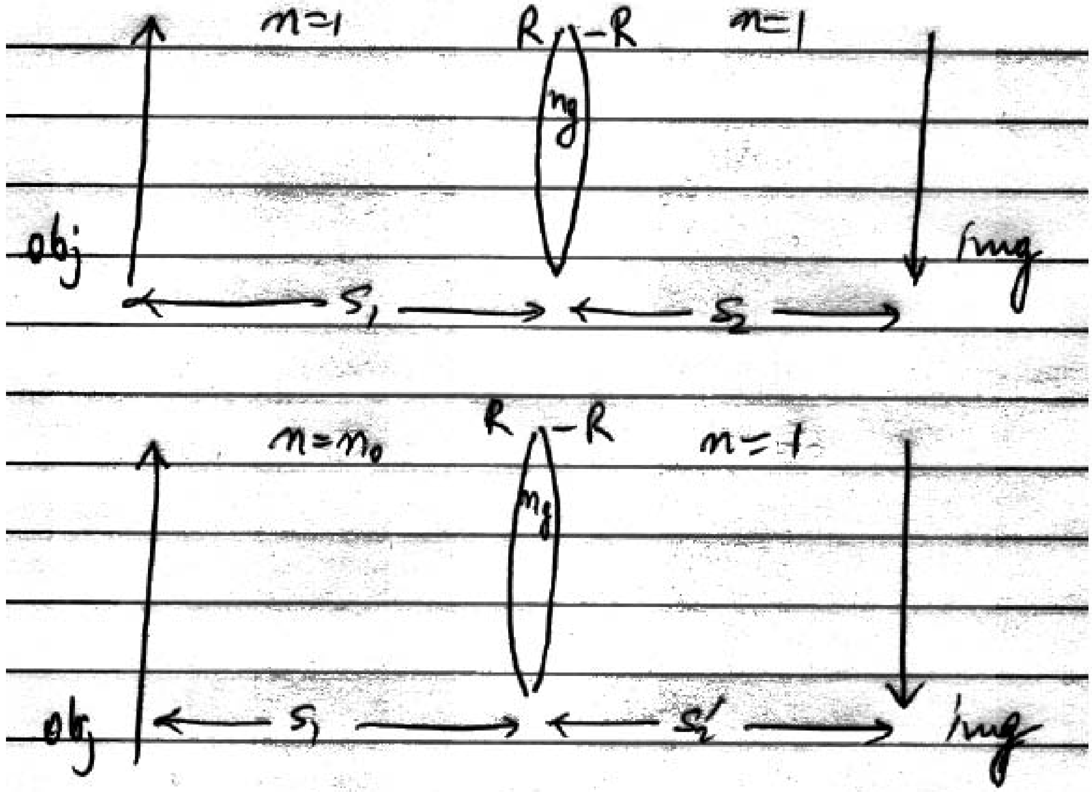
1. A thin bi-convex lens with the same absolute curvature on both faces is used in the two imaging systems shown below. In the first, both object and image are in air, whereas in the second the object is "immersed" in a material of index $n_0 < n_g$, where n_g is the index of the glass used to make the lens. Compare the two imaging systems in terms of imaging condition and magnification.



Solution: The first system is the usual single-lens imaging system, so it satisfies:

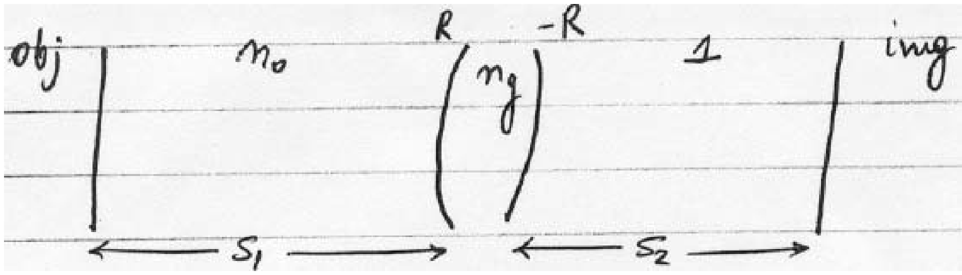
imaging condition: $$\frac{1}{S_1} + \frac{1}{S_2} = \frac{1}{f}$$

lateral magnification: $$m = -\frac{S_2}{S_1}$$

where, from the way the lens is described,

$$\frac{1}{f} = (n_g - 1)\left(\frac{1}{R} - \frac{1}{-R}\right) = (n_g - 1)\frac{2}{R}$$

The second system is best modeled anew using the matrix formulation for ray propagation:



$$\begin{pmatrix} \alpha_{\text{img}} \\ x_{\text{img}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ S_2' & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1-n_g}{-R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{n_g-n_0}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{S_1}{n_0} & 1 \end{pmatrix} \begin{pmatrix} n_0\ \alpha_{\text{obj}} \\ x_{\text{obj}} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ S_2' & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{f'} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{S_1}{n_0} & 1 \end{pmatrix} \begin{pmatrix} n_0\ \alpha_{\text{obj}} \\ x_{\text{obj}} \end{pmatrix}$$
$$= \begin{pmatrix} 1-\frac{S_1}{n_0 f'} & -\frac{1}{f'} \\ S_2' + \frac{S_1}{n_0} - \frac{S_1 S_2'}{n_0 f'} & 1-\frac{S_2'}{f'} \end{pmatrix} \begin{pmatrix} n_0\ \alpha_{\text{obj}} \\ x_{\text{obj}} \end{pmatrix}$$

where $\dfrac{1}{f'} = \left(n_g - \dfrac{n_0+1}{2}\right)\dfrac{2}{R}$ (note $f' > f$)

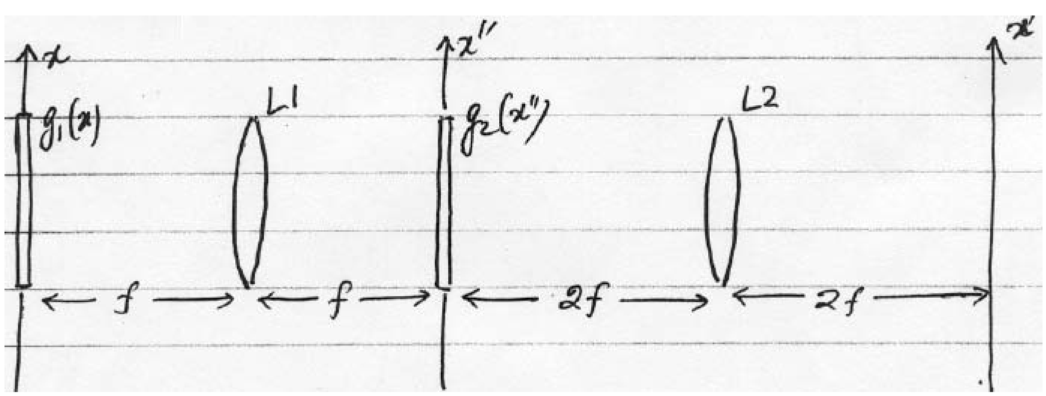
Imaging condition: $\dfrac{\partial x_{\text{img}}}{\partial \alpha_{\text{obj}}} = 0 \Rightarrow S_2' + \dfrac{S_1}{n_0} - \dfrac{S_1 S_2'}{n_0 f'} = 0 \Rightarrow \dfrac{n_0}{S_1} + \dfrac{1}{S_2'} = \dfrac{1}{f'}$

Assuming the imaging condition is satisfied, the system equation becomes:

$$\begin{pmatrix} \alpha_{\text{img}} \\ x_{\text{img}} \end{pmatrix} = \begin{pmatrix} 1-\frac{S_1}{n_0 f'} & -\frac{1}{f'} \\ 0 & 1-\frac{S_2'}{f'} \end{pmatrix} \begin{pmatrix} n_0\ \alpha_{\text{obj}} \\ x_{\text{obj}} \end{pmatrix}$$

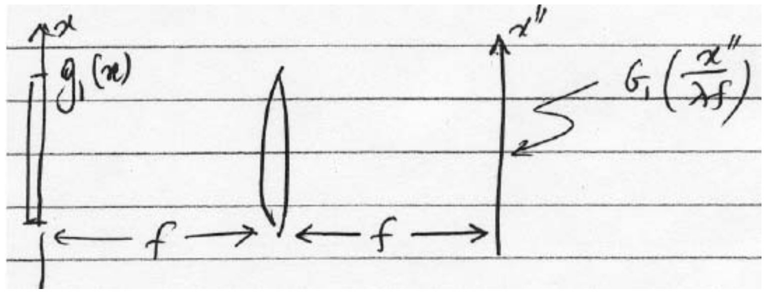
$$\Rightarrow m' = \frac{x_{\text{img}}}{x_{\text{obj}}} = 1 - \frac{S_2'}{f'} = -\frac{n_0 S_2'}{S_1} \text{ (from the imaging condition)}$$

2. In the configuration below, lenses L1 and L2 are identical with focal length f, and we consider them to be infinite aperture. The system is illuminated coherently by an on-axis plane wave.

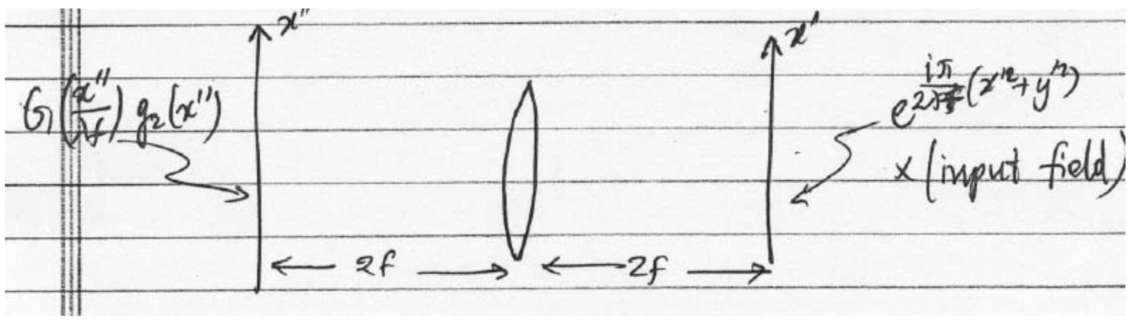


(a) Write an expression for the field at x' in terms of the thin complex transparencies g_1, g_2.

Solution: The first part (to the left of $g_2(x'')$) is a Fourier-transforming system:



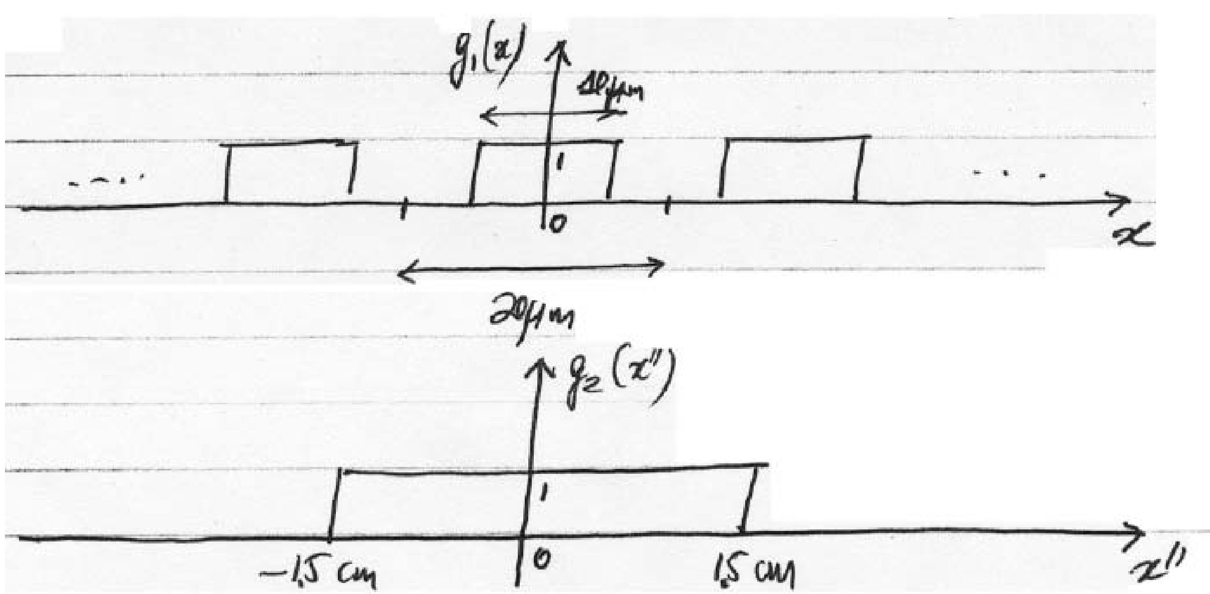
The second part (to the right of g_2) is a single-lens imaging system with unit magnification:



$$\text{The field at } x' = \exp\left(i\frac{\pi}{2\lambda f}(x'^2 + y'^2)\right) G_1\left(\frac{x''}{\lambda f}\right) g_2(x'')$$

(Note that the first exponential term could have been omitted...)

(b) Consider the specific case with $f = 10$ cm and g_1, g_2 defined as:

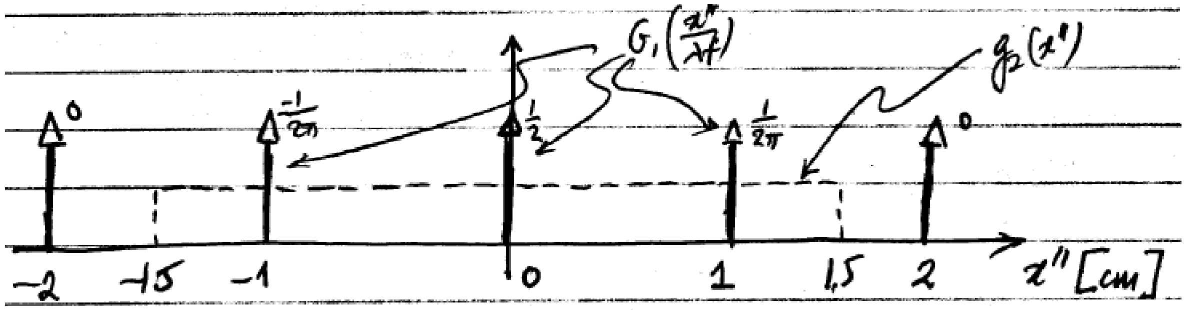


If $\lambda = 1\mu\text{m}$, derive and sketch the intensity at the output plane x'.

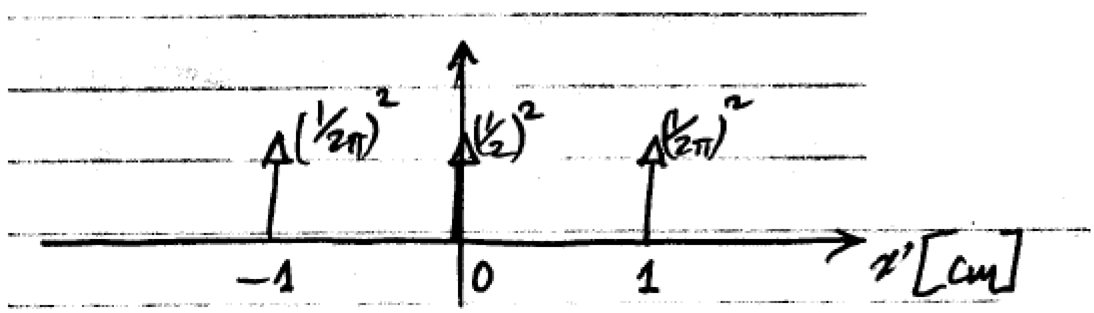
Solution: The binary grating has fundamental period $\Lambda = 20\mu\text{m}$ and duty cycle 50%, i.e. it is of the form:

$$g_1(x) = \frac{1}{2}\left[1 + \operatorname{sgn}\left(\cos\frac{2\pi x}{\Lambda}\right)\right] = \frac{1}{2}\sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n}{2}\right) e^{i2\pi\frac{x}{\Lambda}}$$

$$G_1\left(\frac{x''}{\lambda f}\right) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n}{2}\right) \underbrace{\delta\left(\frac{x''}{\lambda f} - \frac{1}{\Lambda}\right)}_{\text{Mind the coordinates!}}$$

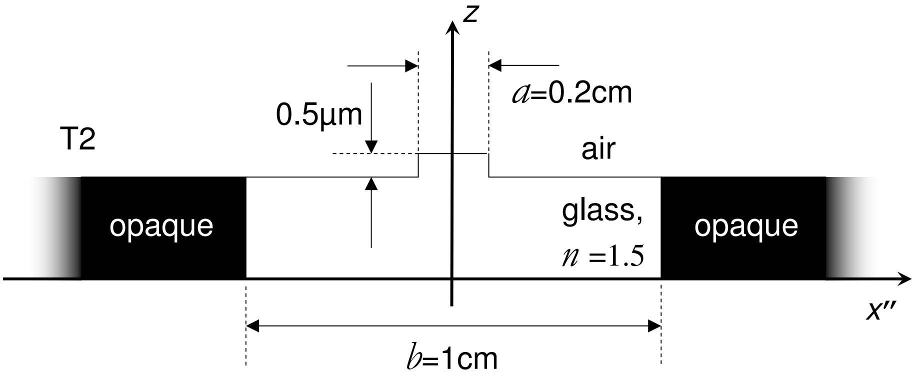


Since $\dfrac{\lambda f}{\Lambda} = \dfrac{1\mu\text{m} \times 20\text{cm}}{20\mu\text{m}} = 1$ cm, the g_2 transparency only lets orders $-1, 0, +1$ to pass through, therefore the field (or intensity) at the output plane is three bright spots.



Example: OTF of the Zernicke phase mask

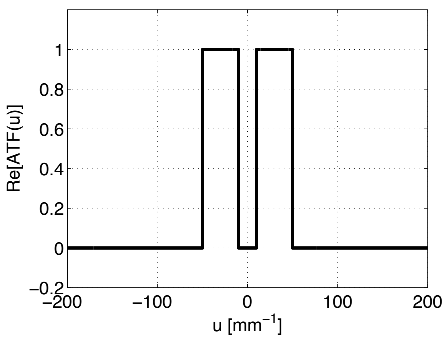
The thin phase transparency whose schematic is given below is placed at the Fourier plane of a unit magnification telescope with focal length $f = 10\text{cm}$. What is the optical transfer function for quasi–monochromatic illumination at wavelength $\lambda = 1\mu\text{m}$?

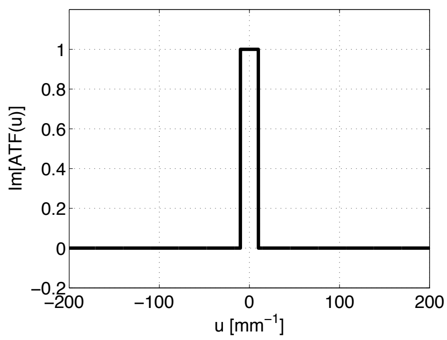


Solution: Since $b/(\lambda f) = 100$ cycles/mm and $a/(\lambda f) = 20$ cycles/mm, and $2\pi\,(1.5 - 1.0)\times 0.5\mu\text{m}/\lambda = \pi/2$, the Amplitude Transfer Function (ATF) *vs.* spatial frequency u is

$$T(u) = \text{rect}\left(\frac{u}{100}\right) + \left(\mathrm{e}^{i\pi/2} - 1\right)\text{rect}\left(\frac{u}{20}\right), \tag{1}$$

whose real and imaginary parts are plotted below.





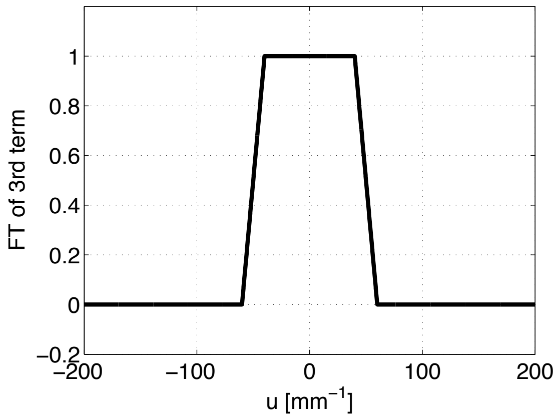
The Optical Transfer Function (OTF) is the autocorrelation of the ATF, and the easiest way to compute it is as follows. First, we obtain the Fourier transform $t(x)$ of the ATF as

$$t(x) = 100\text{sinc}\,(100x) + 20\left(\mathrm{e}^{i\pi/2} - 1\right)\text{sinc}\,(20x)\,. \tag{2}$$

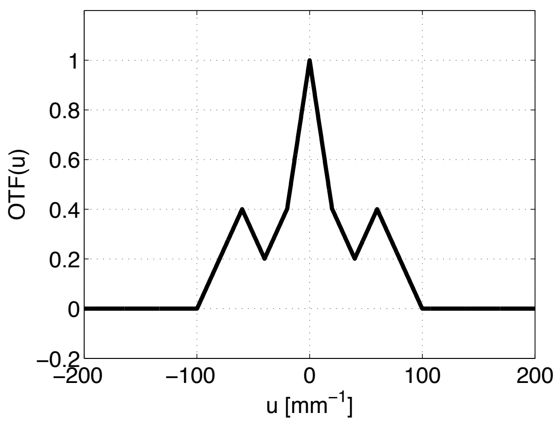
The Fourier transform of the OTF is the modulus–squared of $t(x)$, *i.e.*

$$|t\,(x)|^2 = 10^4\text{sinc}^2\,(100x) + 800\text{sinc}^2\,(20x) - 4\times 10^3\text{sinc}\,(100x)\,\text{sinc}\,(20x)\,. \tag{3}$$

The inverse Fourier transform of the first two terms is easy, yielding triangular functions of full–widths 200 and 40, respectively. The inverse Fourier transform of the third term is computed as the convolution of two rect's of width 100 and 20. Some thought will convince you that this convolution equals the "truncated triangle" function shown below normalized to 1.



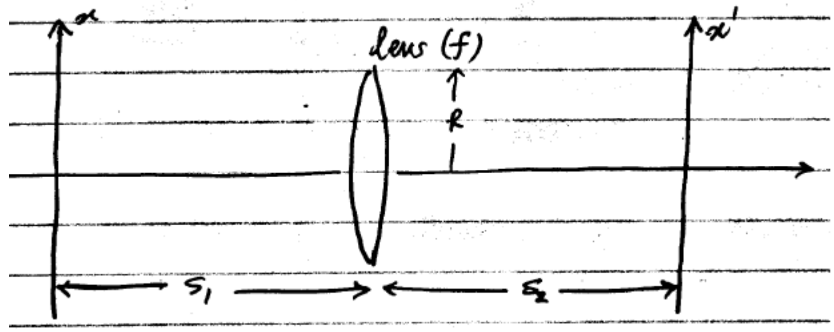
Summing the three inverse Fourier transforms with their appropriate weights and normalizing the DC value to 1, we finally obtain the OTF as shown below.



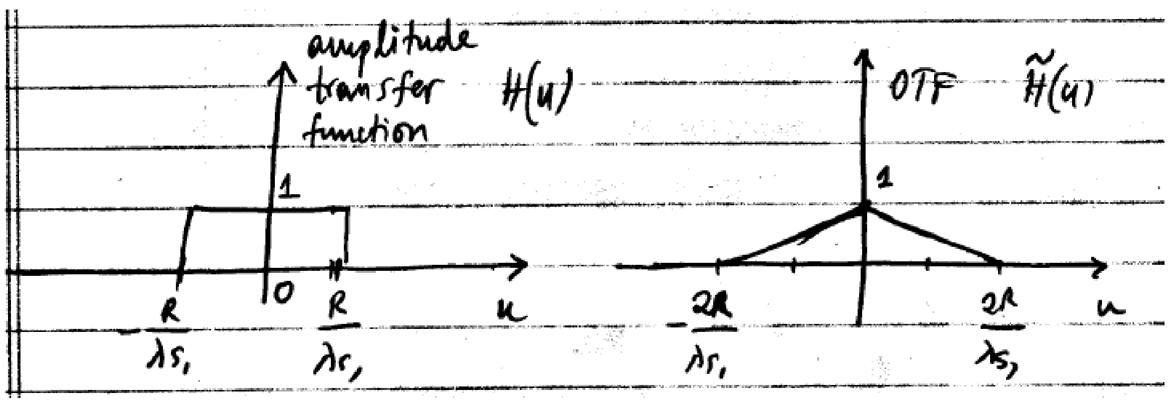
The same result may be obtained by directly computing the autocorrelation of $t(u)$ in the frequency domain, but that would have been much more tedious.

You may have recognized $T(u)$ as a Zernicke phase mask, which is used in microscopy for "phase–contrast imaging," *i.e.* obtaining intensity images of phase features in a transparent object. Some thought will convince you that phase contrast results from the depression in the OTF at intermediate frequencies which acts "sort–of" like a derivative, or better yet like a Hilbert transform. (Hilbert transform is an engine that converts a cosine to a sine, in other words introduces $\pi/2$ phase shift.) This particular phase mask is not so good because a is too large—ideally, a should be as small as possible to obtain the Hilbert transform effect in as large a fraction of the admissible bandwidth as possible!

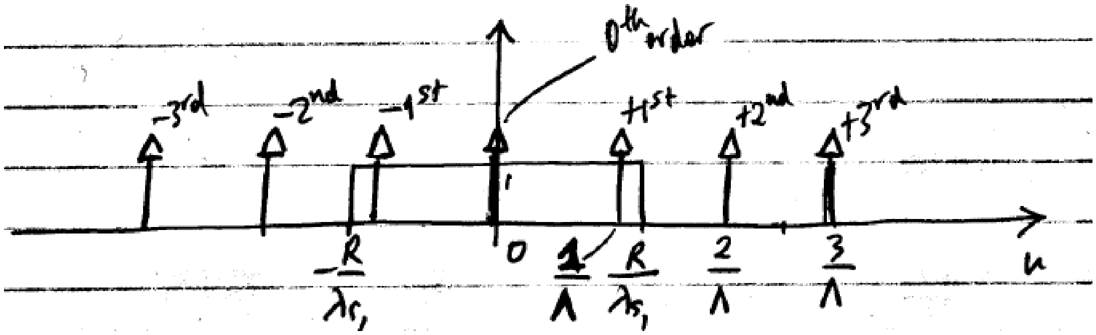
4. Goodman, 6-10



Solution: The imaging condition is $\frac{1}{S_1}+\frac{1}{S_2}=\frac{1}{f}$. We're given $S_1 = 2f \Rightarrow S_2 = 2f$

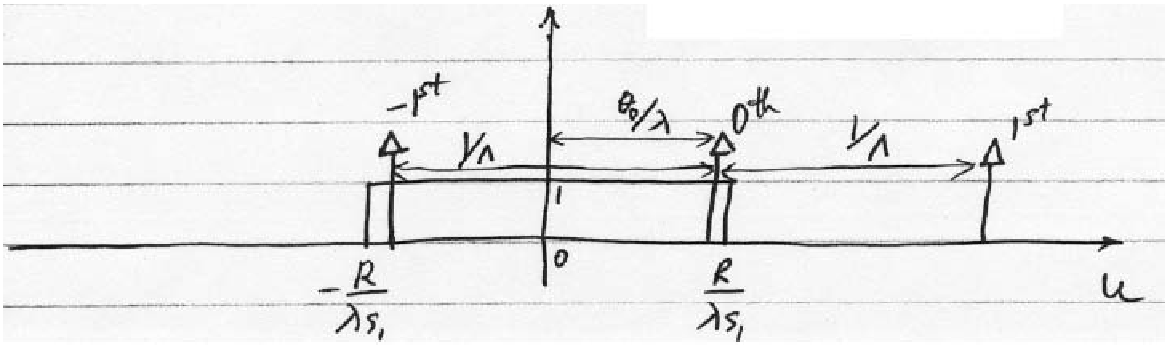


Limitation on coherent <u>on-axis</u> illumination:



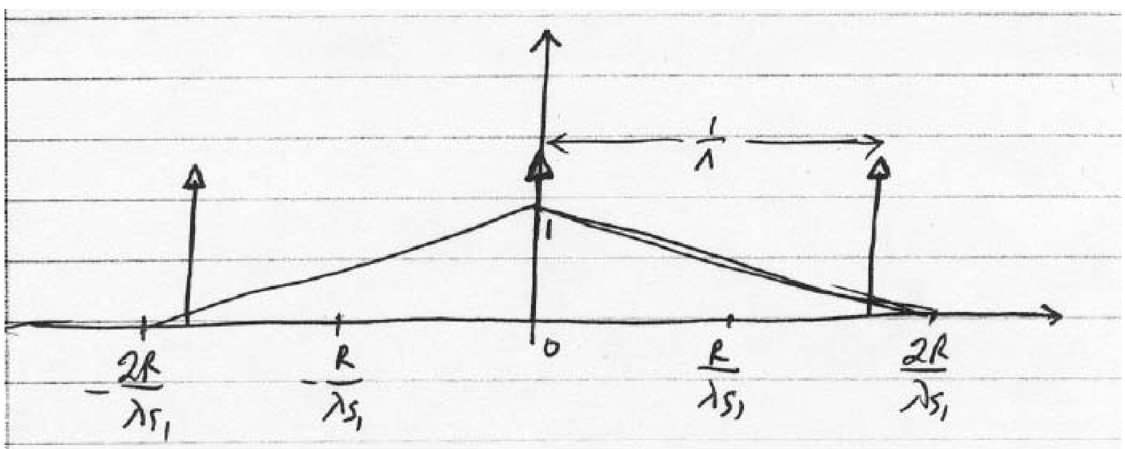
$$\frac{1}{\Lambda} < \frac{R}{\lambda S_1} \Rightarrow R > \frac{\lambda S_1}{\Lambda} = \frac{1\mu\text{m} \times 20\text{cm}}{10\mu\text{m}} = 2\text{cm}$$

Limitation on coherent <u>off-axis</u> (at angle θ_0) illumination:



$$\frac{1}{\Lambda} < \frac{2R}{\lambda S_1} \Rightarrow R > 1\text{cm}$$

Limitation on incoherent illumination:



$$\frac{1}{\Lambda} < \frac{2R}{\lambda S_1} \Rightarrow R > 1\text{cm}$$

5. Calculate and sketch the Fourier transform $\mathcal{F}(u)$ of the function

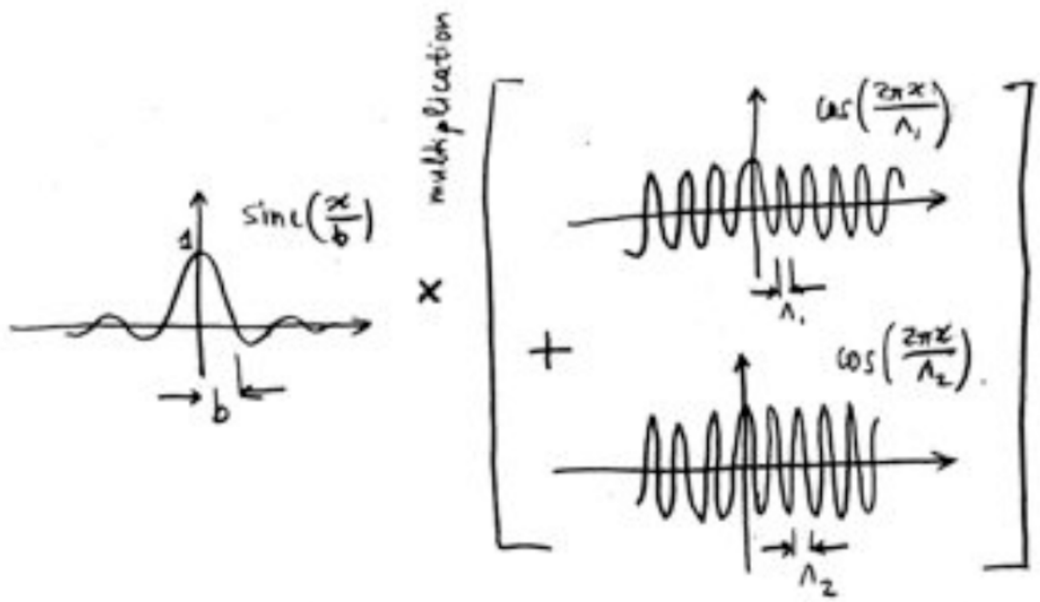
$$f(x) = \operatorname{sinc}\left(\frac{x}{b}\right)\left[\cos\left(\frac{2\pi x}{\Lambda_1}\right) + \cos\left(\frac{2\pi x}{\Lambda_2}\right)\right]$$

Assume that the following condition holds:

$$\frac{1}{b} \ll \frac{1}{\Lambda_1}, \frac{1}{\Lambda_2}, \left|\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}\right|$$

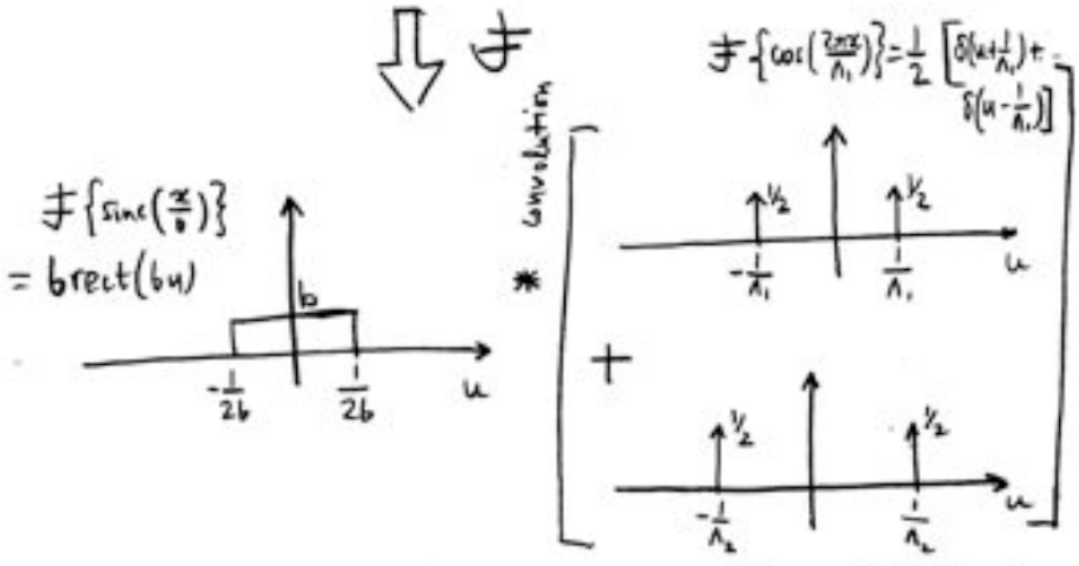
Solution: Space domain

$$f(x) = \operatorname{sinc}\left(\frac{x}{b}\right)\left[\cos\left(\frac{2\pi x}{\Lambda_1}\right) + \cos\left(\frac{2\pi x}{\Lambda_2}\right)\right]$$

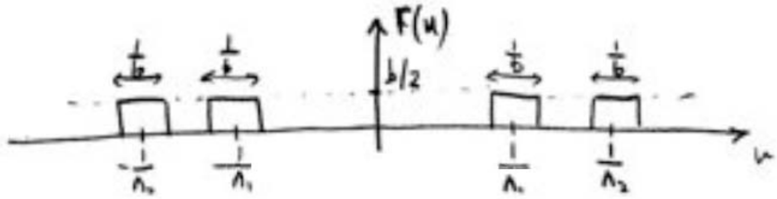


Frequency (Fourier) domain

$$\mathcal{F}(u) = \mathcal{F}\{\mathrm{sinc}\left(\frac{x}{b}\right)\} \otimes \mathcal{F}\left\{\cos\left(\frac{2\pi x}{\Lambda_1}\right) + \cos\left(\frac{2\pi x}{\Lambda_2}\right)\right\}$$
$$= b\mathrm{rect}(bu) \otimes \left(\frac{1}{2}\left[\delta\left(u+\frac{1}{\Lambda_1}\right)+\delta\left(u-\frac{1}{\Lambda_1}\right)\right]+\frac{1}{2}\left[\delta\left(u+\frac{1}{\Lambda_2}\right)+\delta\left(u-\frac{1}{\Lambda_2}\right)\right]\right)$$



(<u>Note:</u> we assumed $\Lambda_2 < \Lambda_1$ for the plots)



Recall the sifting theorem: $h(x) \otimes \delta(x - x_0) = h(x - x_0)$

$$\therefore \mathcal{F}(u) = \tfrac{b}{2}\left[\mathrm{rect}\left(b\left(u - \tfrac{1}{\Lambda_1}\right)\right) + \mathrm{rect}\left(b\left(u + \tfrac{1}{\Lambda_1}\right)\right) + \mathrm{rect}\left(b\left(u - \tfrac{1}{\Lambda_2}\right)\right) + \mathrm{rect}\left(b\left(u + \tfrac{1}{\Lambda_2}\right)\right)\right]$$

6. A very large observation screen (e.g., a blank piece of paper) is placed in the path of a monochromatic light beam (wavelength λ). A sinusoidal interference pattern of the form:

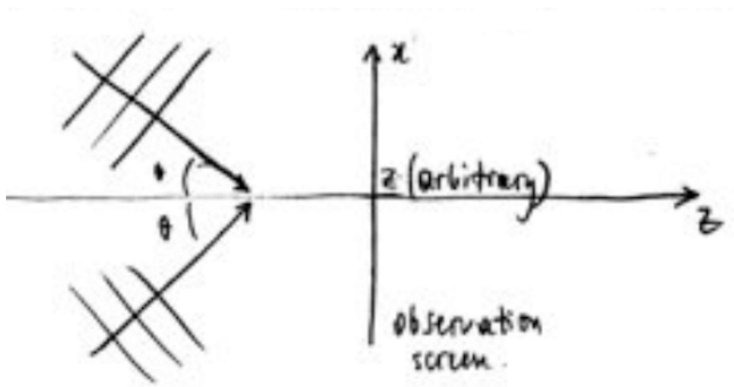
$$I(x) = I_0\left(1 + \cos\frac{2\pi x}{\Lambda}\right)$$

is observed on the screen, where I_0 is a constant with units of optical intensity, Λ is a constant with units of distance, and x is a distance coordinate measured on the observation screen.

   (a) Describe quantitatively two alternative optical fields that could have led to the same measurement on the observation screen.

   Solution: We have seen two occasions of sinusoidal interference patterns arising from optical fields.

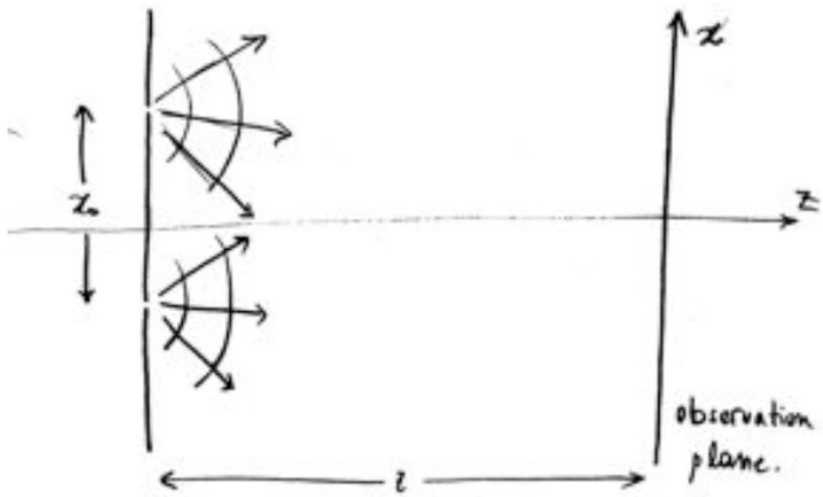
   i. Two plane waves at angle θ with respect to the axis:



$$I(x) = |\exp\left(i(kx\sin\theta + kz\cos\theta)\right) + \exp\left(i(-kx\sin\theta + kz\cos\theta)\right)|^2 \qquad \left(k = \frac{2\pi}{\lambda}\right)$$

$$= |2\cos(kx\sin\theta)|^2 = 2[1 + \cos(2kx\sin\theta)] = 4\left[1 + \cos\frac{2\pi x}{\Lambda}\right] \qquad \text{where } \Lambda = \frac{\lambda}{2\sin\theta}$$

   ii. Two spherical (or cylindrical) waves originating at relative distance x_0, as in Young's interference experiment with two pinholes (or slits):



$$
\begin{aligned}
I(x) &= \left| \frac{\exp\left(i2\pi\frac{z}{\lambda}\right)}{i\lambda z} \exp\left(i\pi \frac{(x-\frac{x_0}{2})^2+y^2}{\lambda z}\right) + \frac{\exp\left(i2\pi\frac{z}{\lambda}\right)}{i\lambda z} \exp\left(i\pi \frac{(x+\frac{x_0}{2})^2+y^2}{\lambda z}\right) \right|^2 \\
&= \frac{1}{(\lambda z)^2} \left| \exp\left(i\pi \frac{x^2+(\frac{x_0}{2})^2-2x(\frac{x_0}{2})}{\lambda z}\right) + \exp\left(i\pi \frac{x^2+(\frac{x_0}{2})^2+2x(\frac{x_0}{2})}{\lambda z}\right) \right|^2 \\
&= \frac{1}{(\lambda z)^2} \left| 2\cos\left(\frac{\pi x x_0}{\lambda z}\right)\right|^2 = \frac{2}{(\lambda z)^2}\left[1+\cos\left(\frac{2\pi x x_0}{\lambda z}\right)\right] \\
&= \frac{2}{(\lambda z)^2}\left[1+\cos\left(\frac{2\pi x}{\Lambda}\right)\right] \qquad \text{where } \Lambda = \frac{\lambda z}{x_0}
\end{aligned}
$$

(b) Describe an experimental procedure by which we can determine which one of the two alternative fields is illuminating the observation screen.

Solution: In case (i), the interference pattern is independent of z (i.e. the location of the observation screen), unlike case (ii). Therefore, by moving the screen in the longitudinal direction we can discriminate between the two cases.

7. Figure 3 below shows the schematic diagram of a simple grating spectrometer. It consists of a sinusoidal amplitude grating of period Λ and lateral size (aperture) a followed by a lens of focal length f and sufficiently large aperture. To analyze this spectrometer, we will assume that it is illuminated from the left in spatially coherent fashion by two plane waves on-axis. One of the plane waves is at wavelength λ and the other is at wavelength $\lambda + \Delta\lambda$, where $|\Delta\lambda| \ll \lambda$. (The two plane waves at different wavelengths are mutually incoherent.) Since the two colors are diffracted by the grating to slightly different angles, the goal of this system is to produce two adjacent but sufficiently well separated bright spots at the output plane, one for each color.

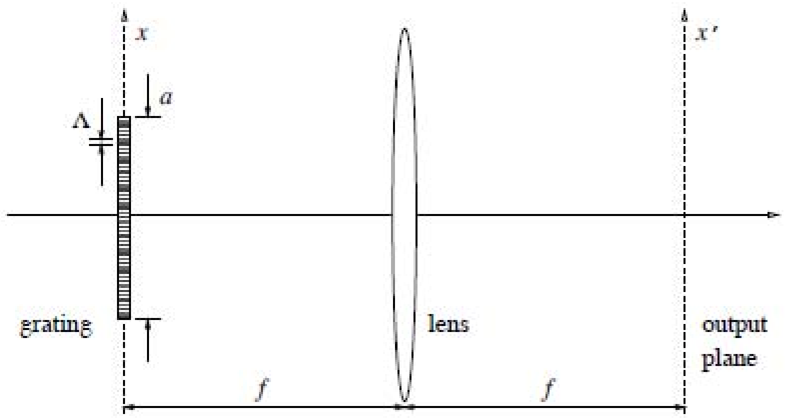
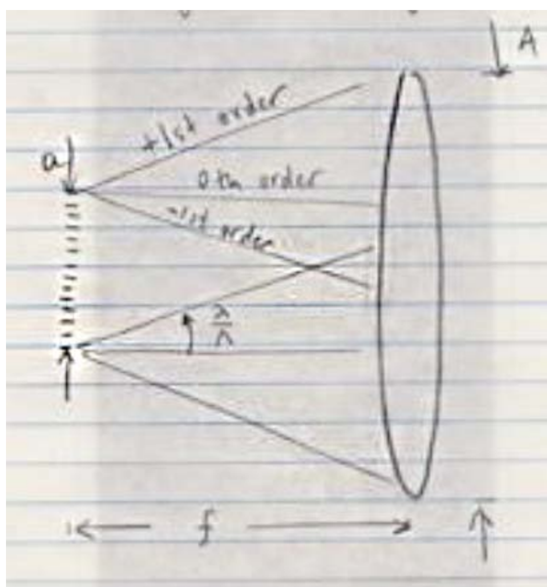


Figure 3

(a) Estimate the minimum aperture size of the lens so that it does not impair the operation of the spectrometer.

Solution: The diffraction angle of light at color λ diffracted by a grating of period Λ is λ/Λ. Therefore, the largest diffraction angle is at the red end of the spectrum (longest wavelength).



The lens aperture A must admit the full size of the $\pm1^{\text{st}}$ diffraction order at the longest wavelength, i.e.

$$\frac{A}{2} > \frac{a}{2} + \frac{\lambda}{\Lambda}\cdot f \Rightarrow A > a + 2\frac{\lambda f}{\Lambda}$$

(b) What is the maximum power efficiency that this spectrometer can achieve?

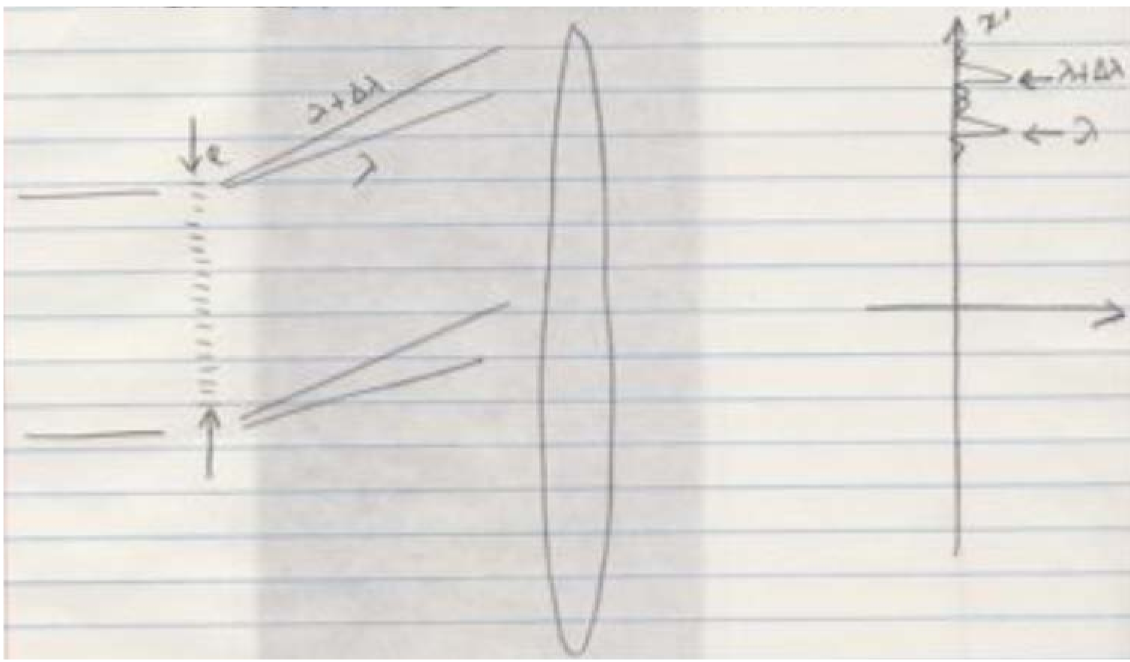
Solution: Since it's an amplitude grating, its maximum efficiency at full contrast is $\frac{1}{16}$. (See Goodman, eq. 4.37, p. 82.)

(c) Show that a condition for the two color spots to be "sufficiently well separated" is:

$$\frac{\lambda}{|\Delta\lambda|} < \frac{a}{\Lambda}$$

This result is often stated in spectroscopy books as follows: *The resolving power of a grating spectrometer, defined as the ratio of the mean wavelength λ to the spectral resolution $|\Delta\lambda|$, equals the number of periods in the grating.*

Solution: Consider the two closely-spaced wavelengths $\lambda, \lambda + \Delta\lambda$, particularly the +1st diffraction order for each.



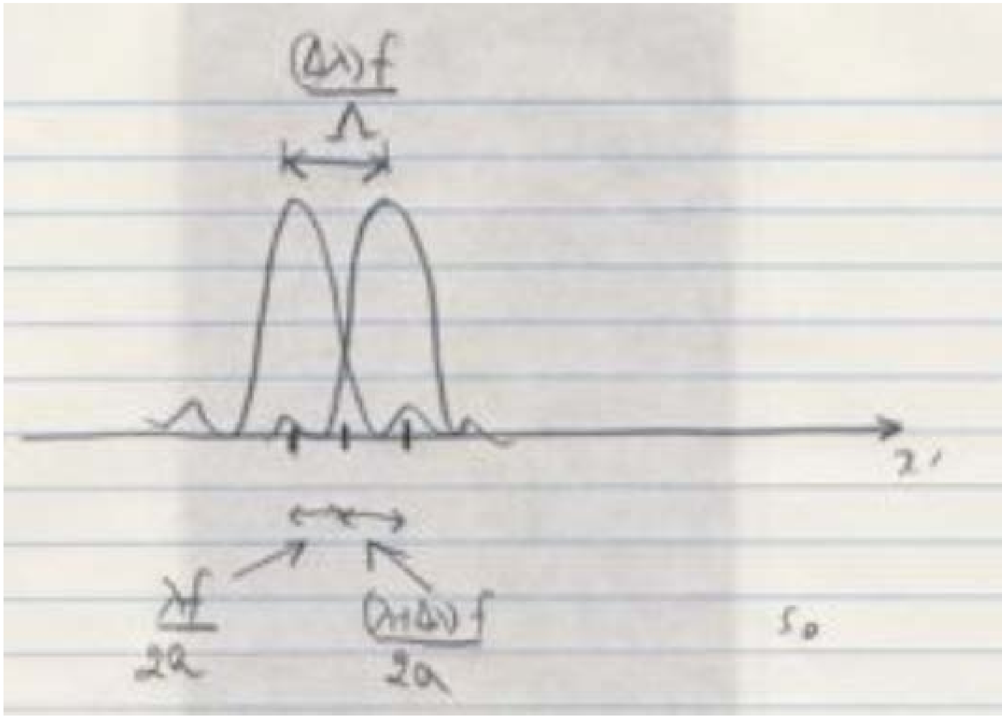
The lens focuses each color to a sinc^2-like spot (in intensity) where the full width of the sinc (main lobe) is:

$$\frac{\lambda f}{a} \text{ for } \lambda, \quad \frac{(\lambda + \Delta\lambda)f}{a} \text{ for } \lambda + \Delta\lambda$$

and the spot locations are:

$$\frac{\lambda f}{\Lambda} \text{ for } \lambda, \quad \frac{(\lambda + \Delta\lambda)f}{\Lambda} \text{ for } \lambda + \Delta\lambda$$

The two color spots are "well resolved" if their spacing exceeds the main lobe size.



$$\frac{\Delta\lambda f}{\Lambda} > \frac{\lambda f}{2a} + \frac{(\lambda+\Delta\lambda) f}{2a} \approx \frac{\lambda f}{a} \qquad \text{(since } a \gg \Lambda\text{)}$$

$$\Rightarrow \frac{\lambda}{\Delta\lambda} < \frac{a}{\Lambda} = \#\text{ grooves in the grating}$$

8. Consider the optical system shown in Figure 1, where lenses L1, L2 are identical with focal length f and half-aperture a. A thin-transparency object is placed $2f$ to the left of L1.

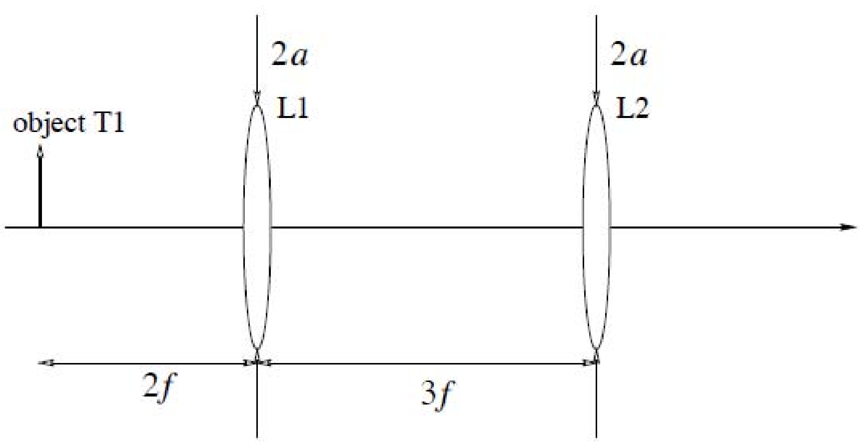
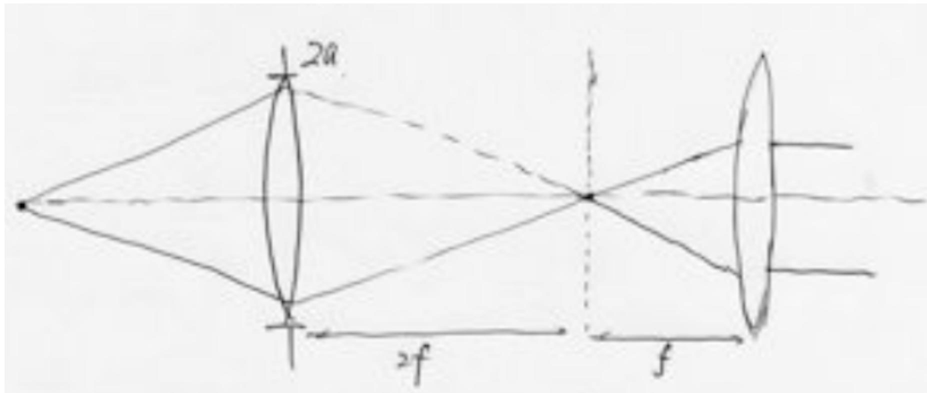


Figure 1

(a) Where is the image formed? Use geometrical optics, ignoring the lens apertures for the moment.

Solution: Using the lens law twice in succession, the image will be at infinity.

(b) If the object T1 is an on-axis point source, describe the Fraunhofer diffraction pattern of the field to the right of L2.



Solution: In order to obtain the field at $2f$ to the right of L1, we can imagine that the system of lens L1 is illuminated by a point source at $2f$ to the left of lens L1, while the object (transparency) is the aperture of $2a$ (the diameter of lens L1) and the lens is infinitely large.

From (5-57) in Goodman, the field at $2f$ to the right of L1 is

$$U_2(x) = \mathcal{F}\left\{\text{rect}\left(\frac{x'}{2a}\right)\right\}\Big|_{u=\frac{x}{2\lambda f}}$$

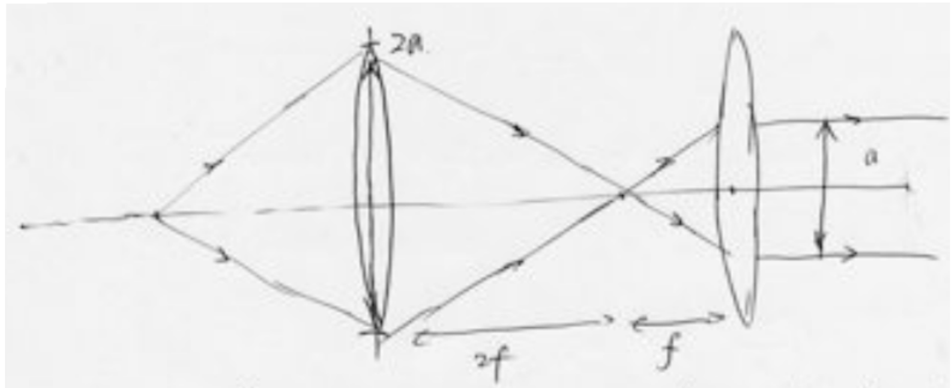
$$\left(\frac{z_1}{z_2(z_1-d)} = \frac{2f}{2f(2f-0)} = \frac{1}{2f}\right)$$

So $U_2(x) \propto \text{sinc}\left(\frac{ax}{\lambda f}\right)$. The Fraunhofer diffraction of the field to the left of lens L2 is

$$\mathcal{F}\{u_2(x)\}|_{\frac{x''}{\lambda f}} = \text{rect}\left(\frac{x''}{a}\right)$$

We can imagine that we have a transparency with function of $U_2(x)$ at f to the left of lens L2 and use a plane wave to illuminate it. Then we find its Fraunhofer diffraction at f to the right of lens L2. What we get is a truncated plane wave with width of a.

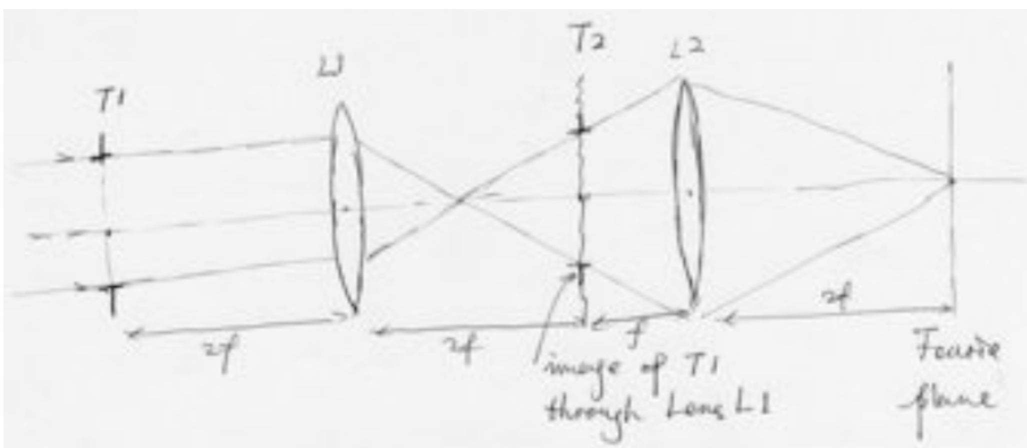
(c) How are your two previous answers consistent within the approximations of paraxial geometrical and wave optics?



Solution: From geometrical optics, we know that lens L1 defines the aperture of the system. We can get the width of the output plane wave easily from the plot above:

$$\frac{f}{2f} \cdot 2a = a$$

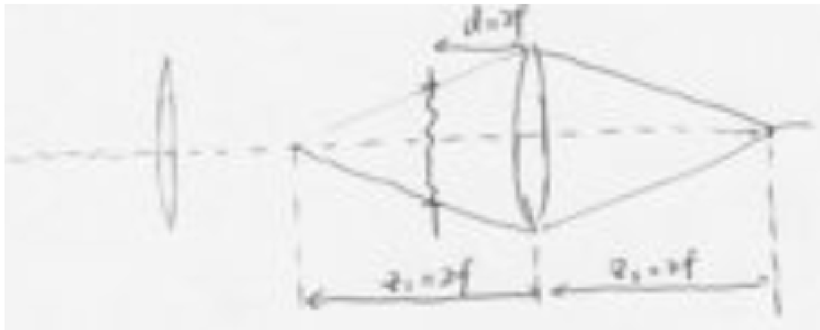
(d) The point source object T1 is replaced by a clear aperture of full width w and a second thin transparency T2 is placed between the two lenses, at distance f to the left of L2. The system is illuminated coherently with a monochromatic on-axis plane wave at wavelength λ. Write an expression for the field at distance $2f$ to the right of L2 and interpret the expression that you found.



Solution: First, without considering T1 and T2, we can find the Fourier plane (the image of the illumination source, which is a plane wave for this case) at $2f$ to the right of L2. Then the image of the aperture T1 through lens L1 is exactly at the same place as the transparency T2. The two objects can be combined by multiplying them together. Now we can predict that at distance $2f$ to the right of lens L2, we will see the Fourier transform of the product of T1 and T2.

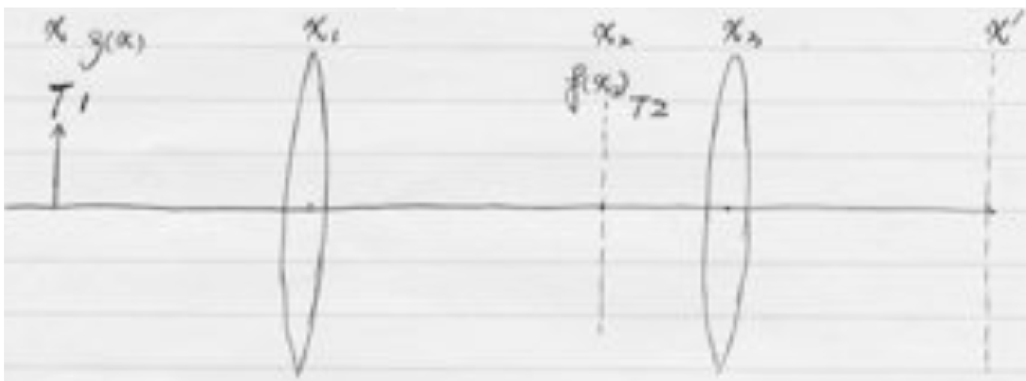
If we assumed T2 with the function of $f(x)$, at the distance $2f$ to the right of L2, the field is

$$U(x') = \mathcal{F}\left\{\text{rect}\left(\frac{x}{w}\right) \cdot f(x)\right\}\Big|_{u=\frac{x'}{\lambda f}}$$



$$\frac{z_1}{z_2(z_1-d)}=\frac{1}{f}$$

We can also obtain the same result from cascade derivation. Let us call object T1 $g(x)$, and transparency T2 $f(x_2)$.



$$\begin{aligned}
U(x') = &\iiiint g(x)\cdot\exp\left\{j\frac{x}{\lambda}\cdot\frac{(x_1-x)^2}{2f}\right\}dx\times\exp\left\{-j\frac{x}{\lambda}\cdot\frac{x_1^2}{f}\right\}\\
&\times\exp\left\{j\frac{x}{\lambda}\cdot\frac{(x_2-x_1)^2}{2f}\right\}\cdot dx_1\cdot f(x_2)\cdot\exp\left\{j\frac{x}{\lambda}\cdot\frac{(x_3-x_2)^2}{f}\right\}dx_2\\
&\times\exp\left\{-j\frac{x}{\lambda}\cdot\frac{x_3^2}{f}\right\}\cdot\exp\left\{j\frac{x}{\lambda}\cdot\frac{(x'-x_3)^2}{z}\right\}dx_3\\
= &\iiiint g(x)f(x_2)\cdot\exp\left\{j\frac{x}{\lambda}\left[\frac{\cancel{x_1^2}-2xx_1+x^2}{2f}-\frac{\cancel{x_1^2}}{f}+\frac{x_2^2-2x_2x_1+\cancel{x_1^2}}{2f}\right.\right.\\
&+\left.\left.\frac{\cancel{x_3^2}-2x_3x_2+x_2^2}{f}-\frac{\cancel{x_3^2}}{f}+\frac{x'^2-2x'x_3+x_3^2}{z}\right]\right\}dx\,dx_1\,dx_2\,dx_3\\
= &\iiiint g(x)f(x_2)\exp\left\{-j\frac{x}{\lambda}\frac{(x+x_2)}{f}x_1\right\}\\
&\times\exp\left\{j\frac{x}{\lambda}\left[\frac{x^2}{2f}+\frac{x_2^2}{2f}+\frac{x_2^2}{f}-\frac{2x_3x_2}{f}+\frac{x'^2-2x'x_3+x_3^2}{z}\right]\right\}dx\,dx_1\,dx_2\,dx_3
\end{aligned}$$

$$
\begin{aligned}
&= \iiint g(x) f(x_2) \delta(x + x_2) \exp\left\{ j\frac{x}{\lambda} \left[\frac{x^2}{2f} + \frac{3x_2^2}{2f} - \frac{2x_3x_2}{f} + \frac{x'^2 - 2x'x_3 + x_3^2}{z} \right] \right\} dx\, dx_2\, dx_3 \\
&= \iint f(x_2) g(-x_2) \exp\left\{ j\frac{x}{\lambda} \left[\frac{2x_2^2}{f} - \frac{2x_3x_2}{f} + \frac{x'^2 - 2x'x_3 + x_3^2}{z} \right] \right\} dx_2\, dx_3 \\
&= \iint f(x_2) g(-x_2) \exp\left\{ j\frac{x}{\lambda} \cdot \frac{2x_2^2}{f} \right\} \exp\left\{ j\frac{x}{\lambda} \cdot \frac{x'^2}{z} \right\} \exp\left\{ j\frac{x}{\lambda} \left[\frac{x_3^2}{z} - \left(\frac{2x_2}{f} + \frac{2x'}{z} \right) x_3 \right] \right\} dx_2\, dx_3 \\
&= \int f(x_2) g(-x_2) \cdot \exp\left\{ j\frac{x}{\lambda} \cdot \frac{2x_2^2}{f} \right\} \exp\left\{ j\frac{x}{\lambda} \cdot \frac{x'^2}{z} \right\} \exp\left\{ -j\frac{x}{\lambda} \cdot z \cdot \left(\frac{x_2}{f} + \frac{x'}{z} \right)^2 \right\} dx_2 \\
&= \int f(x_2) g(-x_2) \exp\left\{ j\frac{x}{\lambda} \left(\frac{2}{f} - \frac{z}{f^2} \right) x_2^2 \right\} \exp\left\{ -j\frac{x}{\lambda} \cdot \frac{2x_2x'}{f} \cdot \right\} dx_2 \\
&= \int f(x_2) g(-x_2) \exp\left\{ -j\frac{x}{\lambda} \cdot \frac{2x_2x'}{f} \cdot \right\} dx_2 \qquad \text{if } z = 2f
\end{aligned}
$$

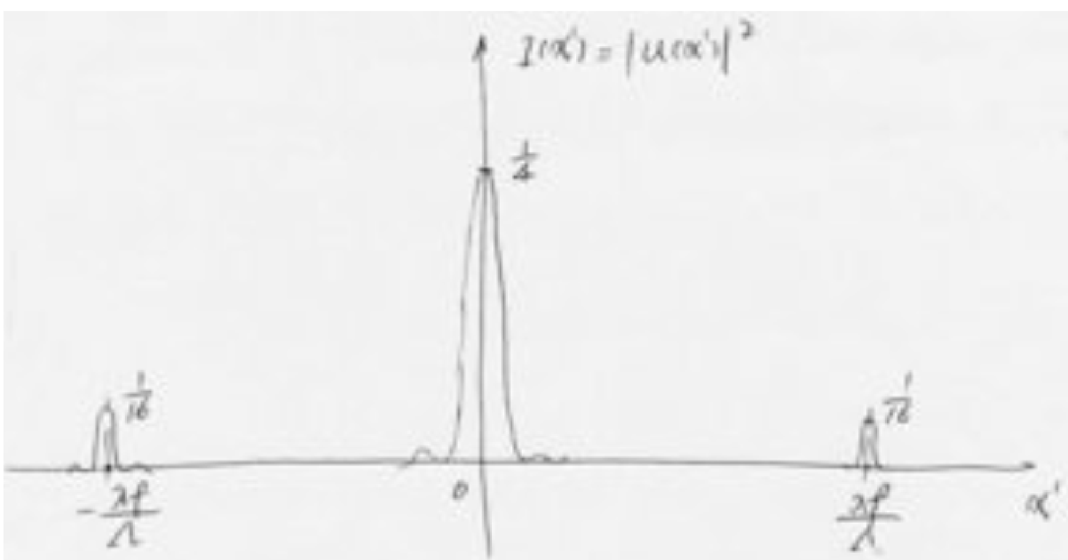
(e) Derive and approximately sketch, with as much quantitative detail as you can, the intensity observed at distance $2f$ to the right of L2 when T2 is an infinite sinusoidal amplitude grating of period Λ, such that $\Lambda \ll a$.

Solution: The transparency can be expressed as:

$$
f(x) = \frac{1}{2}\left(1 + \cos\left(2\pi\frac{x}{\Lambda}\right)\right)
$$

The aperture is rect $\left(\frac{x}{w}\right)$, so:

$$
\begin{aligned}
U(x') &= \mathcal{F}\left\{ f(x) \cdot \mathrm{rect}\left(\frac{x}{w}\right) \right\}\Big|_{u=\frac{x'}{\lambda f}} \\
&= \frac{1}{2}\mathrm{sinc}\left(\frac{wx'}{\lambda f}\right) + \frac{1}{4}\mathrm{sinc}\left[w\left(\frac{x'}{\lambda f} - \frac{1}{\Lambda}\right)\right] + \frac{1}{4}\mathrm{sinc}\left[w\left(\frac{x'}{\lambda f} + \frac{1}{\Lambda}\right)\right]
\end{aligned}
$$



9. An infinite periodic square-wave grating with transmittivity as shown in Figure 3A is placed at the input of the optical system of Figure 3B. Both lenses are positive, $F/1$, and have focal length f. The grating is illuminated with monochromatic, spatially coherent light of wavelength λ and intensity I_0. The spatial period of the grating is $X = 4\lambda$. The element at the Fourier plane of the system is a nonlinear transparency with the intensity transmission function shown in Figure 3C, where the threshold and saturating intensities $I_{\text{thr}} = I_{\text{sat}} = 0.1 I_0$. To calculate the response of this system analytically, we need to make the paraxial approximation; strictly speaking, that is questionable for $F/1$ optics, but we will follow it nevertheless. An additional necessary assumption is discussed in the first question below.

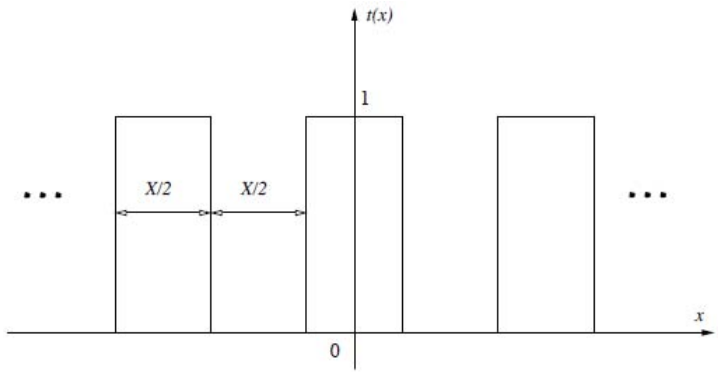


Figure 3A

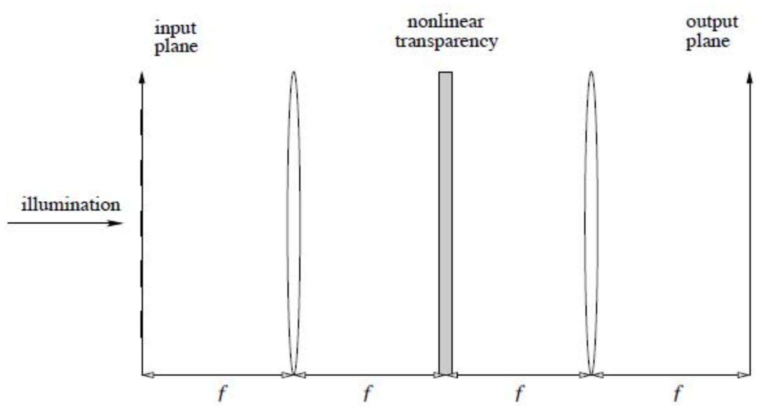


Figure 3B

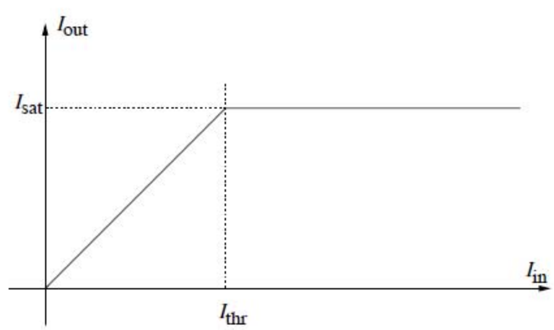
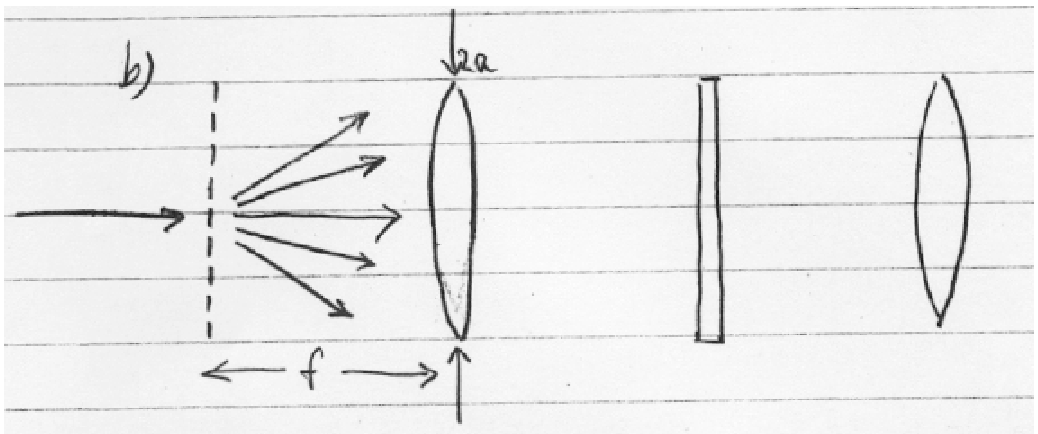


Figure 3C

(a) To answer the second question, we need to neglect the Airy patterns forming at the Fourier plane and pretend they are uniform bright dots. Explain why this assumption is justified and what effects it might have.

Solution: Nonlinearity will be significant only at the peaks of the Airy disks. The system has a low $F/\#$ (high NA), so the Airy disks are very tight and the assumption is justified.

(b) Derive and plot the intensity distribution at the output plane using the above assumption.



Solution: Diffraction angle $\theta = \dfrac{\lambda}{X} = \dfrac{1}{4}$

System aperture $= \dfrac{a}{f} = \dfrac{1}{2}$ $(F/1) \Rightarrow$ system admits orders $0, \pm1, \pm2$

0th order intensity: $(\frac{1}{2})^2 I_0 = \frac{1}{4} I_0 \Rightarrow$ transparency transmits $0.1 I_0$

$\pm$1st order intensity: $\left(\dfrac{1}{2} \cdot \dfrac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}}\right)^2 I_0 = 0.101 I_0 \Rightarrow$ transparency transmits $0.1 I_0$

$\pm$2nd order intensity: $0 \Rightarrow$ output $I(x') = 0.1 I_0 \left[1 + 2\cos\left(\dfrac{2\pi x}{X}\right)\right]$