{\lambda f}\right)}_{\text{Fourier transform of } g(x,y)}$$

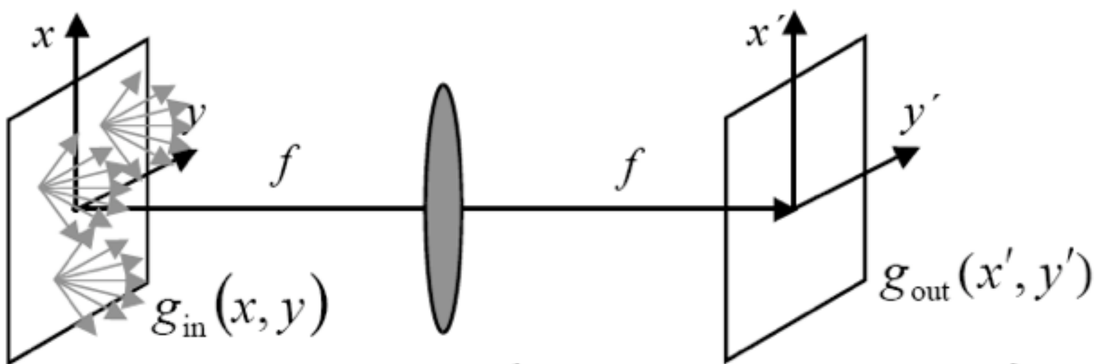
spherical wave-front

Fourier transform of $g(x,y)$

Fourier transform by far field propagation *or* lens

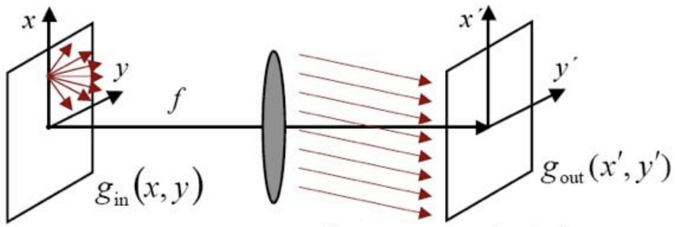


$$g_{\text{out}}(x',y';l) \propto \iint g_{\text{in}}(x,y)\exp\left\{-i2\pi\left[x\left(\frac{x'}{\lambda l}\right)+y\left(\frac{y'}{\lambda l}\right)\right]\right\}\mathrm{d}x\mathrm{d}y$$



$$g_{\text{out}}(x',y';f) \propto \iint g_{\text{in}}(x,y)\exp\left\{-i2\pi\left[x\left(\frac{x'}{\lambda f}\right)+y\left(\frac{y'}{\lambda f}\right)\right]\right\}\mathrm{d}x\mathrm{d}y$$

Spherical-plane wave duality



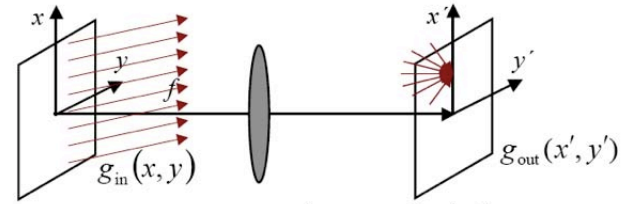
point source at (x,y)
amplitude $g_{in}(x,y)$

plane wave oriented
towards $\left(-\frac{x}{\lambda f}, -\frac{y}{\lambda f}\right)$

... a superposition ... of plane waves corresponding to point sources in the object

each output coordinate (x',y') receives ...

$$g_{out}(x',y') \propto \iint g_{in}(x,y) \exp\left\{ i2\pi \left[\left(-\frac{x}{\lambda f}\right) x' + \left(-\frac{y}{\lambda f}\right) y' \right] \right\} dxdy$$



a plane wave departing
from the transparency
at angle (θ_x, θ_y) has amplitude
equal to the Fourier coefficient
at frequency $(\theta_x/\lambda, \theta_y/\lambda)$ of $g_{in}(x,y)$

produces a spherical wave converging
towards $\left(\frac{\theta_x}{\lambda} \times (\lambda f), \frac{\theta_y}{\lambda} \times (\lambda f)\right) = (\theta_x f, \theta_y f)$

each output coordinate
(x',y') receives amplitude equal
to that of the corresponding
Fourier component

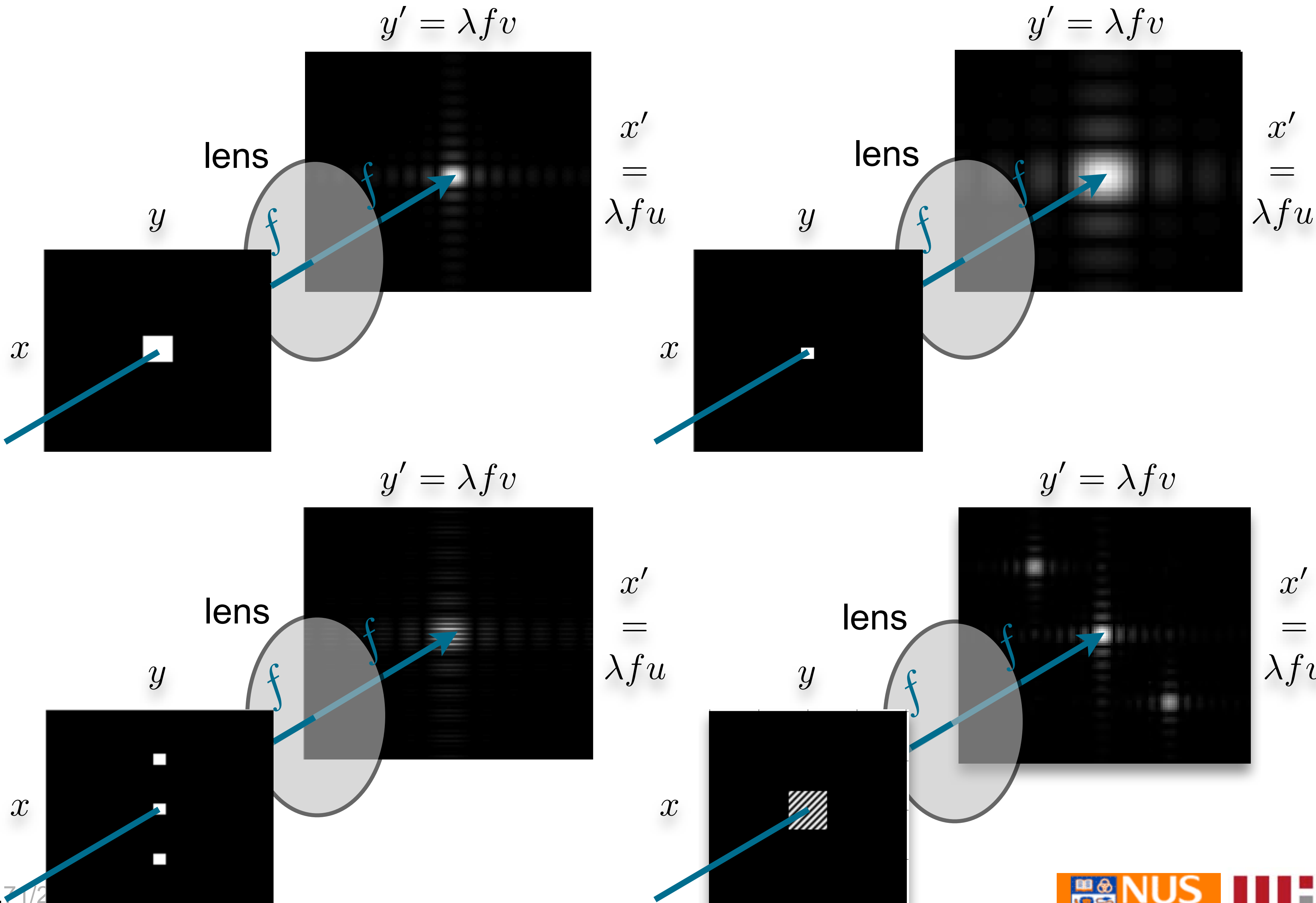
$$g_{out}(x',y') \propto \iint g_{in}(x,y) \exp\left\{ -i2\pi \left[x\left(\frac{x'}{\lambda f}\right) + y\left(\frac{y'}{\lambda f}\right) \right] \right\} dxdy$$

The two pictures above are interpretations of the same physical phenomenon.

On the left, the transparency is interpreted in the Huygens sense as a superposition of "spherical wavelets." Each spherical wavelet is collimated by the lens and contributes to the output a plane wave, propagating at the appropriate angle (scaled by f.)

On the right, the transparency is interpreted in the Fourier sense as a superposition of plane waves ("angular" or "spatial frequencies.") Each plane wave is transformed to a converging spherical wave by the lens and contributes to the output, f to the right of the lens, a point image that carries all the energy that departed from the input at the corresponding spatial frequency.

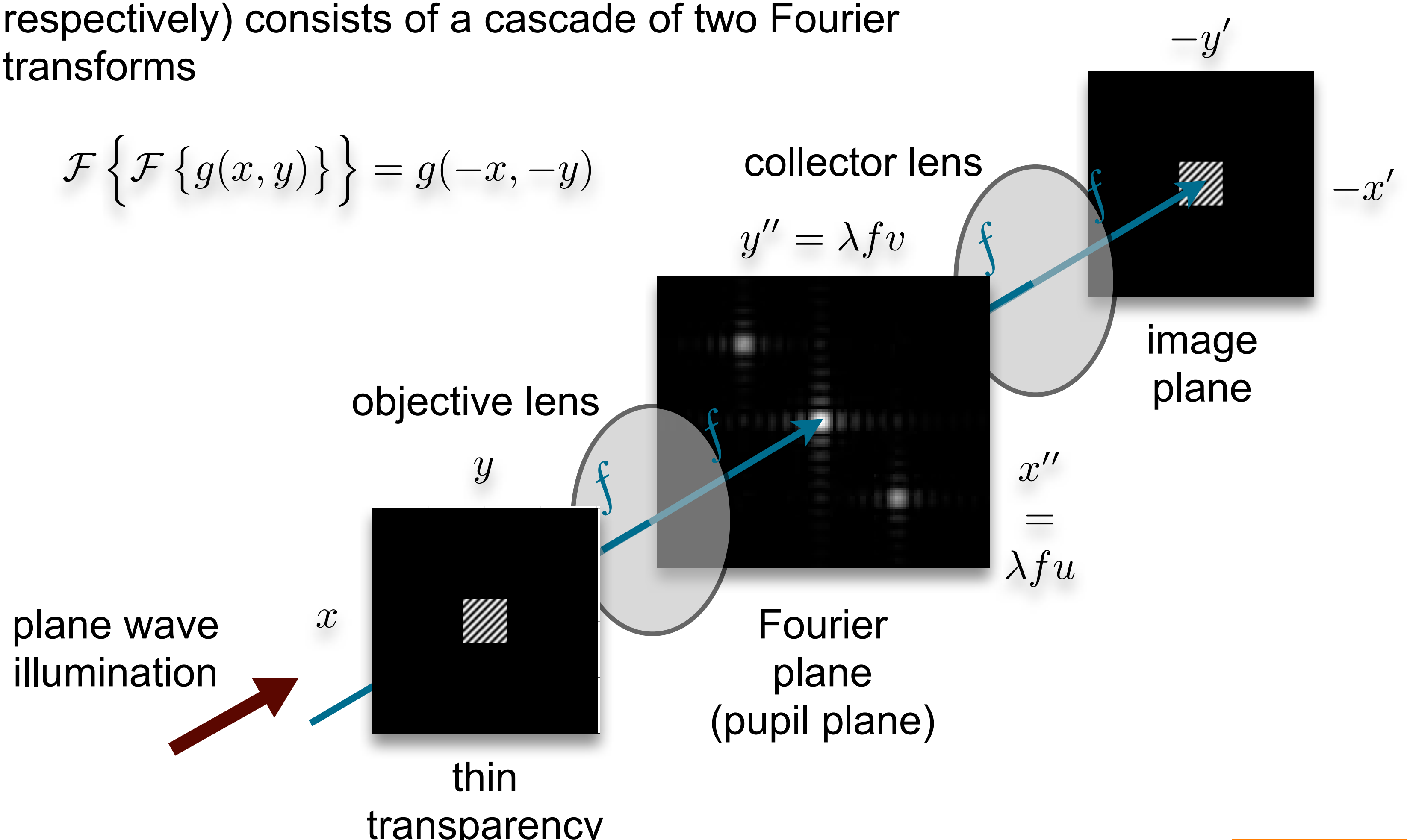
Fourier transforming by lenses



Imaging: the 4F system

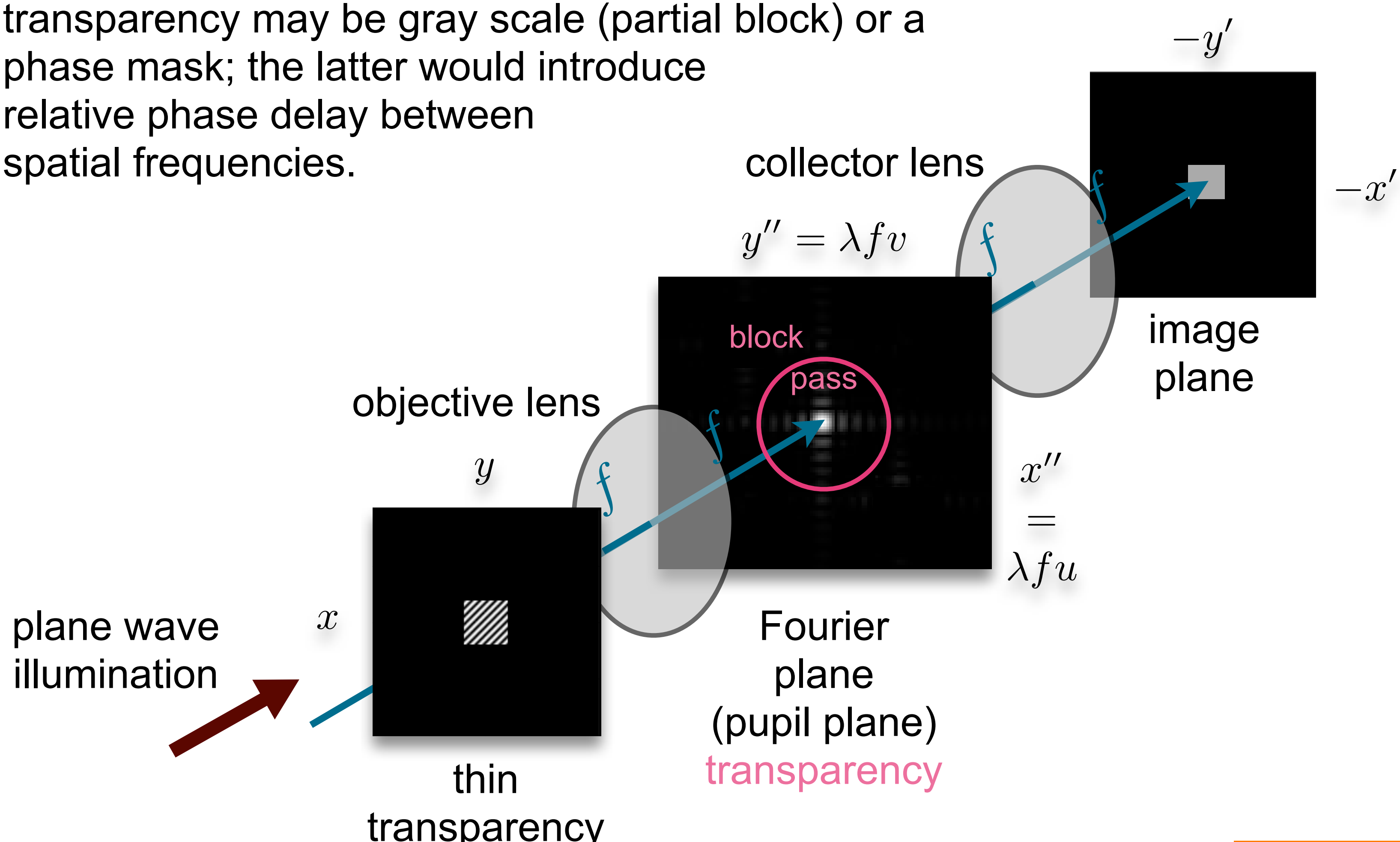
The 4F system (telescope with finite conjugates one focal distance to the left of the objective and one focal distance to the right of the collector, respectively) consists of a cascade of two Fourier transforms

$$\mathcal{F}\left\{\mathcal{F}\left\{g(x,y)\right\}\right\} = g(-x,-y)$$

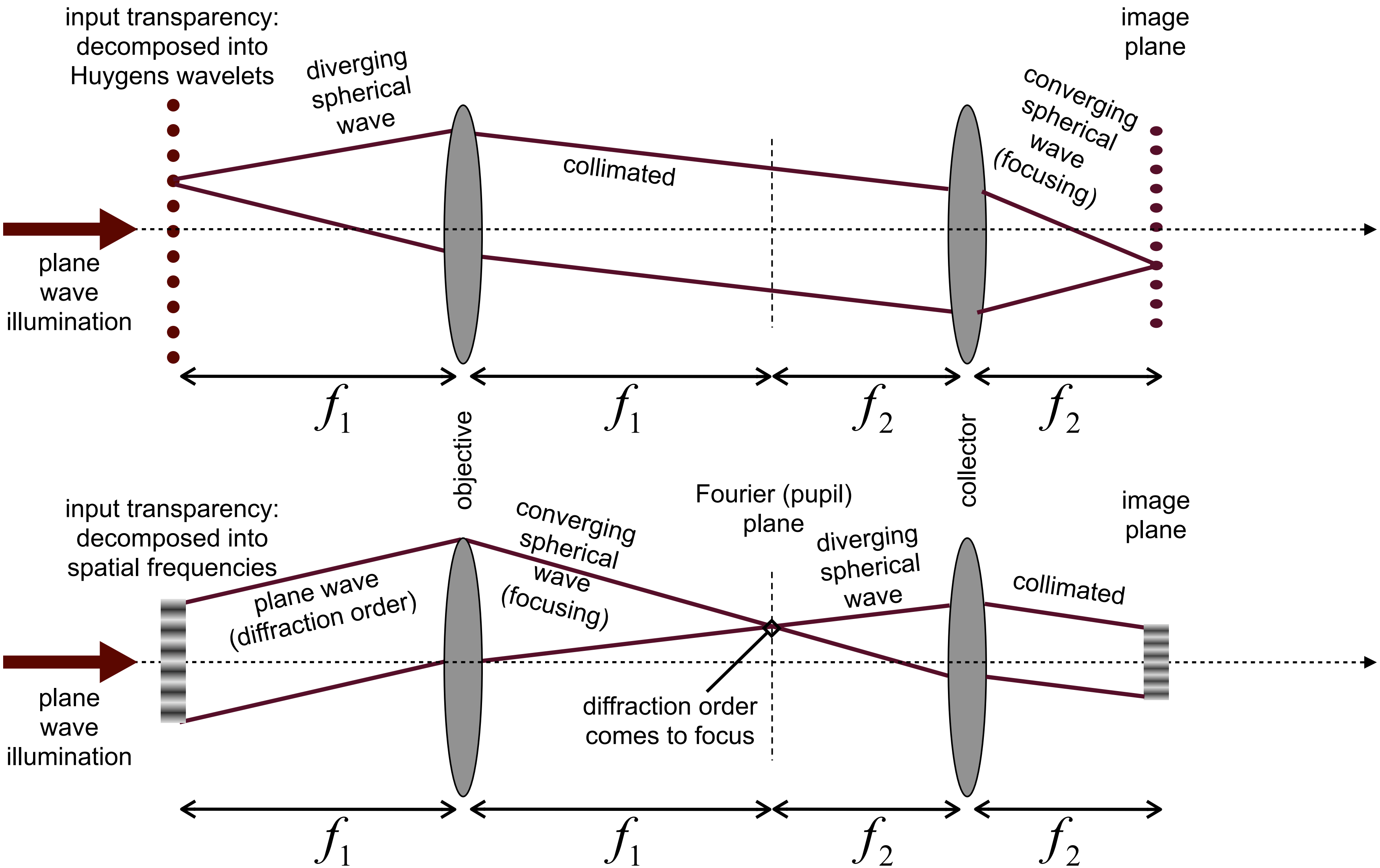


Spatial filtering: the 4F system

Spatial frequencies which have the misfortune of hitting the opaque portions of the pupil plane transparency vanish from the output. Of course the transparency may be gray scale (partial block) or a phase mask; the latter would introduce relative phase delay between spatial frequencies.



Imaging and spatial filtering: physical justification



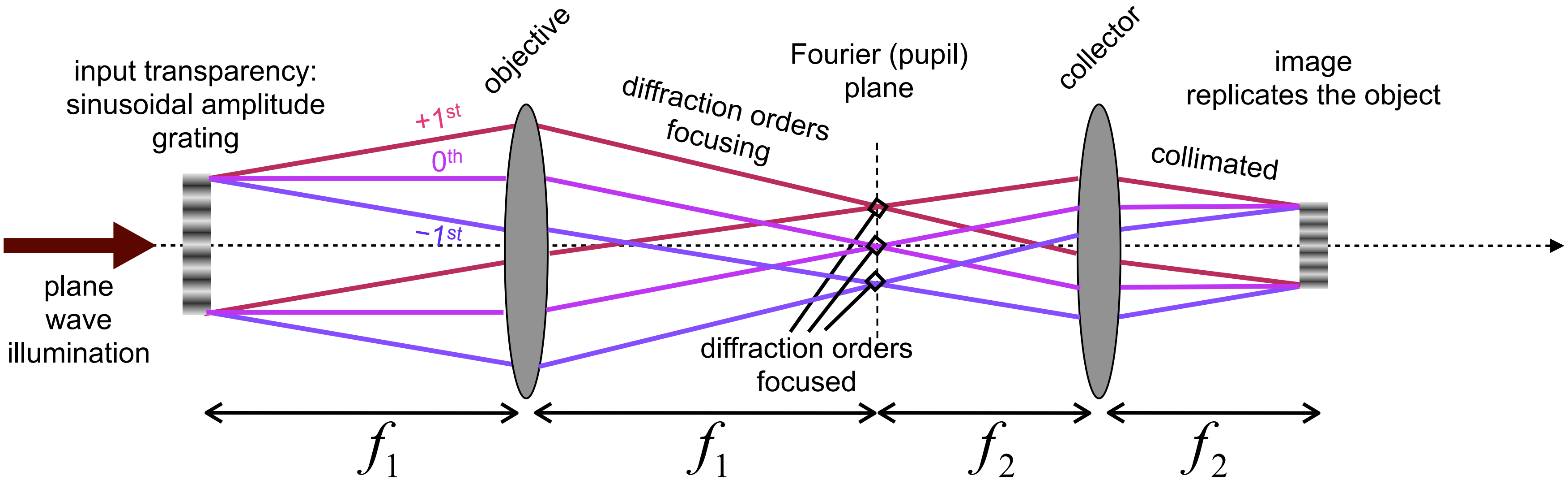
Today

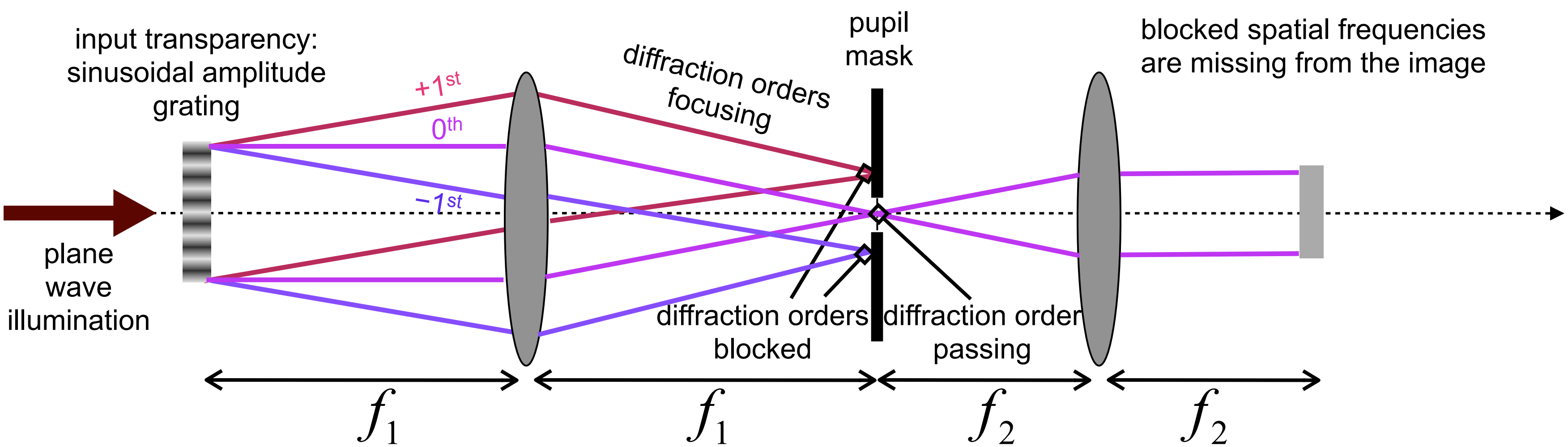
- Spatial filtering in the 4F system
- The Point-Spread Function (PSF) and Amplitude Transfer Function (ATF)

next Wednesday

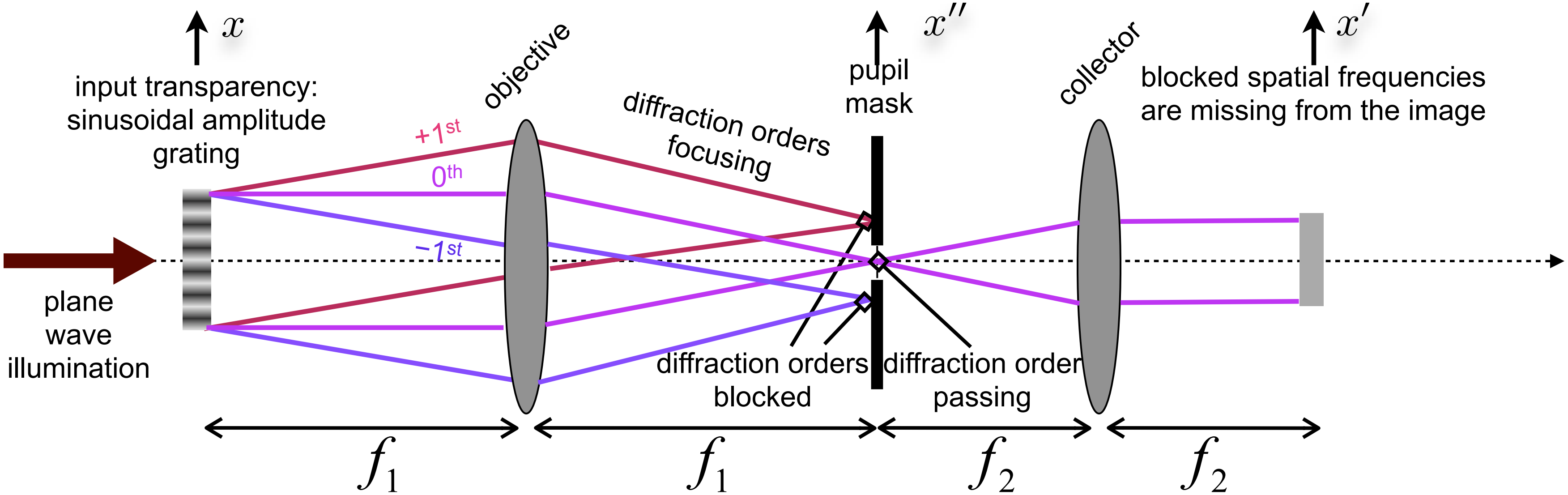
- Lateral and angular magnification
- The Numerical Aperture (NA) revisited
- Sampling the space and frequency domains, and the Space-Bandwidth Product (SBP)
- Pupil engineering

Spatial filtering by a telescope (4F system)





Low-pass filtering: analysis



field after input transparency
$$g_{in}(x) = \frac{1}{2}\left[1 + \cos(2\pi u_0 x)\right] \Rightarrow G_{in}(u) = \frac{1}{2}\left[\delta(u) + \frac{1}{2}\delta(u - u_0) + \frac{1}{2}\delta(u + u_0)\right]$$

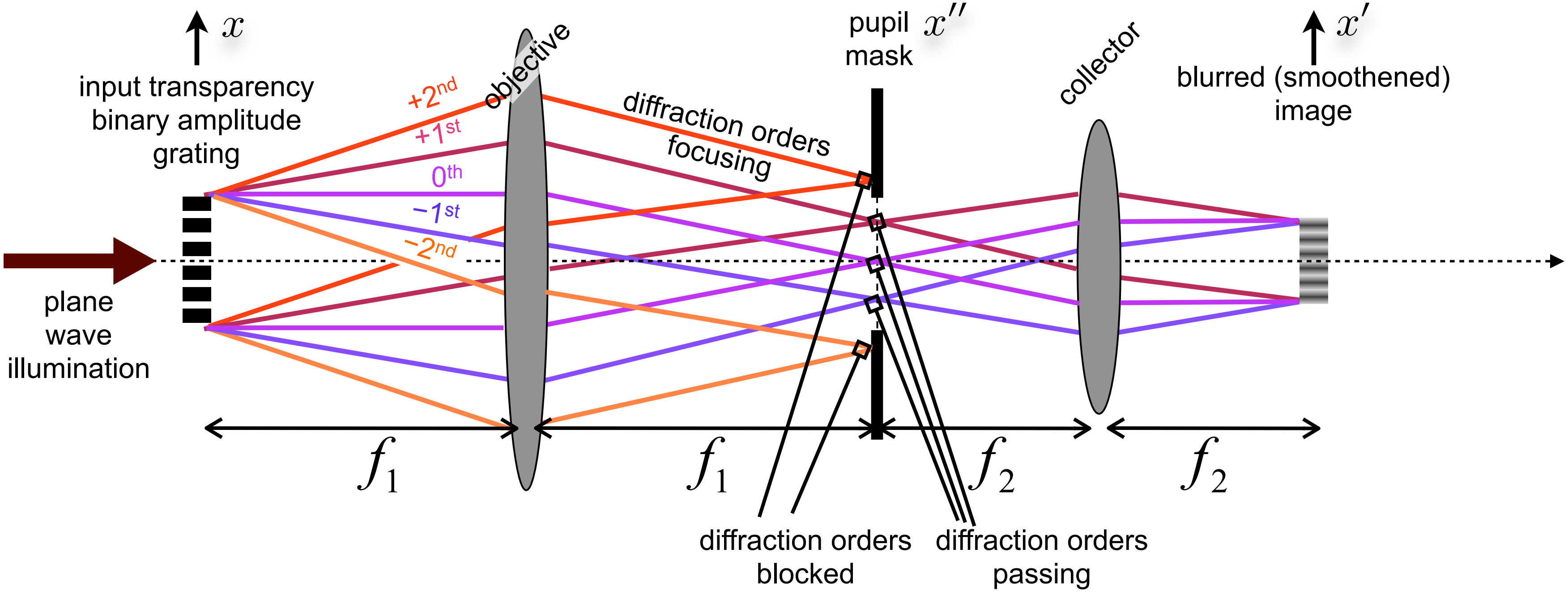
field before pupil mask
$$g_{f-}(x'') = \frac{1}{2}\left[\delta(x'') + \frac{1}{2}\delta(x'' - \lambda f_1 u_0) + \frac{1}{2}\delta(x'' + \lambda f_1 v_0)\right]$$

blocked by the pupil filter

field after pupil mask
$$g_{f+}(x'') = \frac{1}{2}\delta(x'')$$

field at output (image plane)
$$G_{out}(u) = \frac{1}{2}\delta(u) \Rightarrow g_{out}(x') = \frac{1}{2}$$

Example: low-pass filtering a binary amplitude grating



Consider a *binary* amplitude grating, with perfect contrast m=1, period Λ=10μm, duty cycle 1/3 (33.3%), illuminated by an on-axis plane wave at wavelength λ=0.5μm.

The 4F system consists of two identical lenses of focal length f=20cm.

A pupil mask of diameter (aperture) 3cm is placed at the Fourier plane, symmetrically about the optical axis.

What is the intensity observed at the output (image) plane?

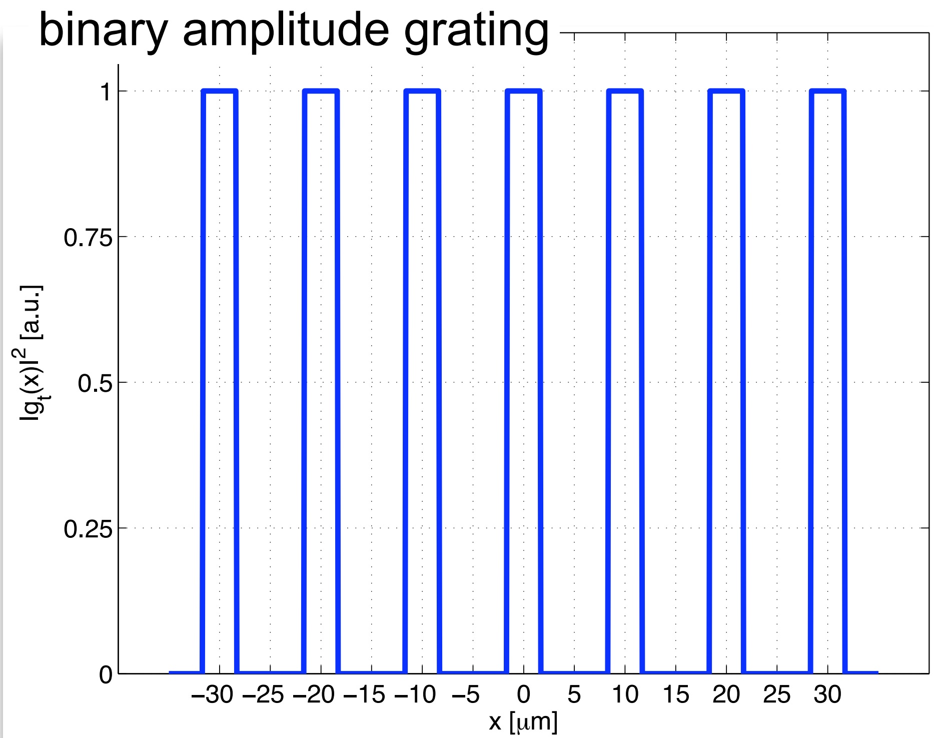
The sequence to solve this kind of problem is:

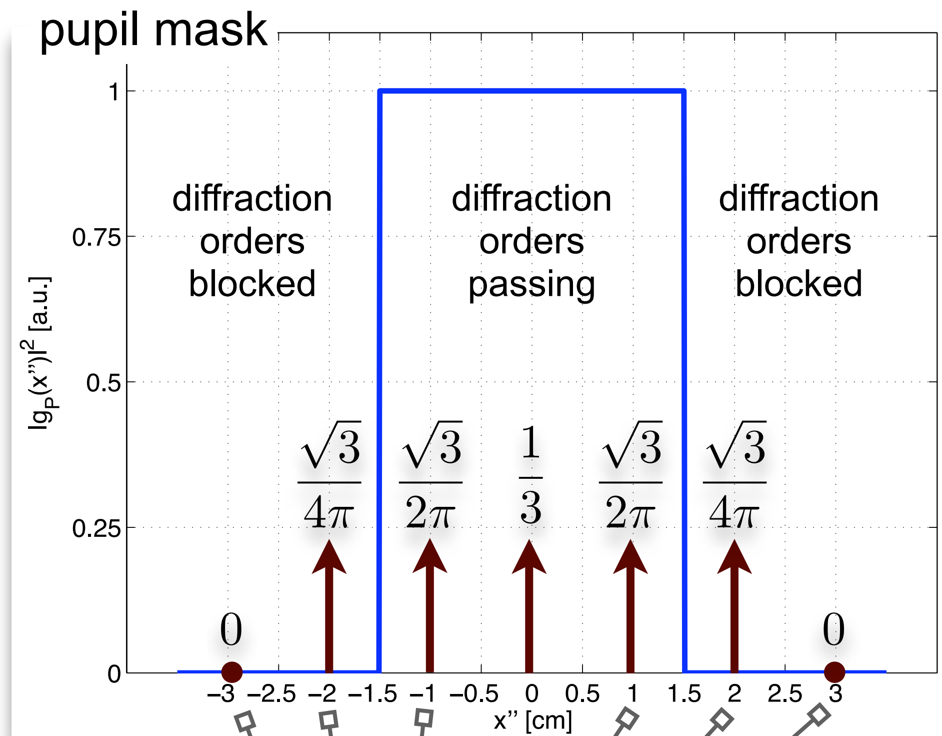
➡ calculate the Fourier transform of the input transparency and scale to the pupil plane coordinates $x''=u\lambda f_1$

➡ multiply by the complex amplitude transmittance of the pupil mask

➡ Fourier transform the product and scale to the output plane coordinates $x'=u\lambda f_2$

Example: low-pass filtering a binary amplitude grating





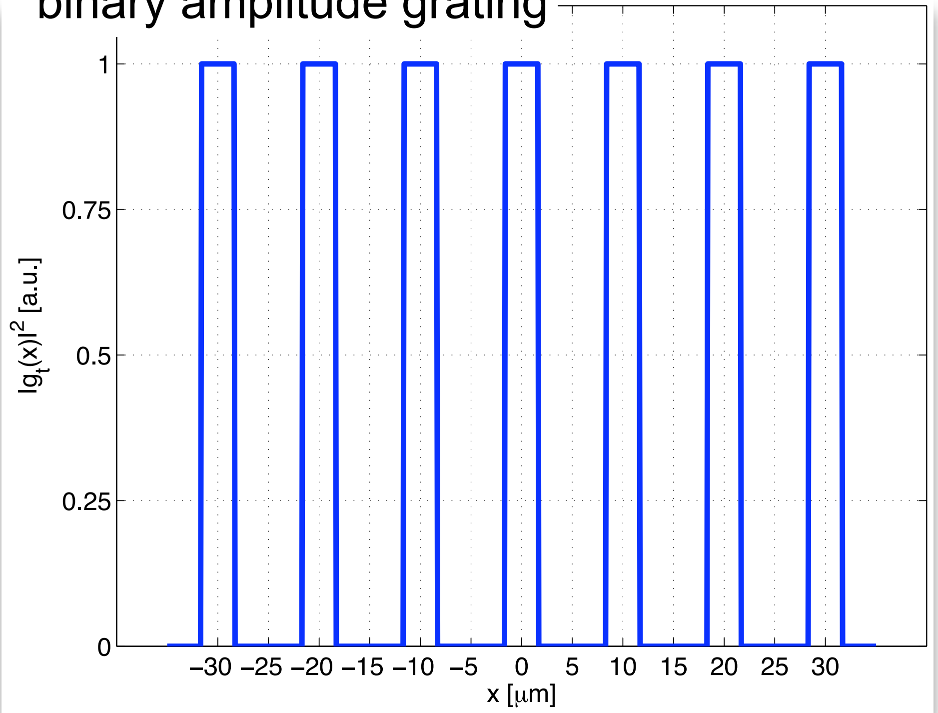
A binary amplitude grating of duty cycle α is expressed in a Fourier series harmonics expansion as

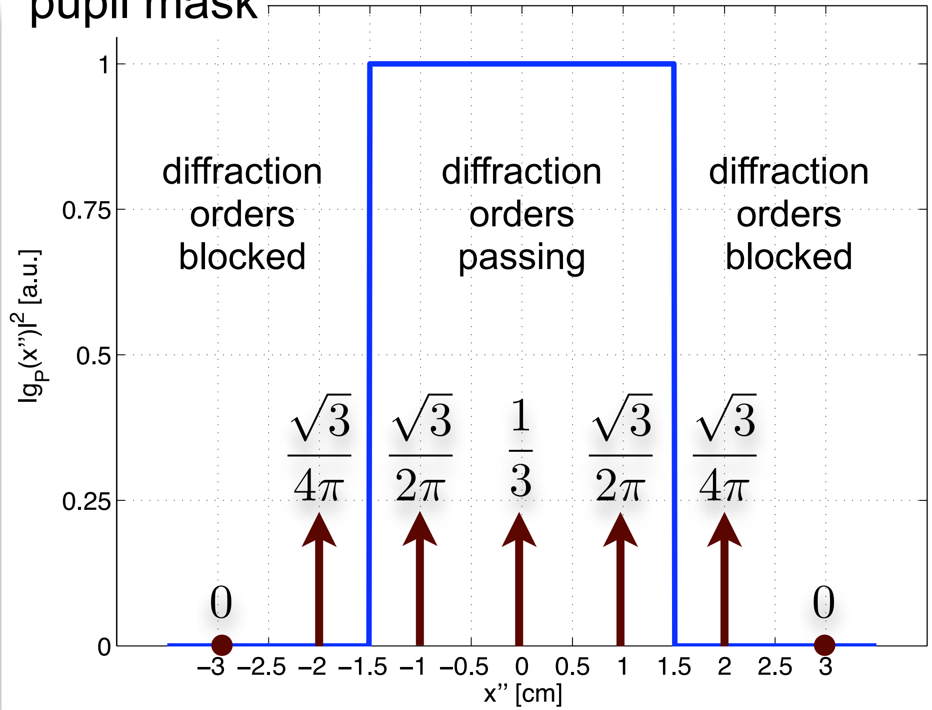
$$g_t(x) = \alpha \sum_{q=-\infty}^{+\infty} \mathrm{sinc}\,(\alpha q) \exp\left\{i2\pi q \frac{x}{\Lambda}\right\} \Rightarrow G_t(u) = \alpha \sum_{q=-\infty}^{+\infty} \mathrm{sinc}\,(\alpha q)\, \delta\left(u - \frac{q}{\Lambda}\right)$$

The field at the pupil plane to the left of the pupil mask is

$$g_{PP-}(x'') = G_t\left(\frac{x''}{\lambda f}\right) = \alpha \sum_{q=-\infty}^{+\infty} \mathrm{sinc}\,(\alpha q)\, \delta\left(\frac{x''}{\lambda f} - \frac{q}{\Lambda}\right) = \frac{1}{3} \sum_{q=-\infty}^{+\infty} \mathrm{sinc}\left(\frac{q}{3}\right) \delta\left(x'' - q \times 1\ \mathrm{cm}\right)$$

Example: low-pass filtering a binary amplitude grating





The pupil mask itself is $g_{\mathrm{PM}}(x'') = \mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right)$ so the field at the pupil plane to the right of the pupil mask is

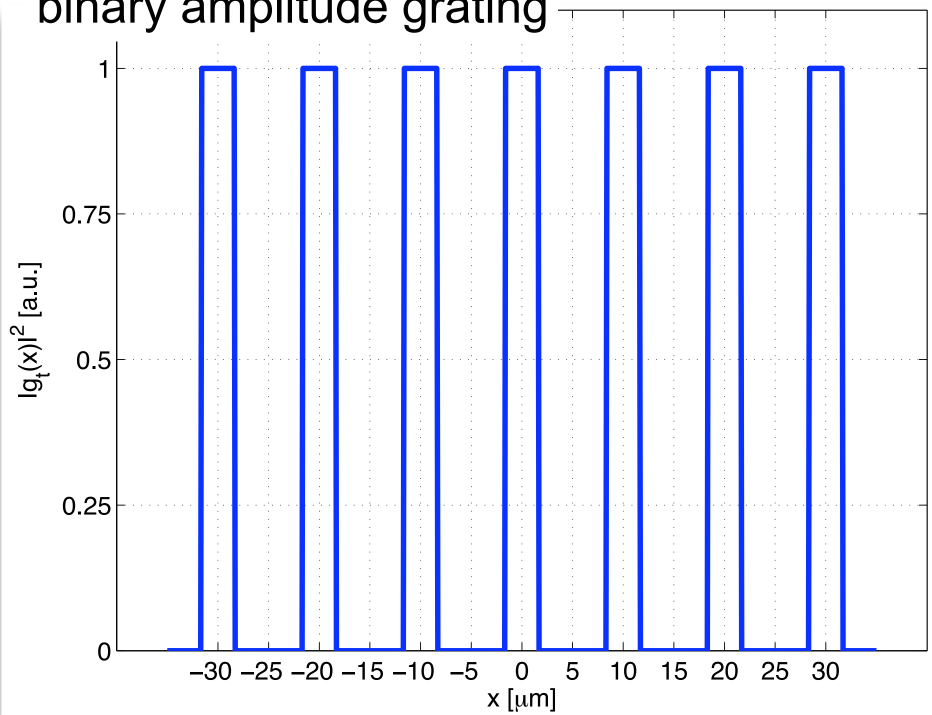
$$g_{\mathrm{PP}+}(x'') = g_{\mathrm{PP}-}(x'')\times g_{\mathrm{PM}}(x'') = \frac{1}{3}\delta\left(x''\right)+\frac{\sqrt{3}}{2\pi}\delta\left(x''-1\mathrm{cm}\right)+\frac{\sqrt{3}}{2\pi}\delta\left(x''+1\mathrm{cm}\right)$$

Its Fourier transform is
$$G_{\mathrm{PP}+}(u) = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}\exp\left\{i2\pi u\times 1\mathrm{cm}\right\} + \frac{\sqrt{3}}{2\pi}\exp\left\{-i2\pi u\times 1\mathrm{cm}\right\}$$

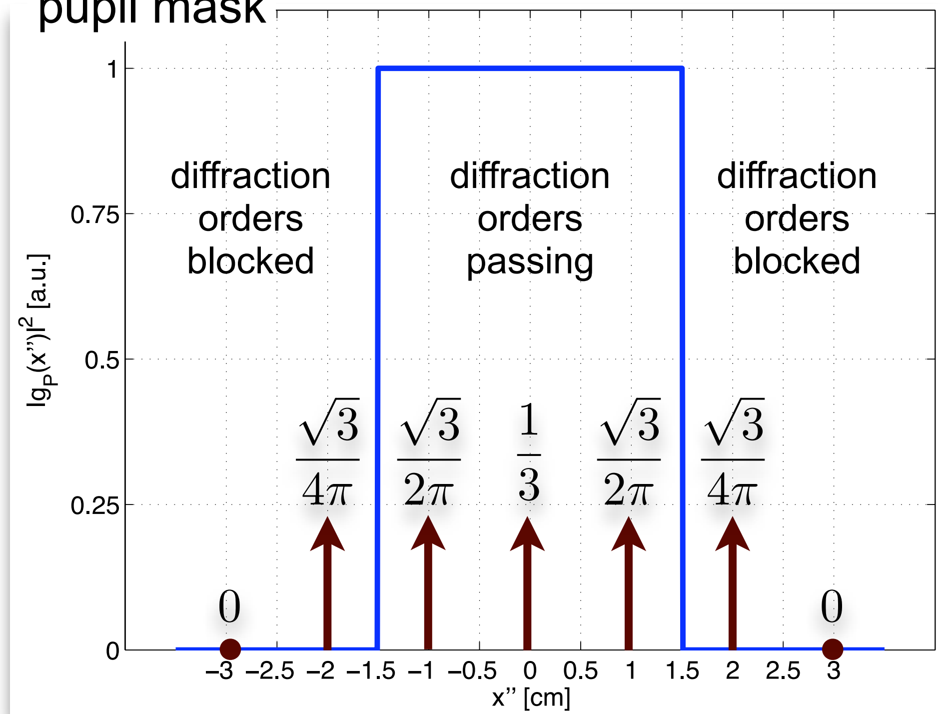
from which we may obtain the output field as
$$g_{\mathrm{out}}(x') = G_{\mathrm{PP}+}\left(\frac{x'}{\lambda f}\right)$$

Example: low-pass filtering a binary amplitude grating

binary amplitude grating



pupil mask



$$g_{\text{out}}(x') = G_{\text{PP+}}\left(\frac{x'}{\lambda f}\right) = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}\exp\left\{i2\pi\frac{x'}{10\mu\text{m}}\right\} + \frac{\sqrt{3}}{2\pi}\exp\left\{-i2\pi\frac{x'}{10\mu\text{m}}\right\}$$

$$= \frac{1}{3} + \frac{\sqrt{3}}{\pi}\cos\left(2\pi\frac{x'}{10\mu\text{m}}\right)$$

Note the 2nd harmonic term in the intensity, due to the magnitude-square operation! This term explains the “ringing” in coherent low-pass filtering systems

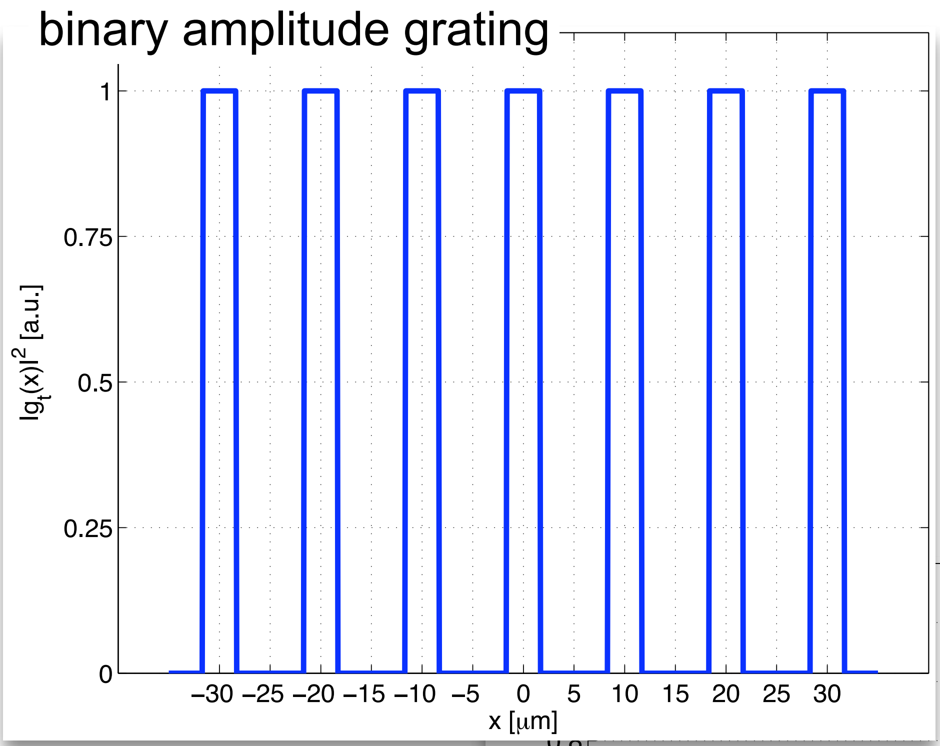
The output intensity is

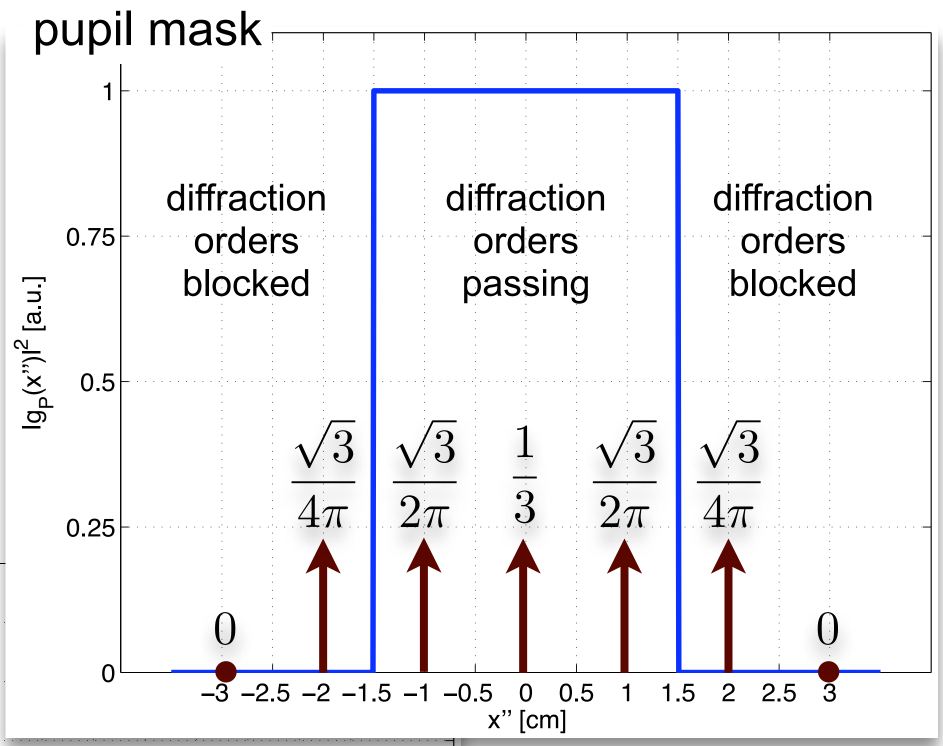
$$I_{\text{out}}(x') = |g_{\text{out}}|^2(x') = \left[\frac{1}{3} + \frac{\sqrt{3}}{\pi}\cos\left(2\pi\frac{x'}{10\mu\text{m}}\right)\right]^2$$

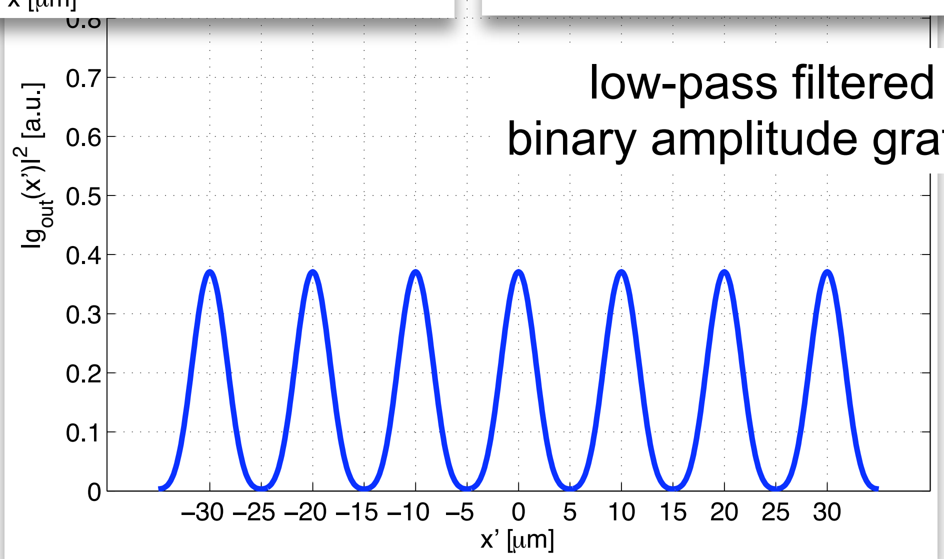
$$= \left(\frac{1}{3}\right)^2 + \frac{2}{\pi\sqrt{3}}\cos\left(2\pi\frac{x'}{10\mu\text{m}}\right) + \frac{3}{\pi^2}\cos^2\left(2\pi\frac{x'}{10\mu\text{m}}\right)$$

$$= \left(\frac{1}{3}\right)^2 + \frac{3}{2\pi^2} + \frac{2}{\pi\sqrt{3}}\cos\left(2\pi\frac{x'}{10\mu\text{m}}\right) + \frac{3}{2\pi^2}\cos\left(2\pi\frac{2x'}{10\mu\text{m}}\right)$$

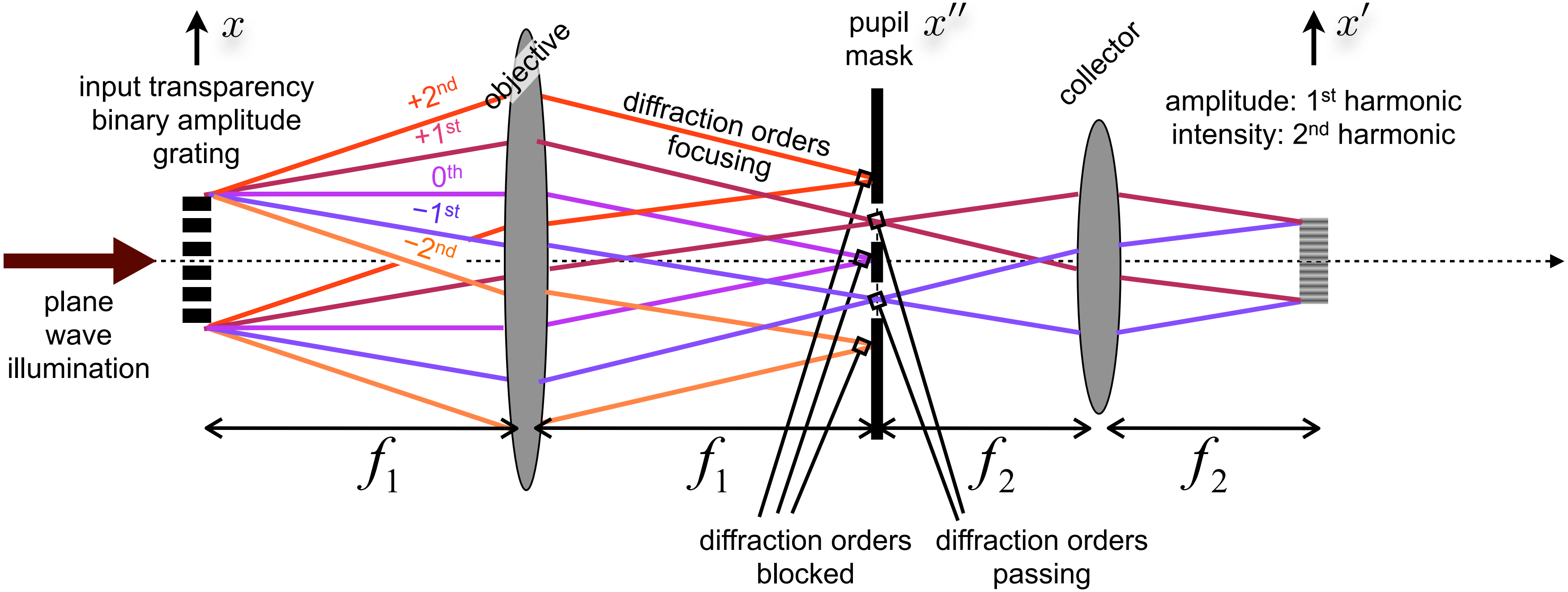
Example: low-pass filtering a binary amplitude grating







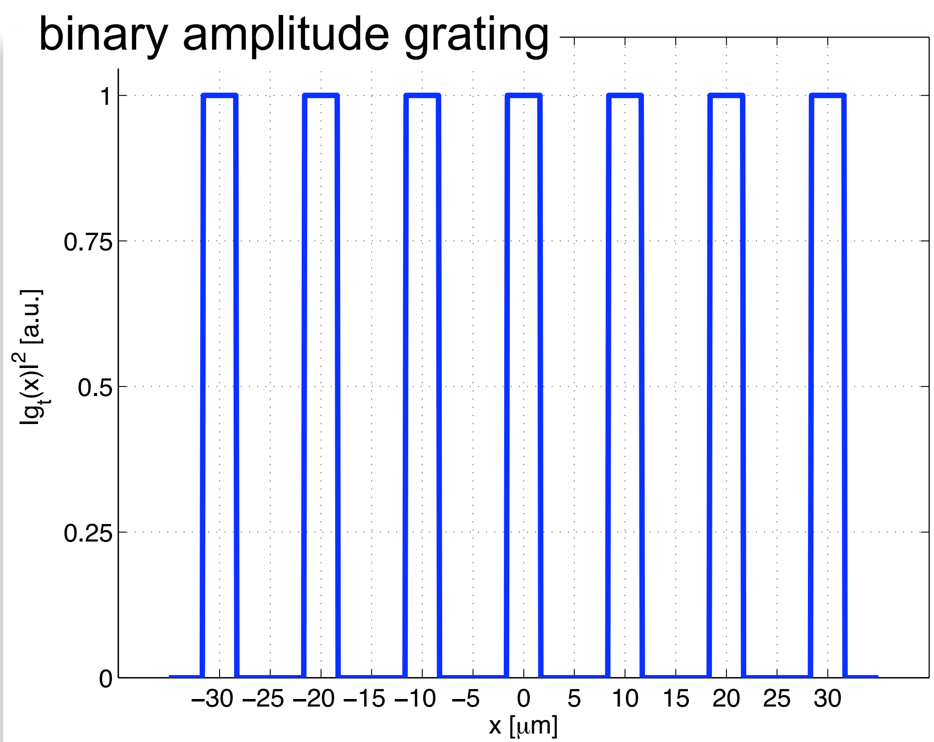
Example: band-pass filtering a binary amplitude grating

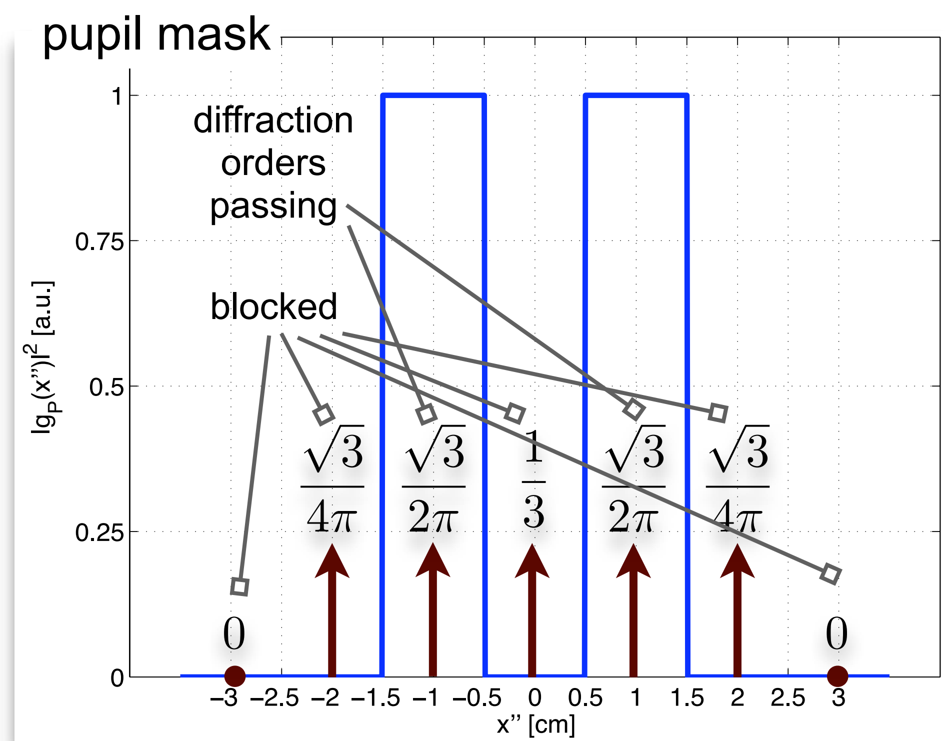


Now consider the same optical system, but with a new pupil mask consisting of two holes, each of diameter (aperture) 1cm and centered at ±1cm from the optical axis, respectively.

What is the intensity observed at the output (image) plane?

Example: band-pass filtering a binary amplitude grating





The new pupil mask is $g_{\mathrm{PM}}(x'') = \mathrm{rect}\left(\frac{x''+1\mathrm{cm}}{1\mathrm{cm}}\right) + \mathrm{rect}\left(\frac{x''-1\mathrm{cm}}{1\mathrm{cm}}\right)$ so the field at the pupil plane to the right of the pupil mask is

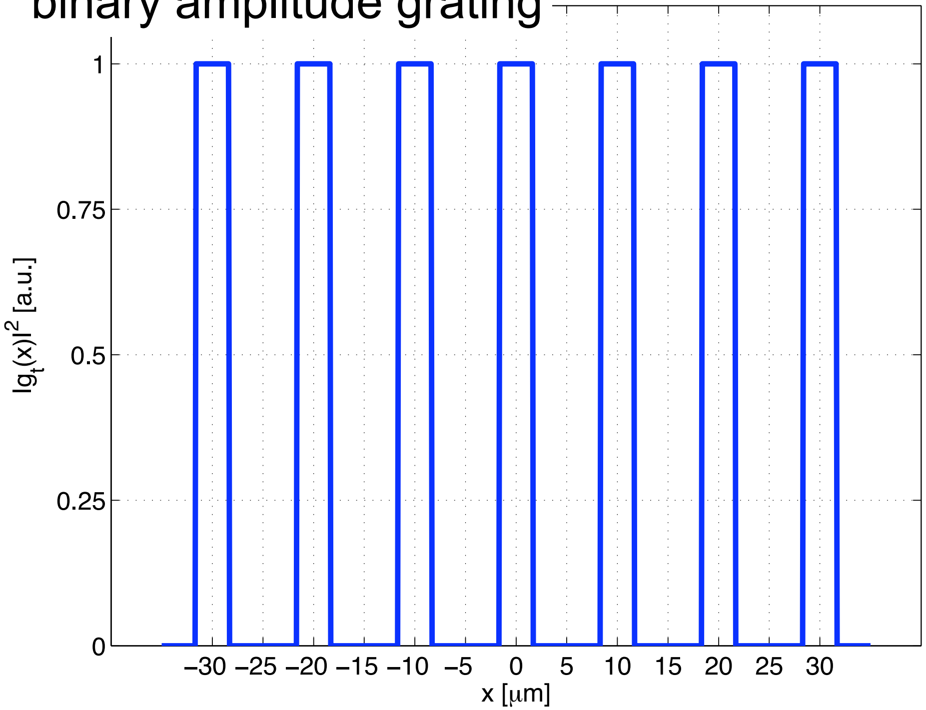
$$g_{\mathrm{PP}+}(x'') = g_{\mathrm{PP}-}(x'') \times g_{\mathrm{PM}}(x'') = \frac{\sqrt{3}}{2\pi}\delta\left(x''-1\mathrm{cm}\right) + \frac{\sqrt{3}}{2\pi}\delta\left(x''+1\mathrm{cm}\right)$$

Its Fourier transform is $G_{\mathrm{PP}+}(u) = \frac{\sqrt{3}}{2\pi}\exp\left\{i2\pi u \times 1\mathrm{cm}\right\} + \frac{\sqrt{3}}{2\pi}\exp\left\{-i2\pi u \times 1\mathrm{cm}\right\}$

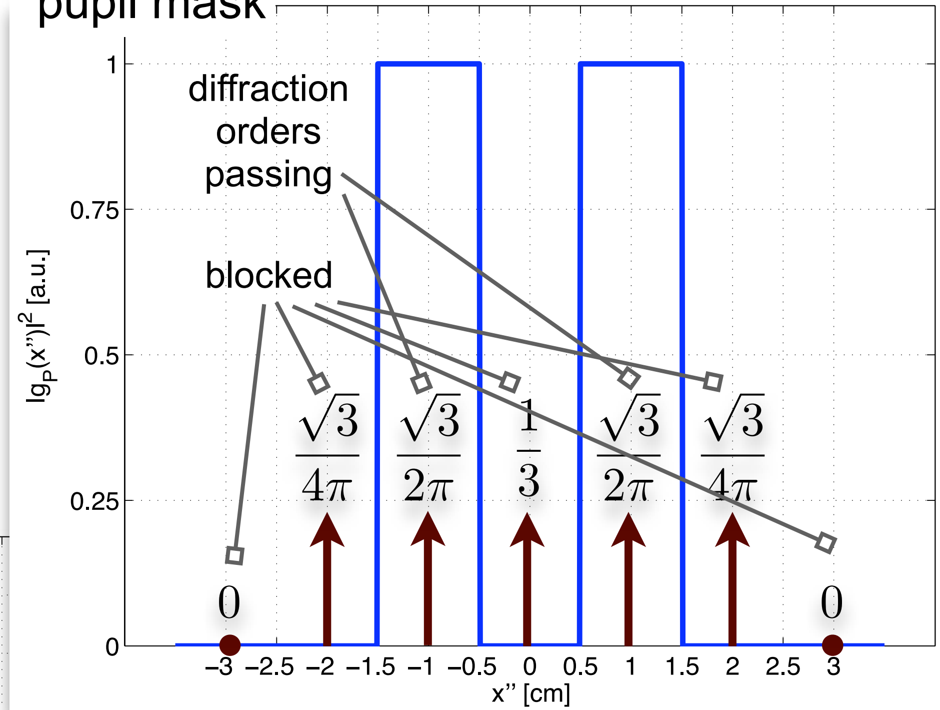
so the output field and intensity are $g_{\mathrm{out}}(x') = \frac{\sqrt{3}}{\pi}\cos\left(2\pi\frac{x'}{10\mu\mathrm{m}}\right)$ $\quad I_{\mathrm{out}}(x') = \frac{3}{2\pi^2} + \frac{3}{2\pi^2}\cos\left(2\pi\frac{2x'}{10\mu\mathrm{m}}\right)$

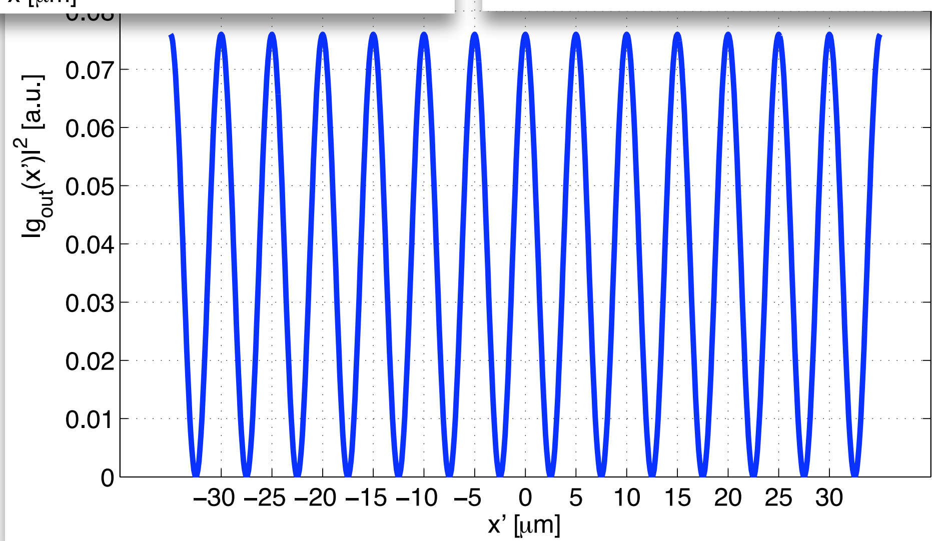
Example: band-pass filtering a binary amplitude grating

binary amplitude grating



pupil mask



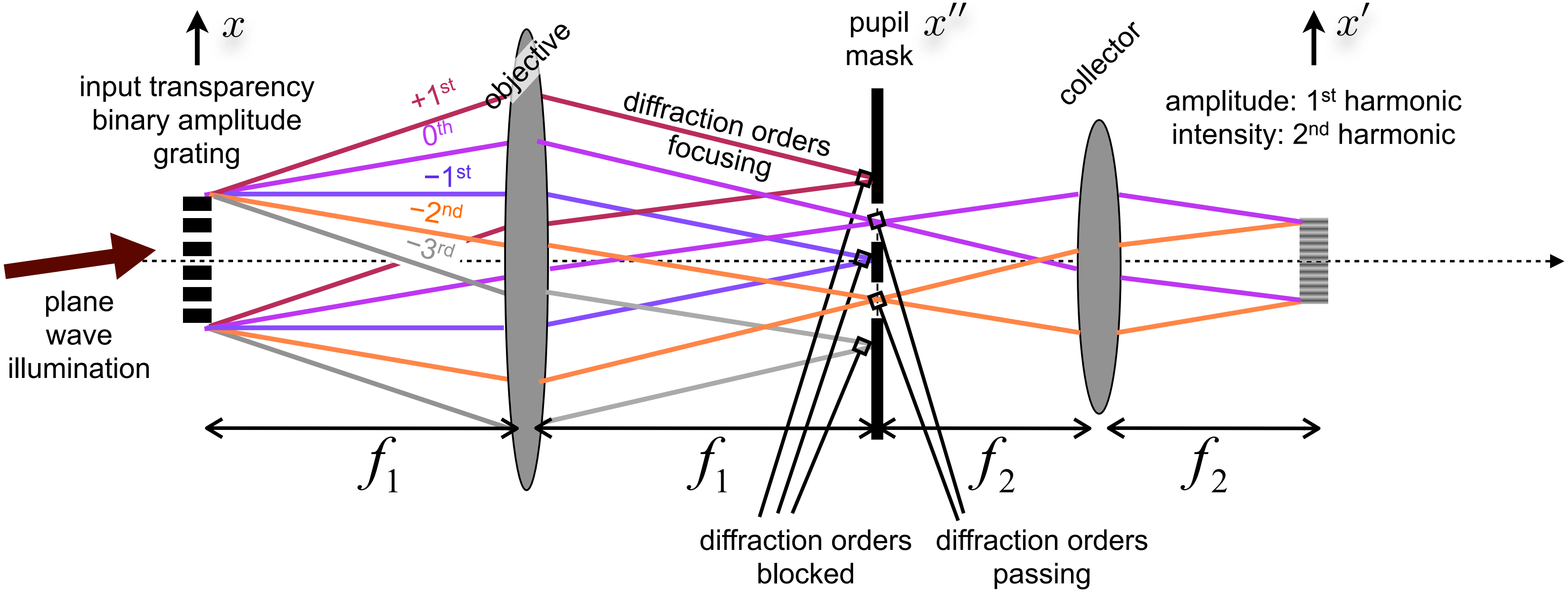


band-pass filtered binary amplitude grating

field: 1st harmonic
intensity: 2nd harmonic (because of squaring)

contrast = 1

Example: band-pass filtering a binary amplitude grating with tilted illumination



Now consider the same optical system, again with the pupil mask consisting of two holes, each of diameter (aperture) 1cm and centered at ±1cm from the optical axis, respectively. We illuminate this grating with an off-axis plane wave at angle θ_0=2.865º.

What is the intensity observed at the output (image) plane?

As you saw in a homework problem, the effect of rotating the input illumination is that the entire diffraction pattern from the grating rotates by the same amount; so in this case the 0th order is propagating at angle θ_0 off-axis, the +1st order at angle $\theta_0+\lambda/\Lambda$, etc.

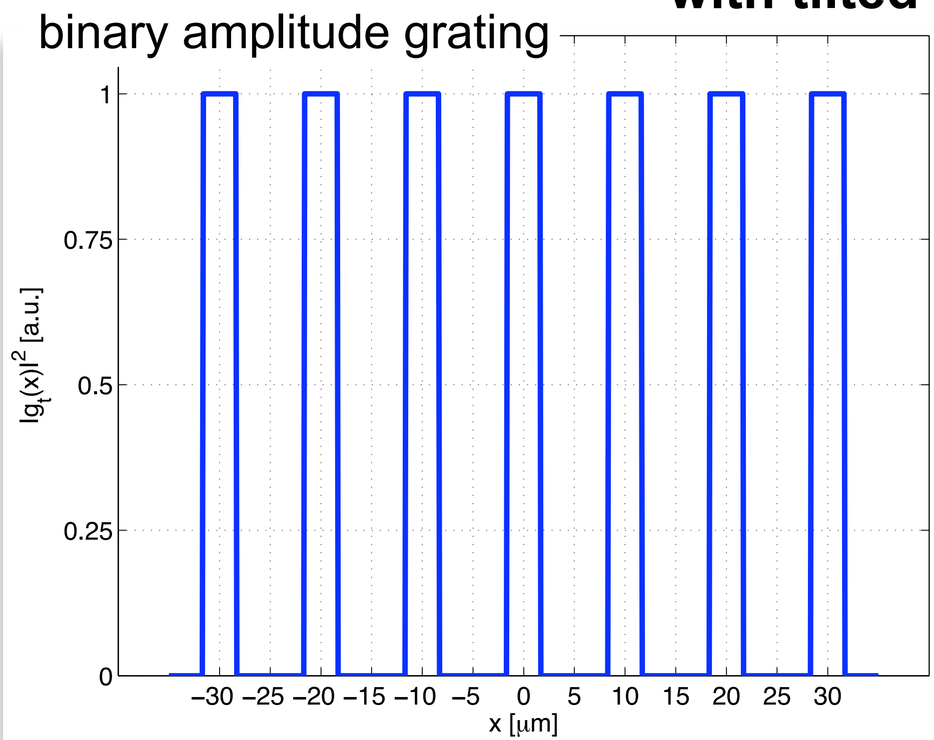
Analytically, we find this by expressing the illuminating plane wave as

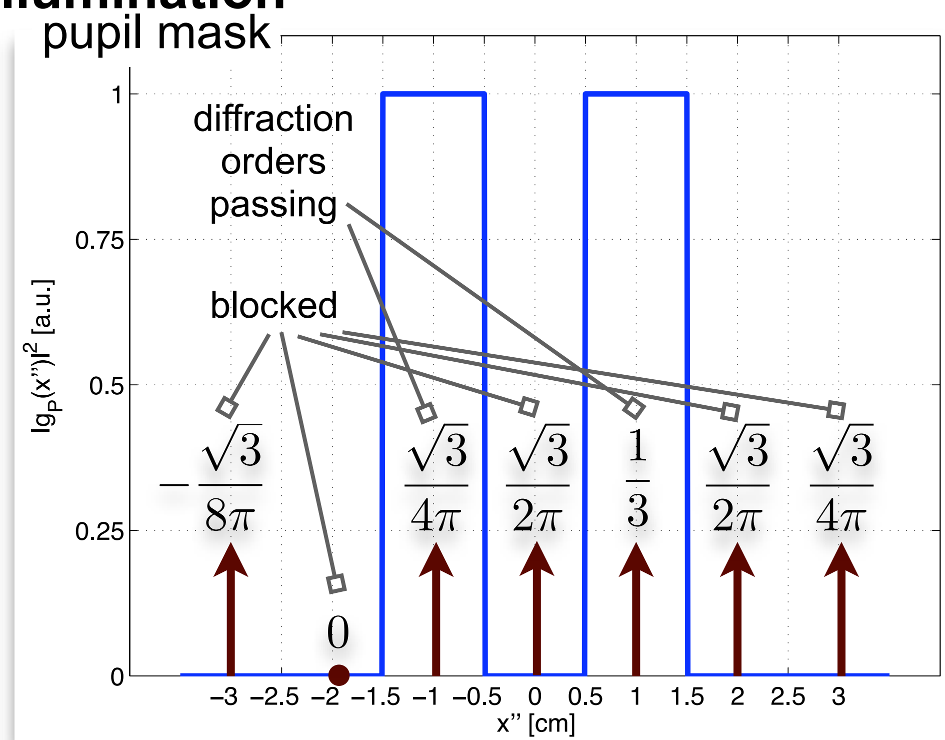
$$g_{\text{illum}}(x) = \exp\left\{ i2\pi \frac{\sin\theta_0}{\lambda} x \right\}$$

and the field after the input transparency $g_t(x)$ as

$$g_{\text{in}}(x) = g_{\text{illum}}(x) \times g_{\text{t}}(x) = \exp\left\{ i2\pi \frac{\sin\theta_0}{\lambda} x \right\} \times g_{\text{t}}(x)$$

Example: band-pass filtering a binary amplitude grating with tilted illumination





The field to the left of the pupil mask is the Fourier transform of $g_{\text{in}}(x)$. Using the shift theorem,

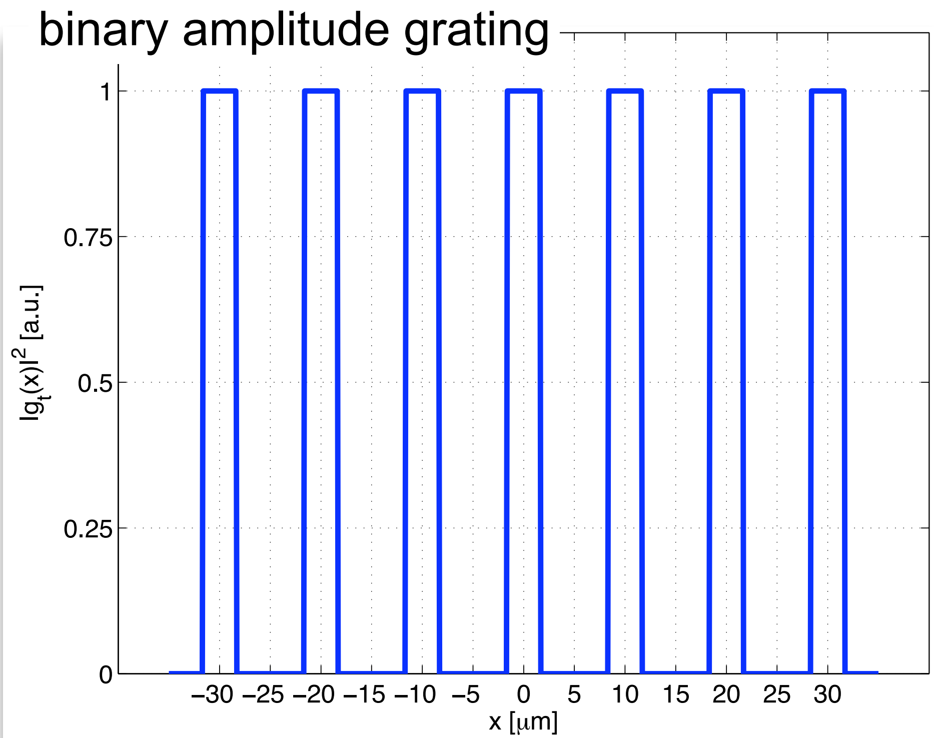
$$G_{\text{in}}(u) = G_{\text{illum}}(u-u_0) \qquad \text{where} \quad u_0 \equiv \frac{\sin\theta_0}{\lambda} \approx 0.1\mu\text{m}^{-1}$$

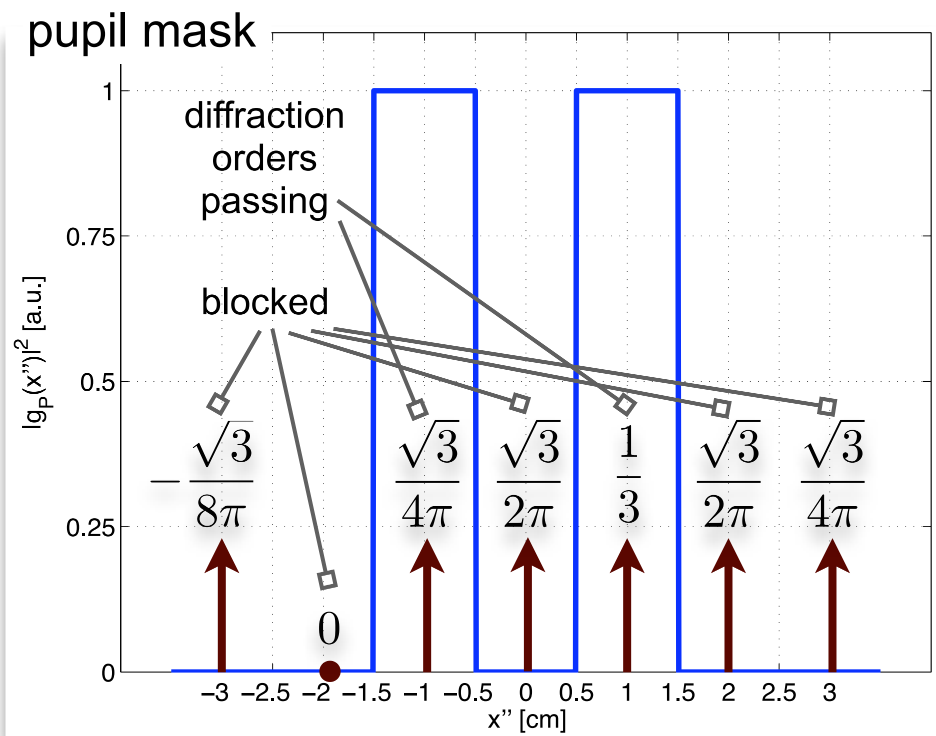
$$= \alpha \sum_{q=-\infty}^{+\infty} \operatorname{sinc}(\alpha q)\, \delta\left(u - u_0 - \frac{q}{\Lambda}\right) \Rightarrow$$

$$g_{\text{PP}-}(x'') = G_{\text{in}}\left(\frac{x''}{\lambda f}\right) = \frac{1}{3} \sum_{q=-\infty}^{+\infty} \operatorname{sinc}\left(\frac{q}{3}\right) \delta\left(x'' - 1\text{cm} - q \times 1\ \text{cm}\right).$$

all diffracted orders are displaced by 1cm in the positive x'' direction

Example: band-pass filtering a binary amplitude grating with tilted illumination





After passing through the pupil mask $g_{\mathrm{PM}}(x'') = \mathrm{rect}\left(\frac{x''+1\mathrm{cm}}{1\mathrm{cm}}\right) + \mathrm{rect}\left(\frac{x''-1\mathrm{cm}}{1\mathrm{cm}}\right)$, the field is

$$g_{\mathrm{PP}+}(x'') = g_{\mathrm{PP}-}(x'') \times g_{\mathrm{PM}}(x'') = \frac{1}{3}\delta(x''-1\mathrm{cm}) + \frac{\sqrt{3}}{4\pi}\delta(x''+1\mathrm{cm}) \Rightarrow$$

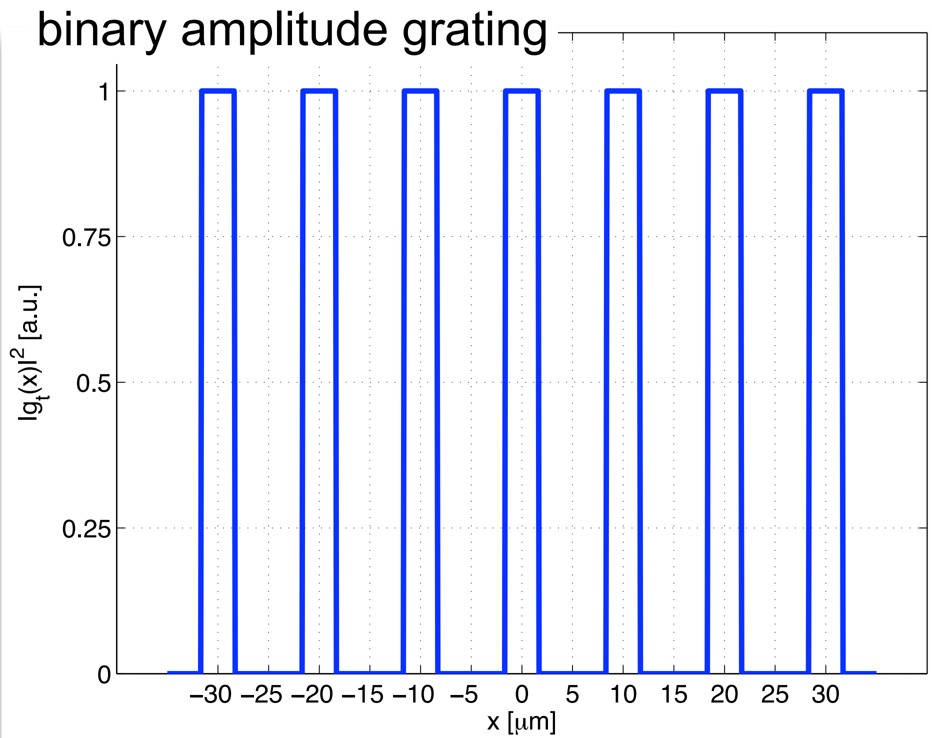
$$G_{\mathrm{PP}+}(u) = \frac{1}{3}\exp\{i2\pi u \times 1\mathrm{cm}\} + \frac{\sqrt{3}}{4\pi}\exp\{-i2\pi u \times 1\mathrm{cm}\}$$

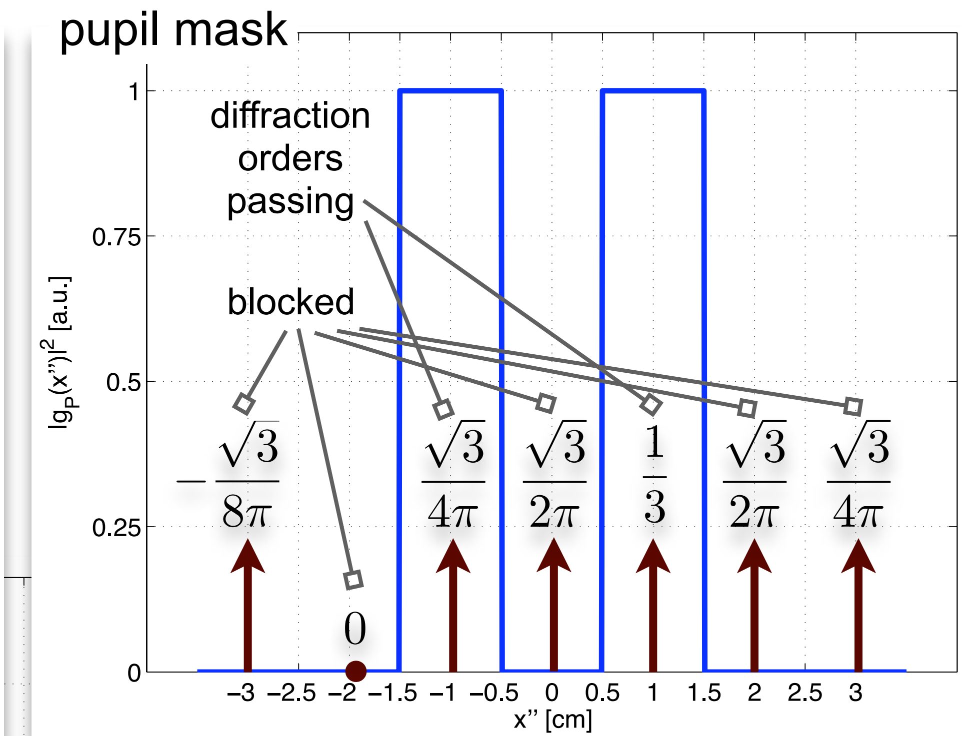
so the output field and intensity are

$$g_{\mathrm{out}}(x') = G_{\mathrm{PP}+}\left(\frac{x'}{\lambda f}\right) = \frac{1}{3}\exp\left\{i2\pi\frac{x'}{10\mu m}\right\} + \frac{\sqrt{3}}{4\pi}\exp\left\{-i2\pi\frac{x'}{10\mu m}\right\}$$

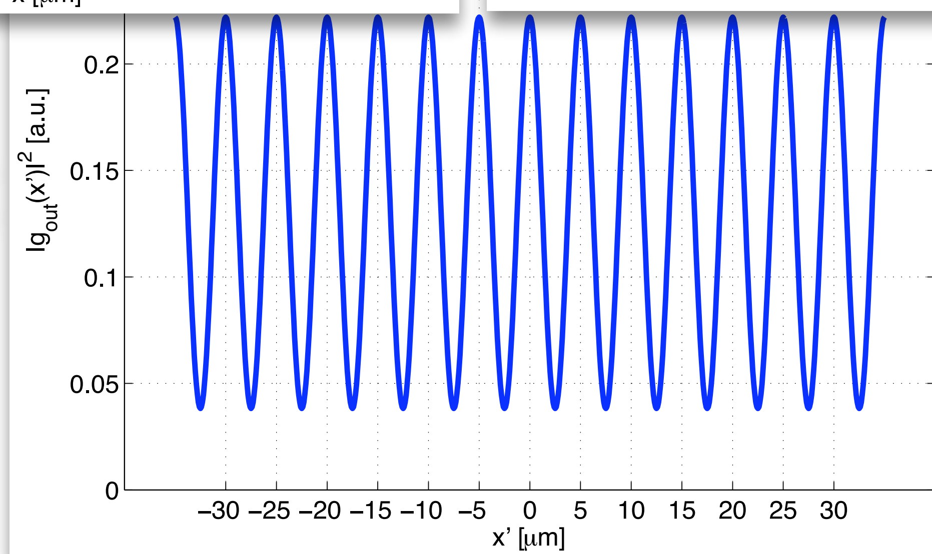
$$I_{\mathrm{out}}(x') = |g_{\mathrm{out}}(x')|^2 = \frac{1}{9} + \frac{3}{16\pi^2} + \frac{1}{2\sqrt{3}\pi}\cos\left(2\pi\frac{2x'}{10\mu m}\right)$$

Example: band-pass filtering a binary amplitude grating with tilted illumination





$$\text{Contrast } V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{0.2220 - 0.0382}{0.2220 + 0.0382} = 0.7062$$

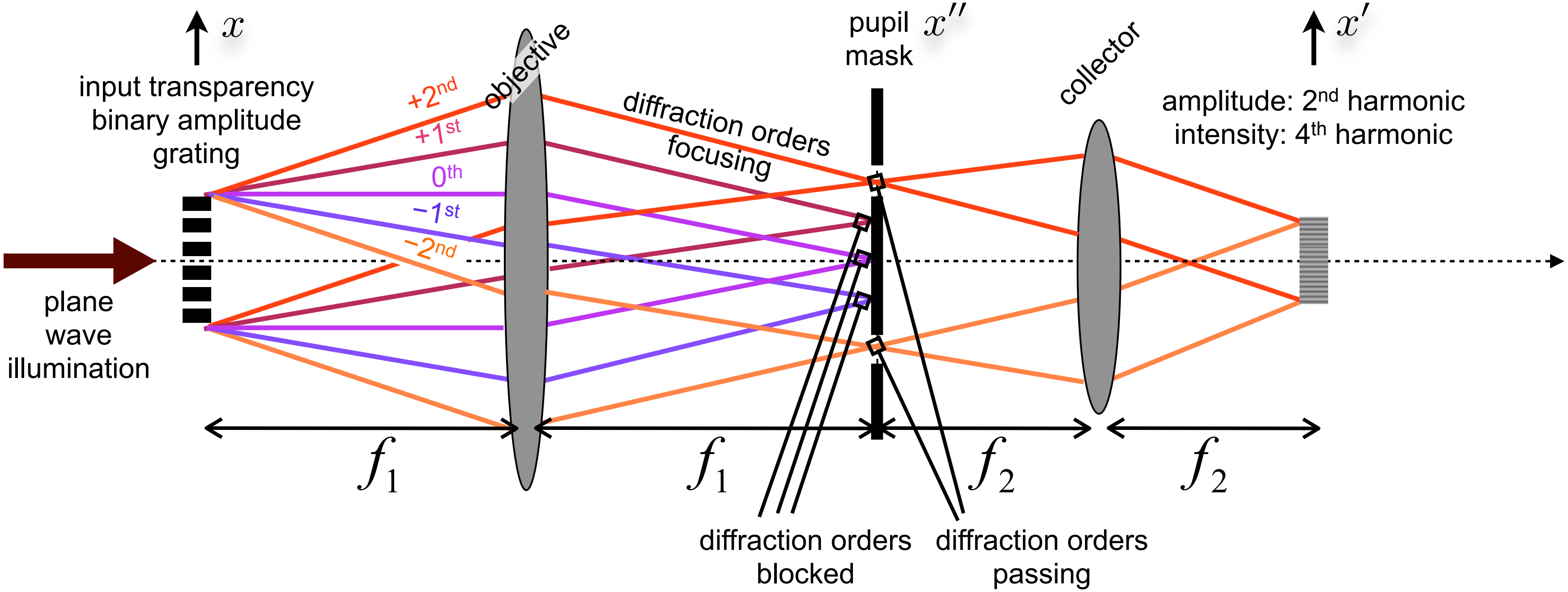


band-pass filtered binary amplitude grating with tilted illumination

field: 1st harmonic
intensity: 2nd harmonic
(because of squaring)

contrast ≈ 0.7

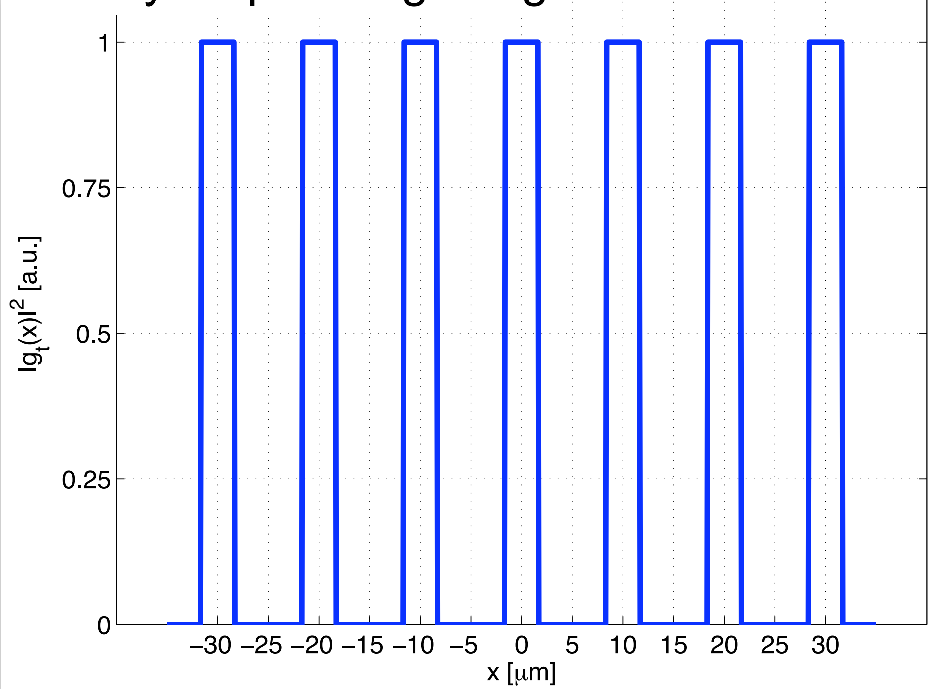
Example: band-pass filtering a binary amplitude grating

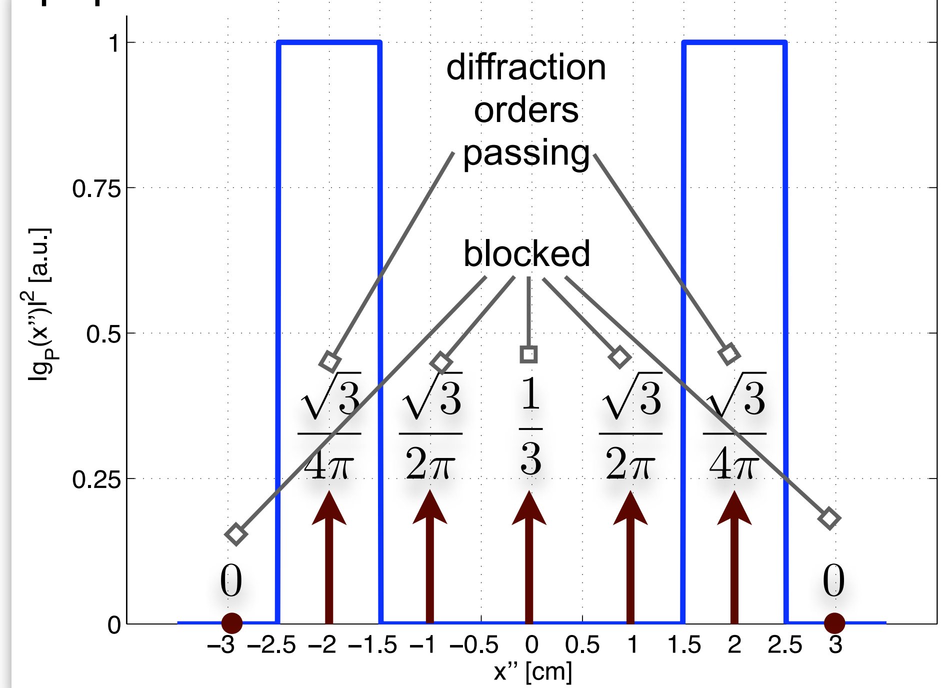


Consider the same optical system yet again, with a new pupil mask consisting of two holes, each of diameter (aperture) $1\mathrm{cm}$ and centered further away from the axis at $\pm 2\mathrm{cm}$ from the optical axis, respectively.

What is the intensity observed at the output (image) plane?

Example: band-pass filtering a binary amplitude grating





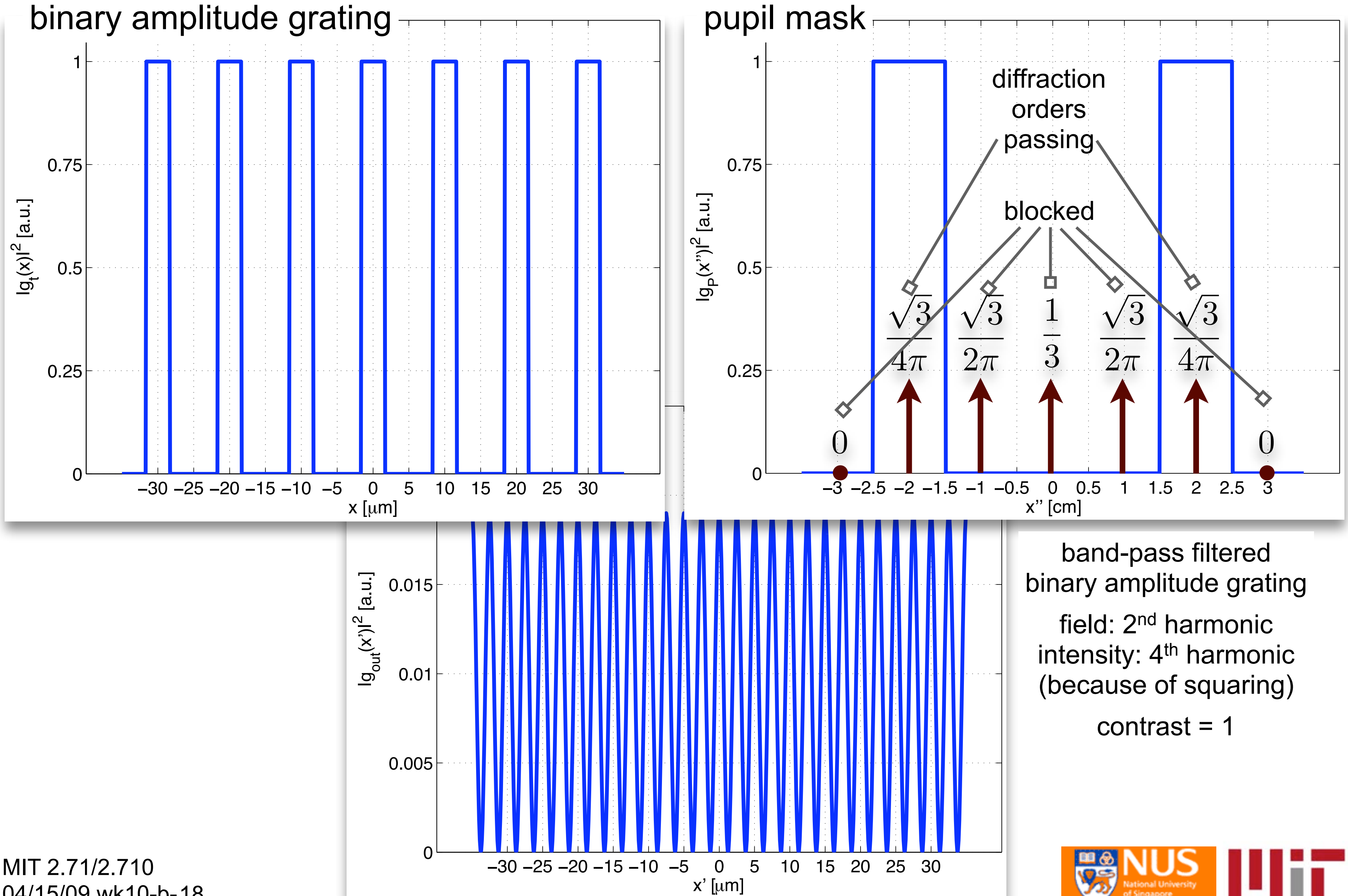
The new pupil mask is $g_{\mathrm{PM}}(x'') = \mathrm{rect}\left(\frac{x''+2\mathrm{cm}}{1\mathrm{cm}}\right) + \mathrm{rect}\left(\frac{x''-2\mathrm{cm}}{1\mathrm{cm}}\right)$ so the field at the pupil plane to the right of the pupil mask is

$$g_{\mathrm{PP}+}(x'') = g_{\mathrm{PP}-}(x'') \times g_{\mathrm{PM}}(x'') = \frac{\sqrt{3}}{4\pi}\delta\left(x''-2\mathrm{cm}\right) + \frac{\sqrt{3}}{4\pi}\delta\left(x''+2\mathrm{cm}\right)$$

Its Fourier transform is $G_{\mathrm{PP}+}(u) = \frac{\sqrt{3}}{4\pi}\exp\left\{i2\pi u \times 2\mathrm{cm}\right\} + \frac{\sqrt{3}}{4\pi}\exp\left\{-i2\pi u \times 2\mathrm{cm}\right\}$

so the output field and intensity are $g_{\mathrm{out}}(x') = \frac{\sqrt{3}}{2\pi}\cos\left(2\pi\frac{x'}{5\mu\mathrm{m}}\right)$ $\quad I_{\mathrm{out}}(x') = \frac{3}{8\pi^2} + \frac{3}{8\pi^2}\cos\left(2\pi\frac{4x'}{10\mu\mathrm{m}}\right)$

Example: band-pass filtering a binary amplitude grating

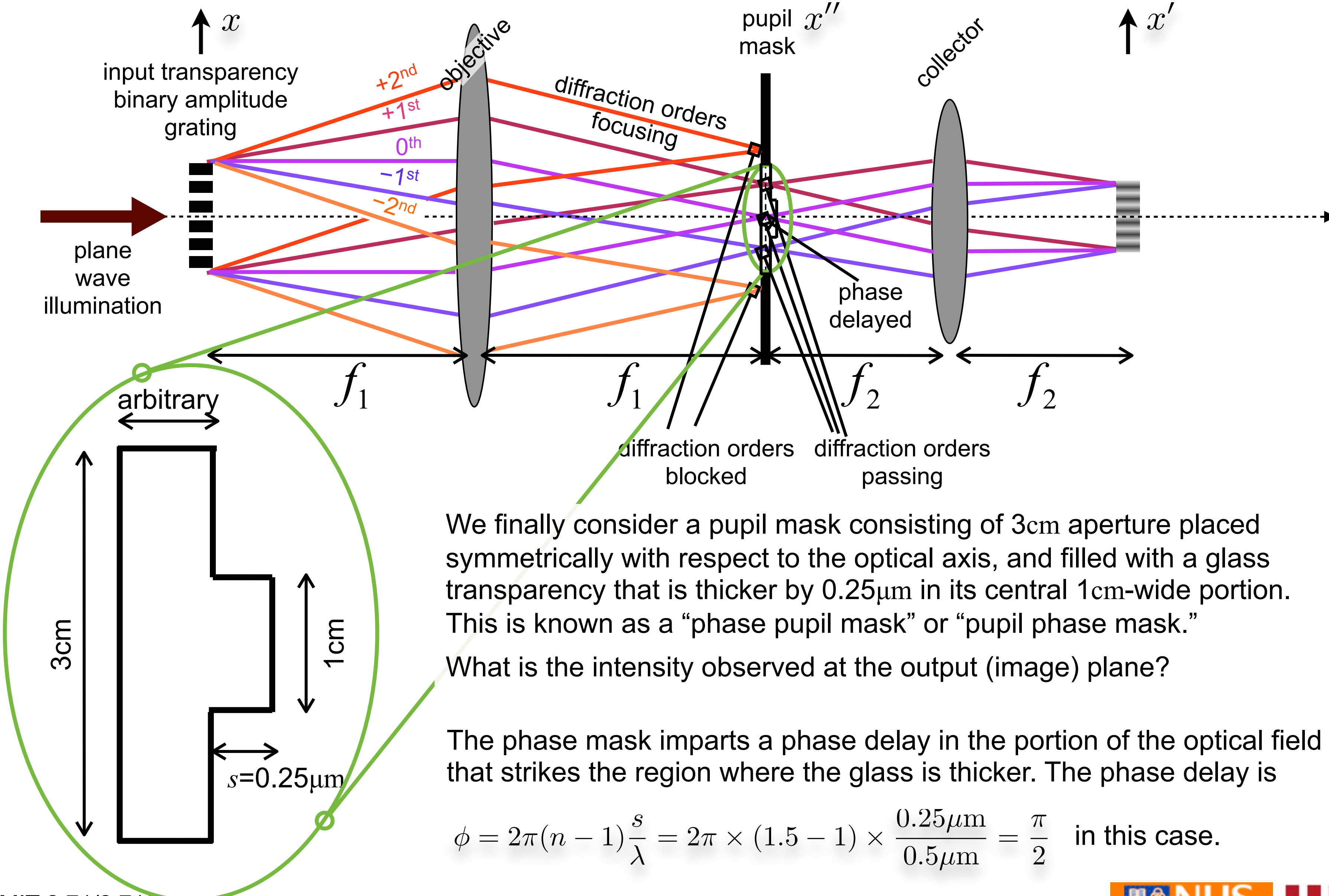


band-pass filtered binary amplitude grating

field: 2nd harmonic
intensity: 4th harmonic
(because of squaring)

contrast = 1

Example: binary amplitude grating through phase pupil mask



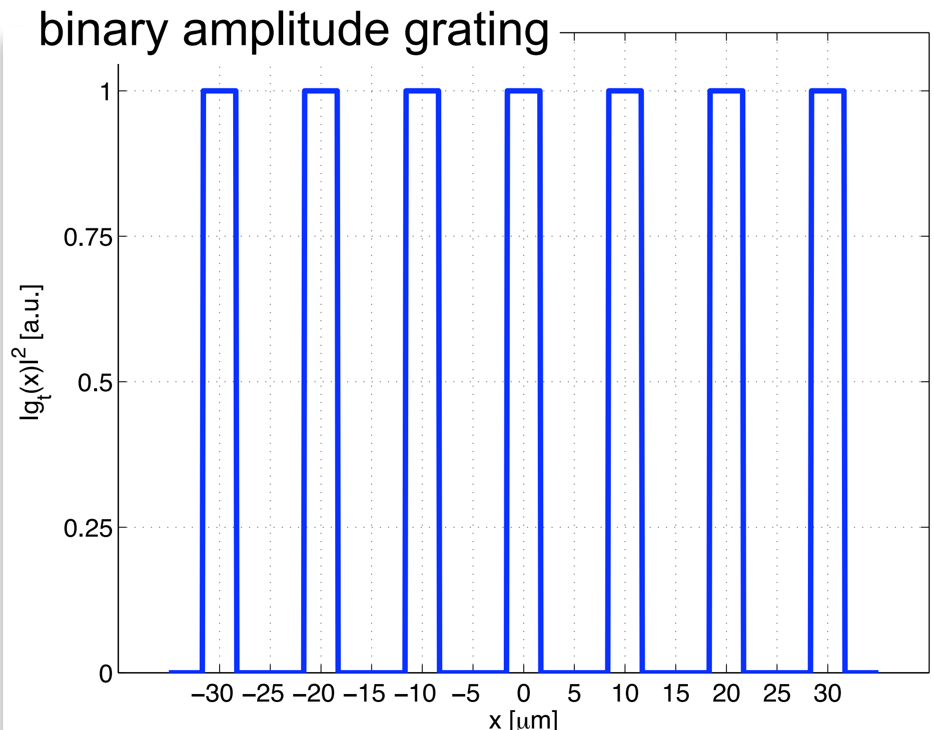
We finally consider a pupil mask consisting of 3cm aperture placed symmetrically with respect to the optical axis, and filled with a glass transparency that is thicker by 0.25µm in its central 1cm-wide portion. This is known as a "phase pupil mask" or "pupil phase mask."

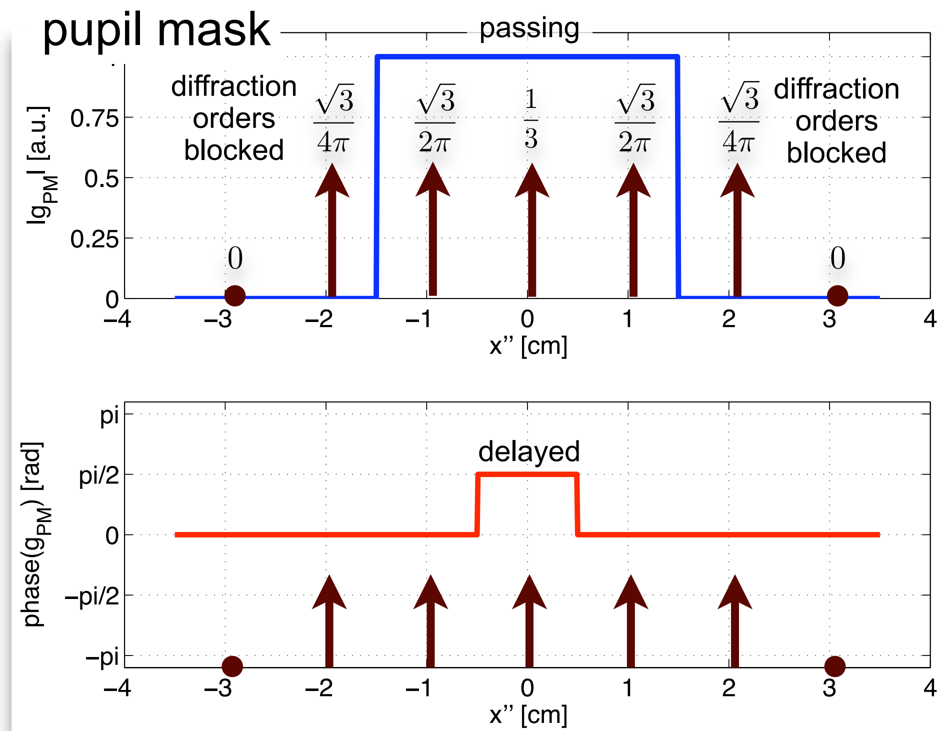
What is the intensity observed at the output (image) plane?

The phase mask imparts a phase delay in the portion of the optical field that strikes the region where the glass is thicker. The phase delay is

$$\phi = 2\pi(n-1)\frac{s}{\lambda} = 2\pi \times (1.5-1) \times \frac{0.25\mu\text{m}}{0.5\mu\text{m}} = \frac{\pi}{2}$$ in this case.

Example: binary amplitude grating through phase pupil mask





This pupil mask is

$$g_{\mathrm{PM}}(x'') = \begin{cases} 0, & \text{if } |x''| > 1.5\mathrm{cm} \\ 1, & \text{if } 1.5\mathrm{cm} \le |x''| < 0.5\mathrm{cm} \\ e^{i\phi}, & \text{if } |x''| \le 0.5\mathrm{cm} \end{cases} = \mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right) + \left(e^{i\phi} - 1\right)\mathrm{rect}\left(\frac{x''}{1\mathrm{cm}}\right)$$

$$\left[\text{since } e^{i\pi/2} = i\right] \quad = \mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right) + (i-1)\,\mathrm{rect}\left(\frac{x''}{1\mathrm{cm}}\right).$$

so the field at the pupil plane to the right of the pupil mask is

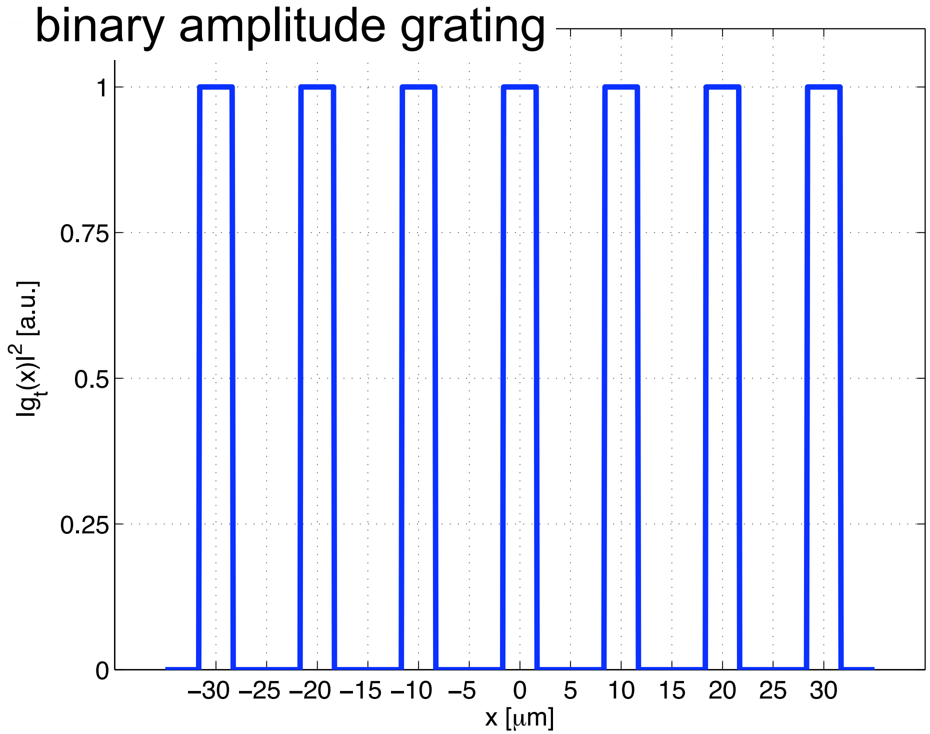
$$g_{\mathrm{PP}+}(x'') = g_{\mathrm{PP}-}(x'') \times g_{\mathrm{PM}}(x'') = \frac{i}{3}\delta(x'') + \frac{\sqrt{3}}{2\pi}\delta(x'' - 1\mathrm{cm}) + \frac{\sqrt{3}}{2\pi}\delta(x'' + 1\mathrm{cm})$$

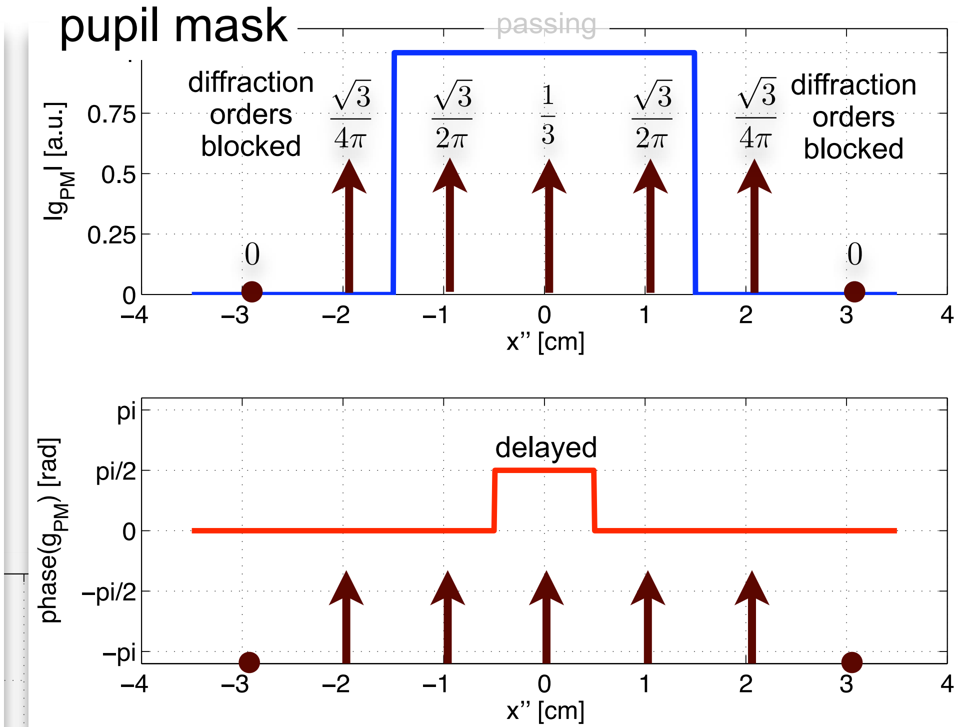
and the output field and intensity are

$$g_{\mathrm{out}}(x') = \frac{i}{3} + \frac{\sqrt{3}}{\pi}\cos\left(2\pi\frac{x'}{10\mu m}\right) \qquad I_{\mathrm{out}}(x') = \left(\frac{1}{3}\right)^2 + \frac{3}{2\pi^2} + \frac{3}{2\pi^2}\cos\left(2\pi\frac{2x'}{10\mu m}\right)$$

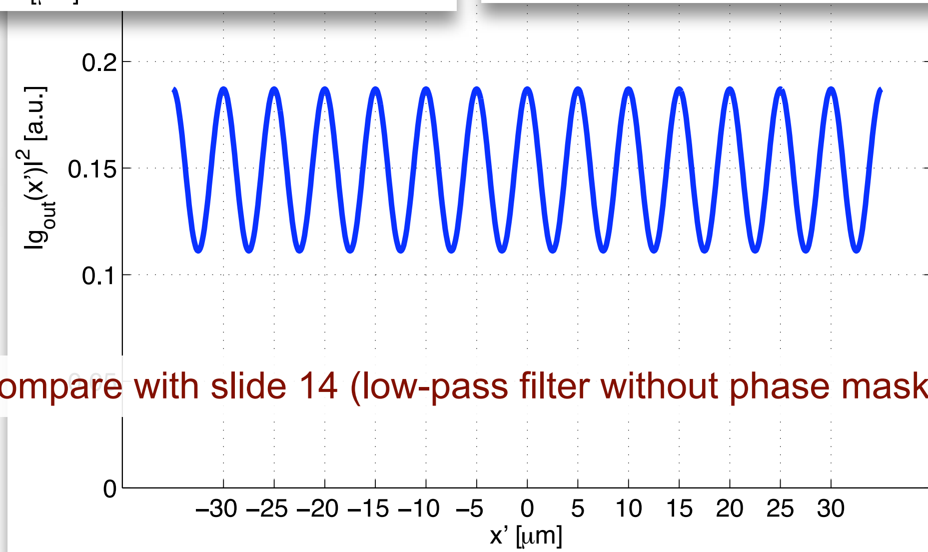
compare with slide 14 (low-pass filter without phase mask)

Example: binary amplitude grating through phase pupil mask





$$\text{Contrast } V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{0.1871 - 0.1111}{0.1871 + 0.1111} = 0.2548$$

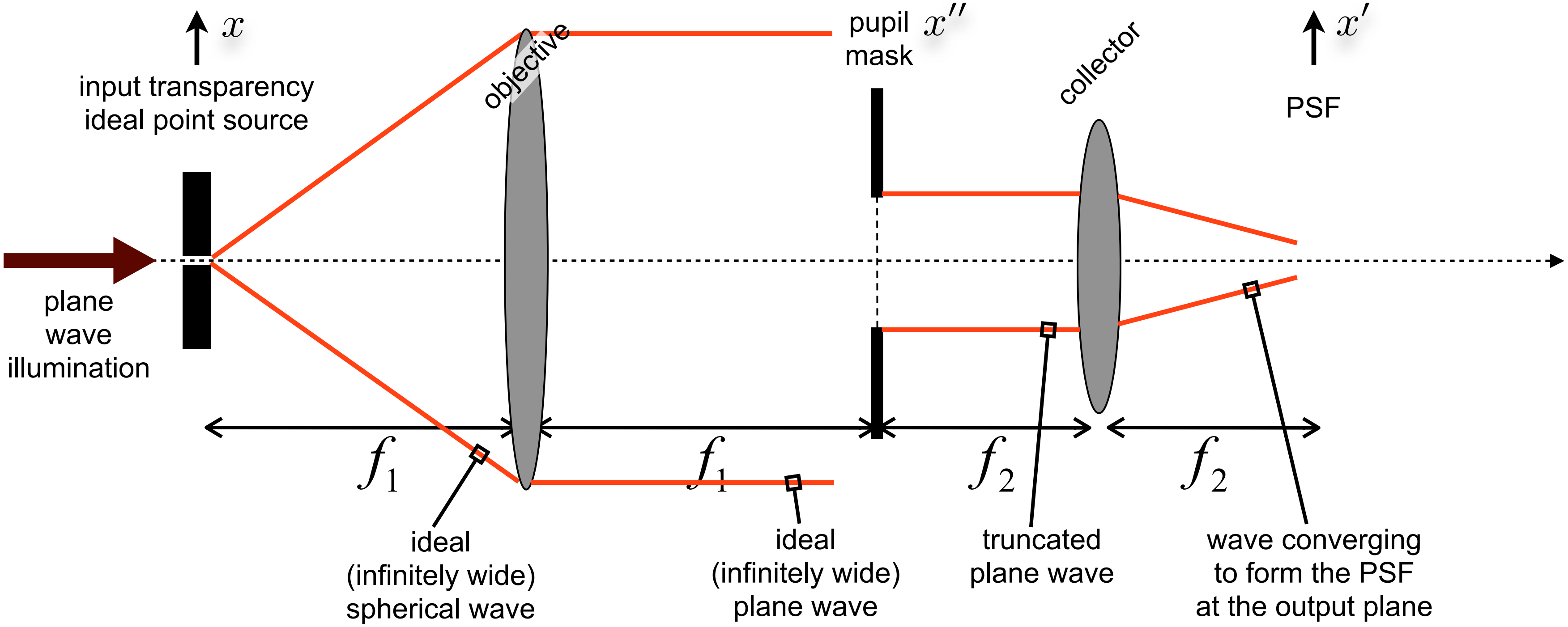


binary amplitude grating filtered by pupil phase mask

field: 1st harmonic
intensity: 2nd harmonic
(because of squaring)

compare with slide 14 (low-pass filter without phase mask)

The Point-Spread Function (PSF) of a low-pass filter



Now consider the same 4F system but replace the input transparency with an ideal point source, implemented as an opaque sheet with an infinitesimally small transparent hole and illuminated with a plane wave on axis (actually, any illumination will result in a point source in this case, according to Huygens.)

In Systems terminology, we are exciting this linear system with an impulse (delta-function); therefore, the response is known as **Impulse Response**.
In Optics terminology, we use instead the term **Point-Spread Function (PSF)** and we denote it as $h(x', y')$.

The sequence to compute the PSF *of a 4F system* is:

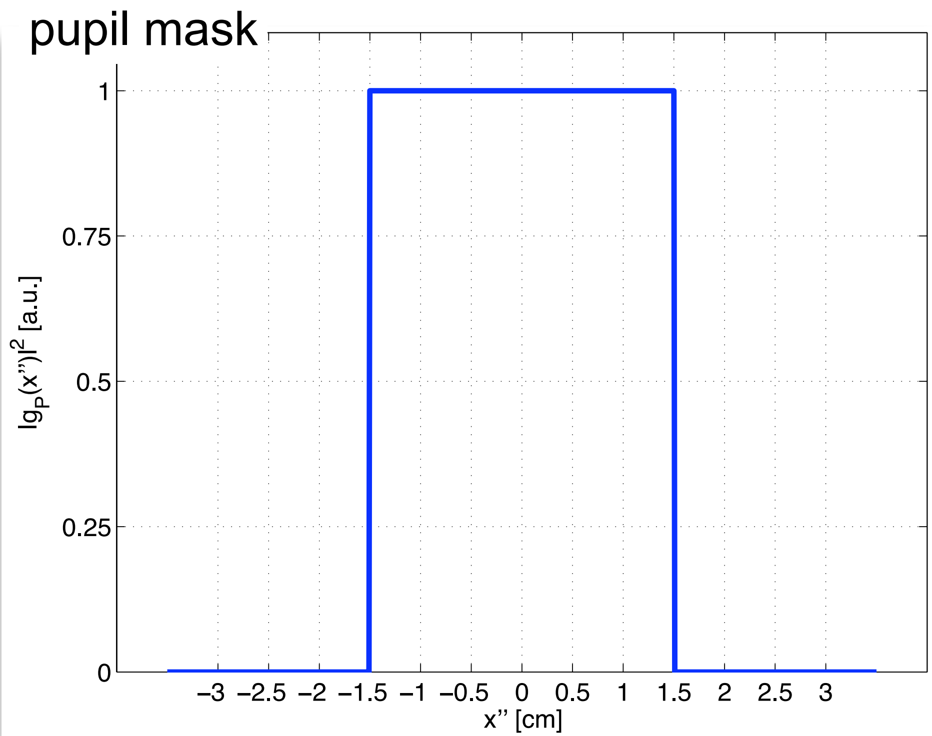
➡ observe that the Fourier transform of the input transparency $\delta(x)$ is simply 1 everywhere at the pupil plane

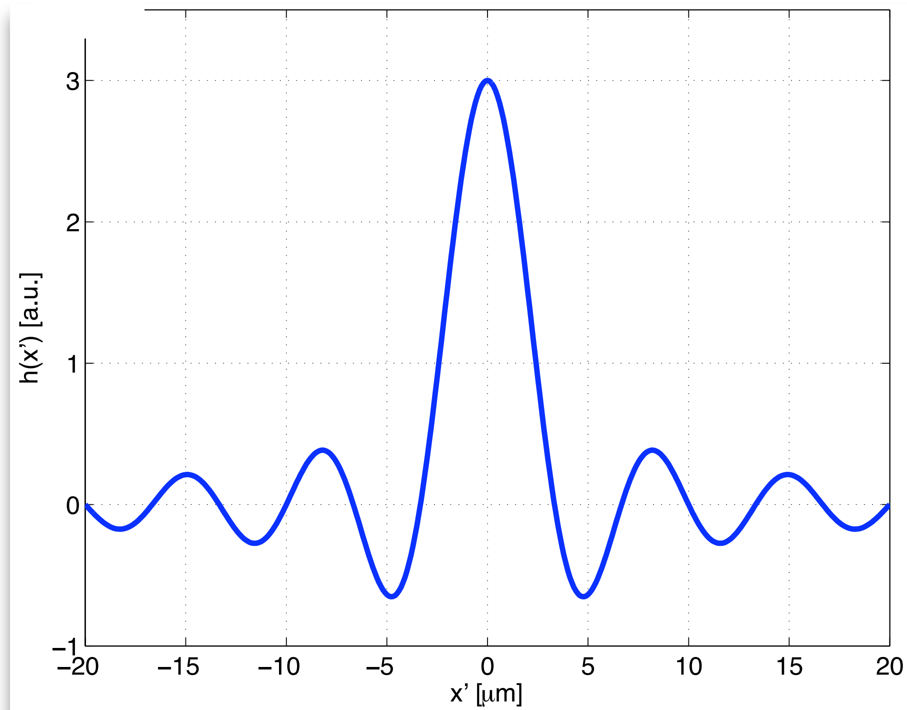
➡ multiply 1 by the complex amplitude transmittance of the pupil mask

➡ Fourier transform the product and scale to the output plane coordinates $x' = u\lambda f_2$.

➡Therefore, the PSF is simply the Fourier transform of the pupil mask, scaled to the output coordinates $x' = u\lambda f_2$

Example: PSF of a low-pass filter





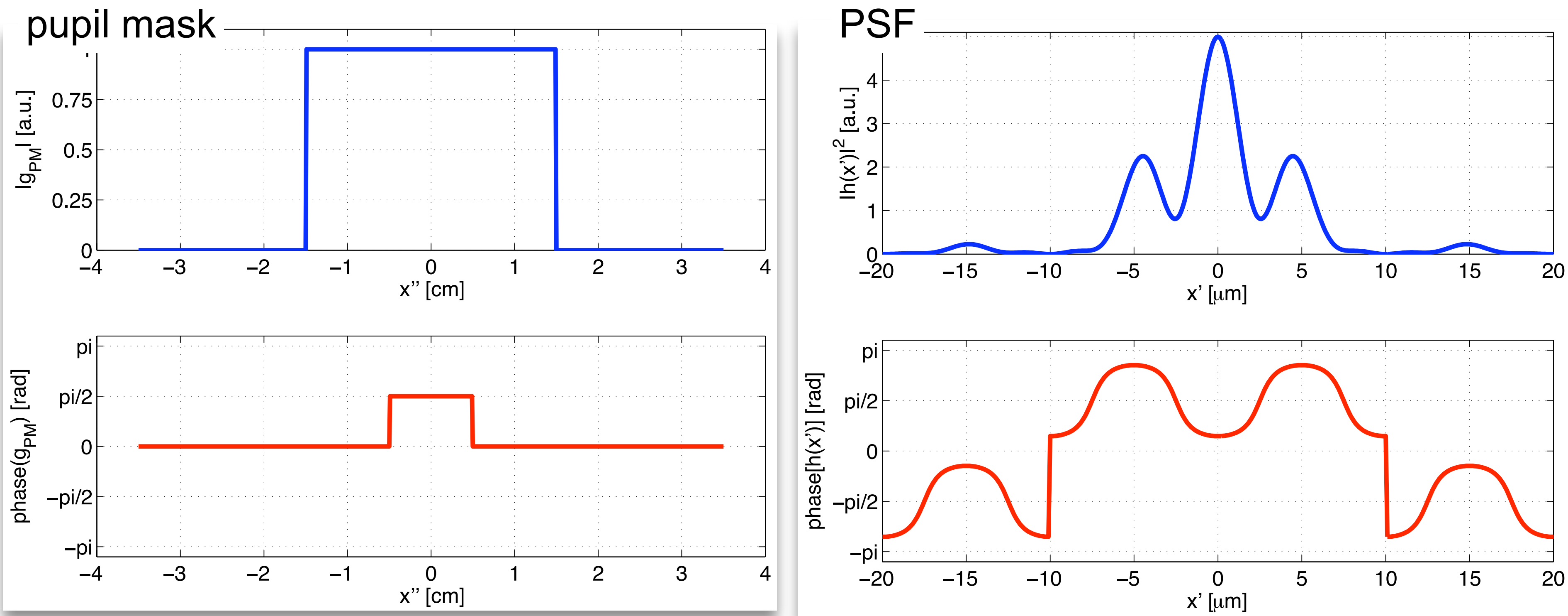
The pupil mask is $g_{\mathrm{PM}}(x'') = \mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right)$. If the input transparency is δ(x), the field at the pupil plane to the right of the pupil mask is $g_{\mathrm{PP}+}(x'') = g_{\mathrm{PP}-}(x'') \times g_{\mathrm{PM}}(x'') = 1 \times \mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right) \Rightarrow G_{\mathrm{PP}+}(u) = (3\ \mathrm{cm})\,\mathrm{sinc}\,(u \times 3\ \mathrm{cm})$.

The output field, i.e. the PSF is

$$g_{\mathrm{out}}(x') \equiv h(x') = G_{\mathrm{PM}}\left(\frac{x'}{\lambda f}\right) = (3\ \mathrm{cm})\,\mathrm{sinc}\left(\frac{x' \times 3\ \mathrm{cm}}{0.5\mu\mathrm{m} \times 20\mathrm{cm}}\right) = (3\ \mathrm{cm})\,\mathrm{sinc}\left(\frac{x'}{3.33\mu\mathrm{m}}\right).$$

The scaling factor (3×) in the PSF ensures that the integral $\int |h(x')|^2 \mathrm{d}x$ equals the portion of the input energy transmitted through the system

Example: PSF of a phase pupil filter

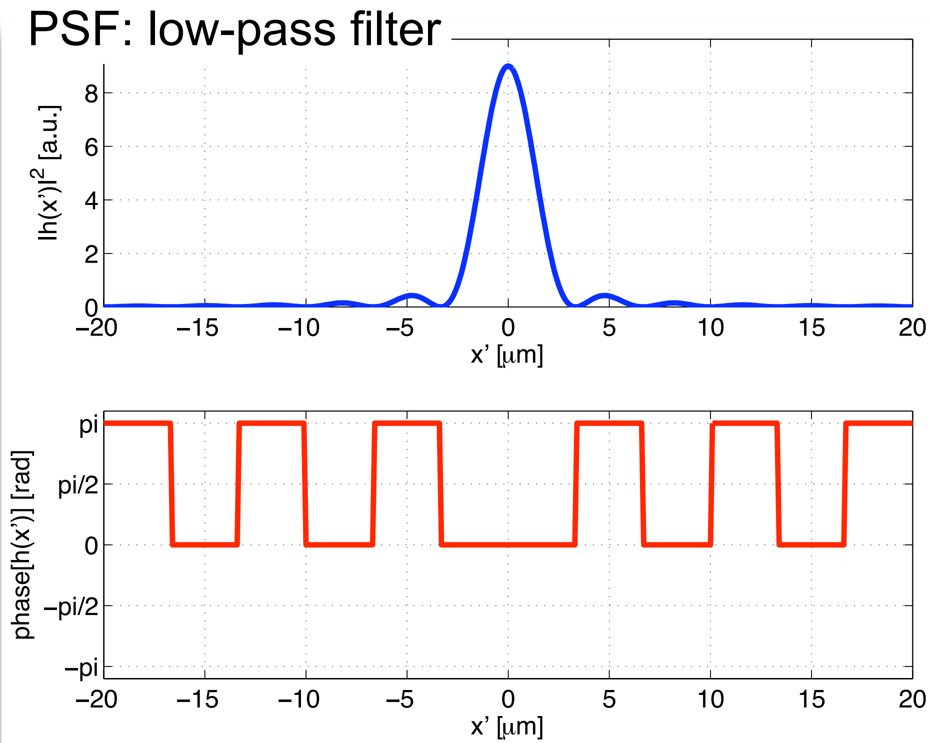


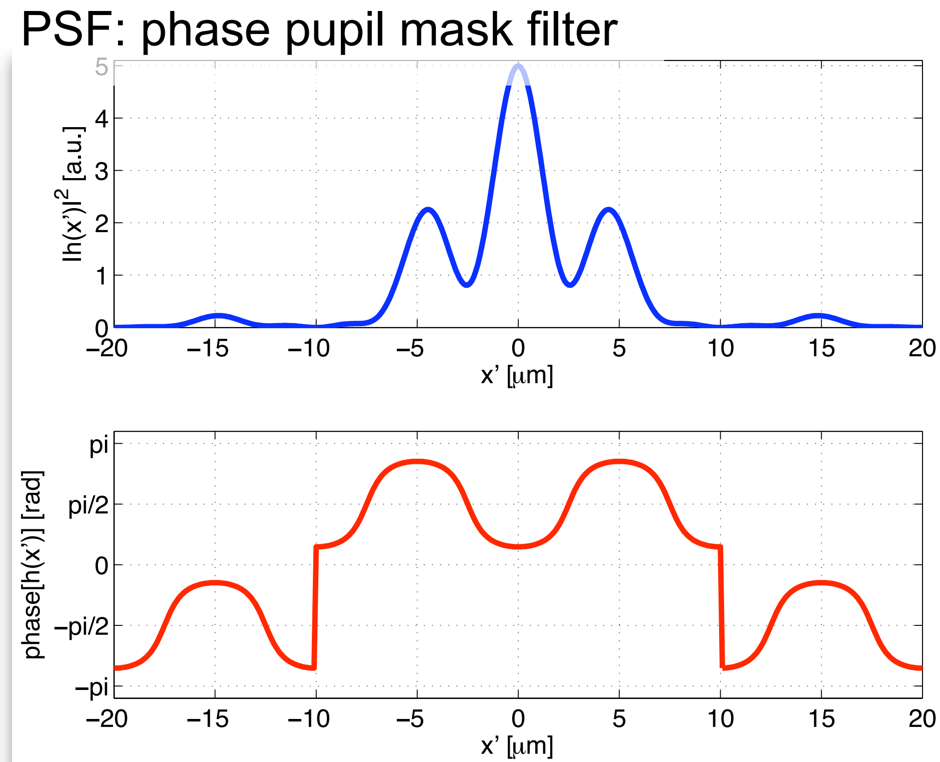
The pupil mask is $g_{\mathrm{PM}}(x'') = \mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right) + (i-1)\,\mathrm{rect}\left(\frac{x''}{1\mathrm{cm}}\right)$.

The PSF is

$$
\begin{aligned}
h(x') &= G_{\mathrm{PM}}\left(\frac{x'}{\lambda f}\right) = (3\ \mathrm{cm})\,\mathrm{sinc}\left(\frac{x'}{3.33\mu\mathrm{m}}\right) + (i-1)\times(1\ \mathrm{cm})\,\mathrm{sinc}\left(\frac{x'}{1\mu\mathrm{m}}\right); \\
|h(x')|^2 &= \left[3\,\mathrm{sinc}\left(\frac{x'}{3.33}\right) - \mathrm{sinc}\left(\frac{x'}{1}\right)\right]^2 + \left[\mathrm{sinc}\left(\frac{x'}{1}\right)\right]^2 \\
\angle h(x') &= \arctan\frac{\mathrm{sinc}\,(x')}{3\,\mathrm{sinc}\,(0.3\,x') - \mathrm{sinc}\,(x')}.
\end{aligned}
$$

Comparison: low-pass filter vs phase pupil mask filter





$$\begin{aligned} h(x') &= (3\ \text{cm})\,\text{sinc}\left(\frac{x'}{3.33\mu m}\right); \\ |h(x')|^2 &= 9\,\text{sinc}^2\left(\frac{x'}{3.33}\right) \\ \angle h(x') &= \begin{cases} 0, & \text{if sinc}\,(0.3\,x') > 0 \\ \pi, & \text{if sinc}\,(0.3\,x') < 0 \end{cases}. \end{aligned}$$

$$\begin{aligned} h(x') &= (3\ \text{cm})\,\text{sinc}\left(\frac{x'}{3.33\mu m}\right) + (i-1)\times(1\ \text{cm})\,\text{sinc}\left(\frac{x'}{1\mu m}\right); \\ |h(x')|^2 &= \left[3\,\text{sinc}\left(\frac{x'}{3.33}\right) - \text{sinc}\left(\frac{x'}{1}\right)\right]^2 + \left[\text{sinc}\left(\frac{x'}{1}\right)\right]^2 \\ \angle h(x') &= \arctan\frac{\text{sinc}\,(x')}{3\,\text{sinc}\,(0.3\,x') - \text{sinc}\,(x')}. \end{aligned}$$