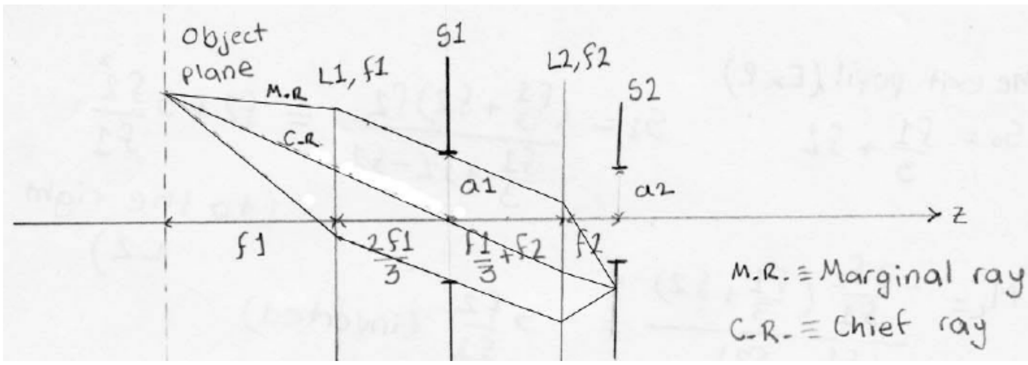
1. Consider the following system.



(a) If we position an on-axis point source at the center of the object plane (front focal plane of L1), a collimated ray bundle will emerge to the right of L1 and its diameter is set by S1; therefore, S1 is the aperture stop (A.S.). Similarly, S2 limits the lateral extent of an imaged object (consider an off-axis point source) and thus, it's our field stop (F.S.).

(b) The entrance pupil is the image of the A.S. by the preceding optical components. To find its location we use the imaging condition,

$$\frac{1}{S_o} + \frac{1}{S_i} = \frac{1}{f_1} \qquad \Rightarrow \quad S_i = \frac{S_o f_1}{S_o - f_1} = \frac{2f_1^2/3}{f_1(\frac{2}{3}-1)}$$

$$S_o = \frac{2f_1}{3} \qquad \Rightarrow \quad S_i = -2f_1 \text{ (virtual)}$$

So the entrance pupil is located at $2f_1$ to the right of L1. To find its radius, we compute the lateral magnification,

$$M_L = -\frac{S_i}{S_o} = 3 \rightarrow r_{\text{EnP}} = 3a_1$$

For the exit pupil (Ex.P.),

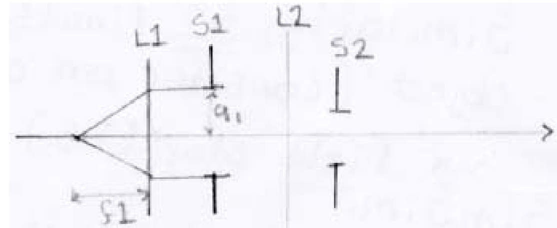
$$S_o = \frac{f_1}{3} + f_2, \quad S_i = \frac{(\frac{f_1}{3} + f_2)f_2}{\frac{f_1}{3} + \cancel{f_2} - \cancel{f_2}} = f_2 + 3\frac{f_2^2}{f_1} \text{ (to the right of L2)}$$

$$M_L = \frac{-\frac{3f_2}{f_1}(\frac{f_1}{3} + f_2)}{(\frac{f_1}{3} + f_2)} = -3\frac{f_2}{f_1} \text{ (inverted)} \quad \rightarrow \quad r_{\text{ExP}} = 3\frac{f_2}{f_1}a_1$$

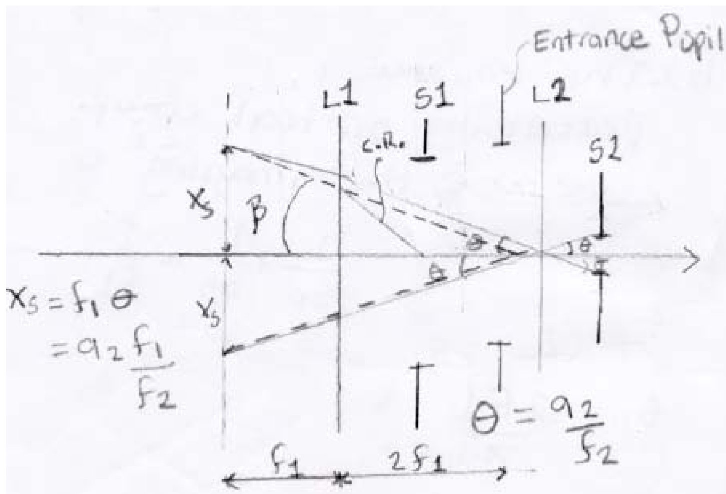
- The exit window is the same as S2.
- The entrance window is the image of S2 through the preceding optical elements (i.e. combination of L1 and L2). It is f_1 to the left of L1.

(c) Solution:

The numerical aperture is:
$\tan\alpha \approx \alpha \approx \sin\alpha \approx \text{NA} \approx \frac{a_1}{f_1}$



The field of view (FOV) is: $\text{FOV} = 2\beta = \dfrac{2X_s}{3f_1} = \dfrac{2}{3}\dfrac{f_1 a_2}{f_1 f_2} = \dfrac{2}{3}\dfrac{a_2}{f_2}$



(d) The location of S1 limits the FOV because of the requirement for the C.R. to go through the center of the aperture stop (A.S.). It can be seen that the least restrictive A.S. location is at the Fourier plane (f_1 to the right of L1 $\Longleftrightarrow$ f_2 to the left of L2).

2. Given the following GRIN medium,

$$n(r) = \begin{cases} \sqrt{2-r^2} & 0 < r < 1 \\ 1 & r \geq 1 \end{cases}, \qquad r = \sqrt{x^2+z^2}$$

(a) The Hamiltonian equations are ($r < 1$):

$$\begin{aligned}
\frac{dx}{ds} &= \frac{\partial H}{\partial P_x} &&= -\frac{1}{2}\frac{2P_x}{\sqrt{P_x^2+P_z^2}} &&= -\frac{P_x}{n} &&= -\frac{P_x}{\sqrt{2-x^2-z^2}} \\
\frac{dz}{ds} &= \frac{\partial H}{\partial P_z} &&= -\frac{1}{2}\frac{2P_z}{\sqrt{P_x^2+P_z^2}} &&= -\frac{P_z}{n} &&= -\frac{P_z}{\sqrt{2-x^2-z^2}} \\
\frac{dP_x}{ds} &= -\frac{\partial H}{\partial x} &&= -\frac{1}{2}\frac{-2x}{n} &&= \frac{x}{n} &&= \frac{x}{\sqrt{2-x^2-z^2}} \\
\frac{dP_z}{ds} &= -\frac{\partial H}{\partial z} &&= -\frac{1}{2}\frac{-2z}{n} &&= \frac{z}{n} &&= \frac{z}{\sqrt{2-x^2-z^2}}
\end{aligned}$$

where,

$$\begin{aligned}
H &= n(q) - [P_x^2 + P_z^2]^{1/2} = 0 \\
&= [2 - x^2 - z^2]^{1/2} - [P_x^2 + P_z^2]^{1/2} = 0 \\
n &= \sqrt{P_x^2 + P_z^2}
\end{aligned}$$

(b) Recall that we just found $n^2 = P_x^2 + P_z^2$.

$$\left(\frac{dP_x}{ds}\right)^2 + \left(\frac{dP_z}{ds}\right)^2 = \left(\frac{x}{n}\right)^2 + \left(\frac{z}{n}\right)^2 = \frac{x^2+z^2}{n^2} = \frac{x^2+z^2}{P_x^2+P_z^2}$$
$$= \frac{r^2}{P_x^2+P_z^2} = \frac{2-n^2}{P_x^2+P_z^2} = \frac{2}{P_x^2+P_z^2} - 1$$

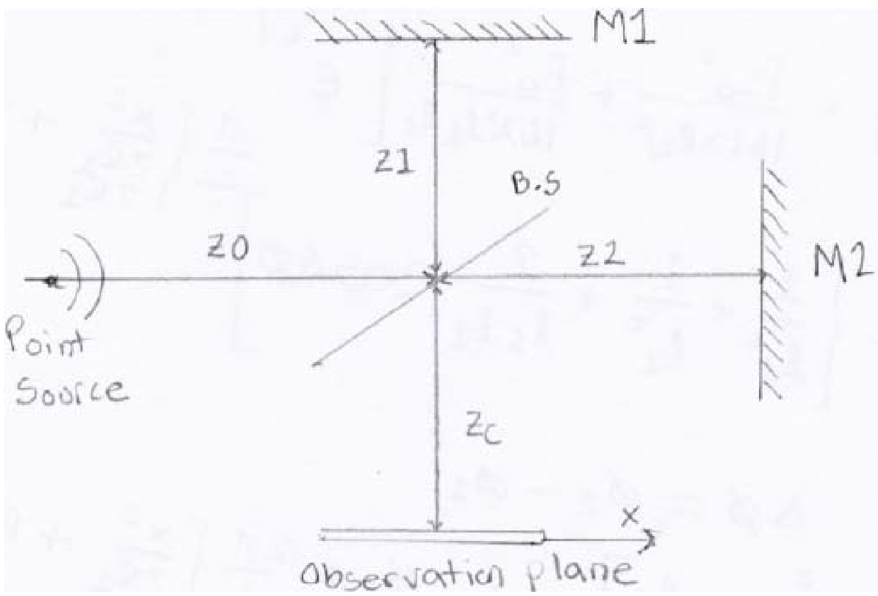
(c) Since $\frac{\partial n}{\partial z} \neq 0$, the Screen Hamiltonian is not conserved. This may be verified by direct substitution:

$$h = -\sqrt{n^2 - P_x^2} = -\sqrt{2 - x^2 - z^2 - P_x^2}$$

and we see that,

$$\frac{\partial h}{\partial z} = \frac{z}{\sqrt{2 - x^2 - z^2 - P_x^2}} \neq 0$$

3. Consider the Michelson interferometer shown below.



(a) We begin by writing the analytic expression (in phasor form) of a spherical wave with origin at (x_p, z_p), using the paraxial approximation:

$$E_S = \frac{E_0 e^{i\frac{2\pi}{\lambda}(z-z_p)}}{i\lambda(z-z_p)} e^{i\frac{\pi}{\lambda}\frac{(x-x_p)^2}{(z-z_p)}}$$

For simplicity we take $x_p = z_p = 0$. At the observation plane, the interference pattern is given by:

$$I = |E_{S_1} + E_{S_2}|^2 = |E_{S_1}|^2 + |E_{S_2}|^2 + E_{S_1}E_{S_2}^* + E_{S_1}^*E_{S_2}$$

where

$$E_{S_1} = \frac{E_0}{4}\frac{e^{i\frac{2\pi}{\lambda}\ell_1}}{i\lambda\ell_1}e^{i\frac{\pi}{\lambda\ell_1}x^2} \qquad E_{S_2} = \frac{E_0}{4}\frac{e^{i\frac{2\pi}{\lambda}\ell_1}}{i\lambda\ell_2}e^{i\frac{\pi}{\lambda\ell_2}x^2}$$

$$\ell_1 = z_0 + 2z_1 + z_c \qquad \ell_2 = z_0 + 2z_2 + z_c$$

$$|E_{S_1}|^2 = \frac{{E_0}^2}{16(\lambda\ell_1)^2} \qquad |E_{S_2}|^2 = \frac{{E_0}^2}{16(\lambda\ell_2)^2}$$

So,

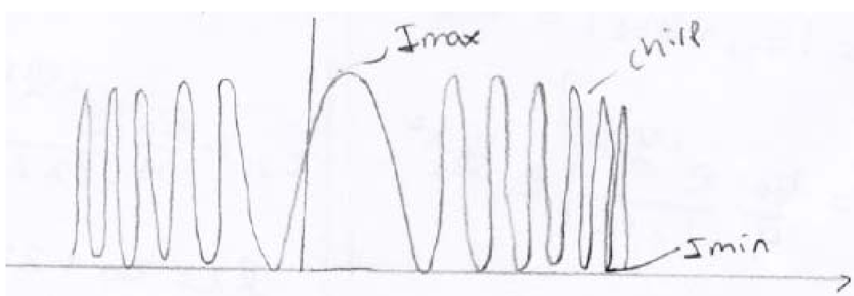
$$\begin{aligned} I &= \frac{{E_0}^2}{16(\lambda\ell_1)^2} + \frac{{E_0}^2}{16(\lambda\ell_2)^2} + \frac{{E_0}^2}{16\lambda^2\ell_1\ell_2}[e^{i\Delta\phi} + e^{-i\Delta\phi}] \\ &= \frac{{E_0}^2}{16\lambda^2}\left[\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} + \frac{2}{\ell_1\ell_2}\cos\Delta\phi\right], \quad \Delta\phi = \phi_2 - \phi_1 \end{aligned}$$

$$\phi_2 = \frac{2\pi}{\lambda}\left[\frac{x^2}{2\ell_2} + \ell_2\right]; \quad \phi_1 = \frac{2\pi}{\lambda}\left[\frac{x^2}{2\ell_1} + \ell_1\right]$$

$$\Rightarrow \Delta\phi = \frac{2\pi}{\lambda}\left[x^2\left(\frac{1}{2\ell_2} - \frac{1}{2\ell_1}\right) + 2\Delta z\right], \quad \Delta z = z_2 - z_1$$

$$\Rightarrow \boxed{I = \frac{{E_0}^2}{16\lambda^2}\left[\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} + \frac{2}{\ell_1\ell_2}\cos\left(\frac{2\pi}{\lambda}\left[x^2\left(\frac{\Delta\ell}{2\ell_1\ell_2} - \Delta\ell\right)\right]\right)\right]}$$

$$\Delta\ell = 2\Delta z = \ell_2 - \ell_1$$



$$I_{\max} = \frac{{E_0}^2}{16\lambda^2}\left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} + \frac{2}{\ell_1\ell_2}\right)$$

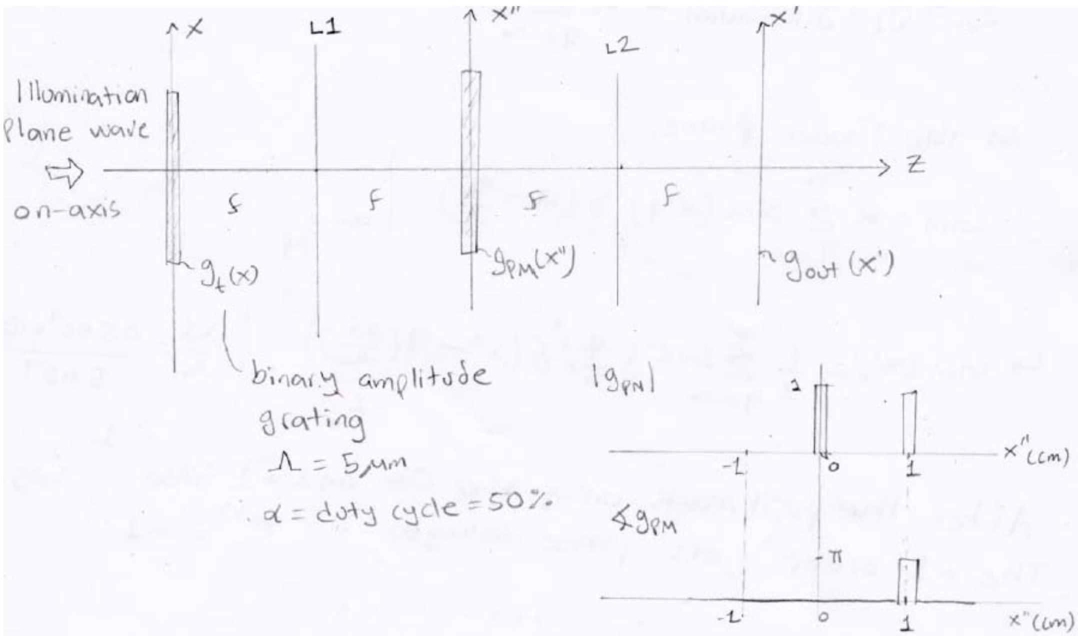
$$I_{\min} = \frac{{E_0}^2}{16\lambda^2}\left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} - \frac{2}{\ell_1\ell_2}\right)$$

(b) If the flat mirror M2 is replaced by a convex spherical mirror of radius $2(z_0 + z_2)$, the spherical wave gets collimated and the interference pattern becomes:

$$I = \left|E_{S_1} + \underbrace{\frac{E_0}{4\lambda\ell'}e^{i\phi'}}_{\text{Plane Wave}}\right|^2 = \underbrace{\frac{{E_0}^2}{16(\lambda\ell')^2} + \frac{{E_0}^2}{16(\lambda\ell_1)^2} + \frac{{E_0}^2}{8\lambda^2\ell'\ell_1}\sin\Delta\phi'}_{\text{Chirp Function}}$$

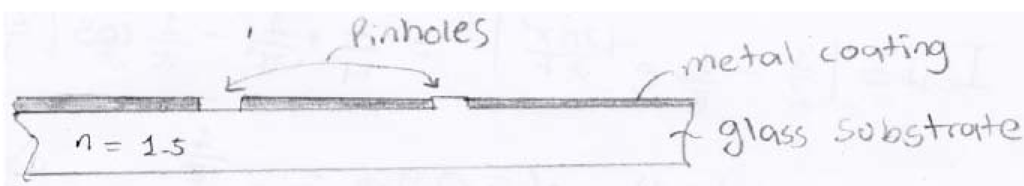
$$\ell' = z_0 + z_2$$
$$\phi' = \frac{2\pi}{\lambda}\ell_2$$
$$\Delta\phi' = \phi_1 - \phi'$$

4. Consider the 4-f system shown below,



(a) The pupil mask can be implemented by placing two pinholes (small apertures), one centered with respect to the optical axis and the second one at 1 cm off-axis. The 2nd pinhole is phase delayed by a piece of glass of thickness t, where

$$\phi = \pi = \frac{2\pi}{\lambda}t(1.5 - 1) \Rightarrow t = \lambda$$



(b) The input transparency is

$$g_{\text{in}} = g_t \cdot \cancelto{1}{g_{\text{illumination}}} = \alpha \sum_{q=-\infty}^{\infty} \text{sinc}(\alpha q) e^{i2\pi\frac{qx}{\Lambda}}$$

At the Fourier plane,

$$G_{\text{in}} = \alpha \sum_{q=-\infty}^{\infty} \text{sinc}(\alpha q)\delta\left(u - \frac{q}{\Lambda}\right)\bigg|_{u=\frac{x''}{\lambda f}}$$

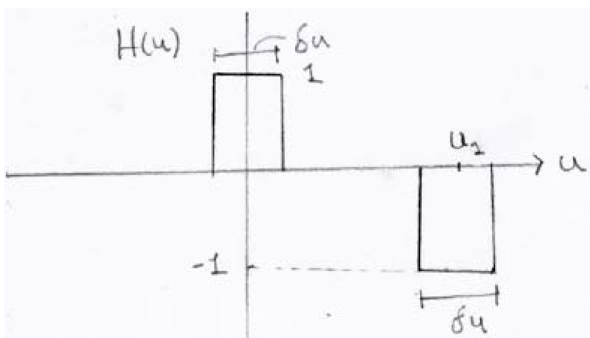
$$G_{\text{in}}(x'') = \frac{1}{2}\sum_{q=-\infty}^{\infty} \text{sinc}\left(\frac{q}{2}\right)\delta\left(x'' - q\left(\frac{\lambda f}{\Lambda}\right)\right), \quad \frac{\lambda f}{\Lambda} = \frac{0.5\times10^{-4}\cdot 10}{5\times10^{-4}} = 1$$

After the pupil mask, only the 0th and +1 orders pass. The +1 order gets phase delayed by $e^{i\pi} = -1$.

$$\begin{aligned} G_{\text{out}}(x'') &= \frac{1}{2}\delta(x'') - \frac{1}{2}\text{sinc}\left(\frac{1}{2}\right)\delta(x''-1) \\ &= \frac{1}{2}\delta(x'') - \frac{1}{\pi}\delta(x''-1) \\ g_{\text{out}}(u') &= \frac{1}{2} - \frac{1}{\pi}e^{-i2\pi u'}\bigg|_{u'=\frac{x'}{\lambda f}} \\ I_{\text{out}} &= \left|\frac{1}{2} - \frac{1}{\pi}e^{-i\frac{2\pi x'}{\lambda f}}\right|^2 = \frac{1}{4} + \frac{1}{\pi^2} - \frac{1}{\pi}\cos\left(\frac{2\pi}{\lambda}\frac{x'}{f}\right) \end{aligned}$$

The contrast is $v = 0.906 = \dfrac{\frac{1}{\pi}}{\frac{1}{4}+\frac{1}{\pi^2}} = \dfrac{4\pi}{4+\pi^2}$.

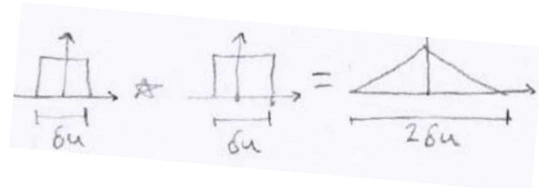
5. (a) To compute the OTF, we first need the ATF:



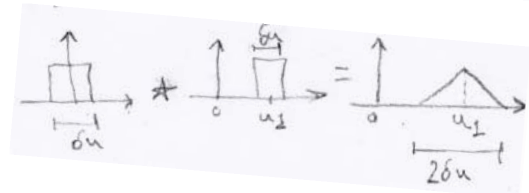
$$\begin{aligned} u_1 &= \frac{x''}{\lambda f} = \frac{1\text{cm}}{0.5\mu\text{m} \times 10\text{cm}} = 0.2\mu\text{m}^{-1} = 200\text{mm}^{-1} \\ \delta u &= \frac{\delta x''}{\lambda f} \approx \frac{\overbrace{0.2\text{cm}}^{\text{est.}}}{0.5\mu\text{m} \times 10\text{cm}} = 0.025\mu\text{m}^{-1} = 25\text{mm}^{-2} \end{aligned}$$

$$\begin{aligned} \mathcal{H}(u) &= H(u) \otimes H(u) \qquad \text{(autocorrelation)} \\ &= \int \left[\text{rect}\left(\frac{u'}{\delta u}\right) - \text{rect}\left(\frac{u'-u_1}{\delta u}\right)\right] \left[\text{rect}\left(\frac{u'-u}{\delta u}\right) - \text{rect}\left(\frac{u'-u_1-u}{\delta u}\right)\right] du' \end{aligned}$$

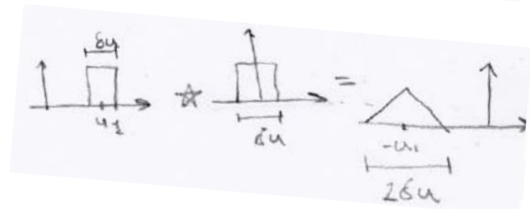
$$= \int \text{rect}\left(\frac{u'}{\delta u}\right) \text{rect}\left(\frac{u'-u}{\delta u}\right) du' \rightarrow$$



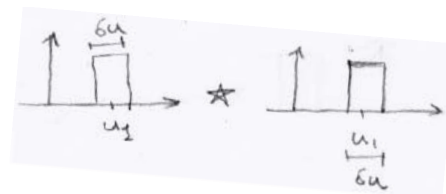
$$- \int \text{rect}\left(\frac{u'}{\delta u}\right) - \text{rect}\left(\frac{u'-u_1-u}{\delta u}\right) du' \rightarrow$$

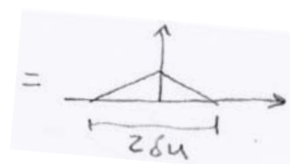


$$- \int \text{rect}\left(\frac{u'-u_1}{\delta u}\right) \text{rect}\left(\frac{u'-u}{\delta u}\right) du' \rightarrow$$

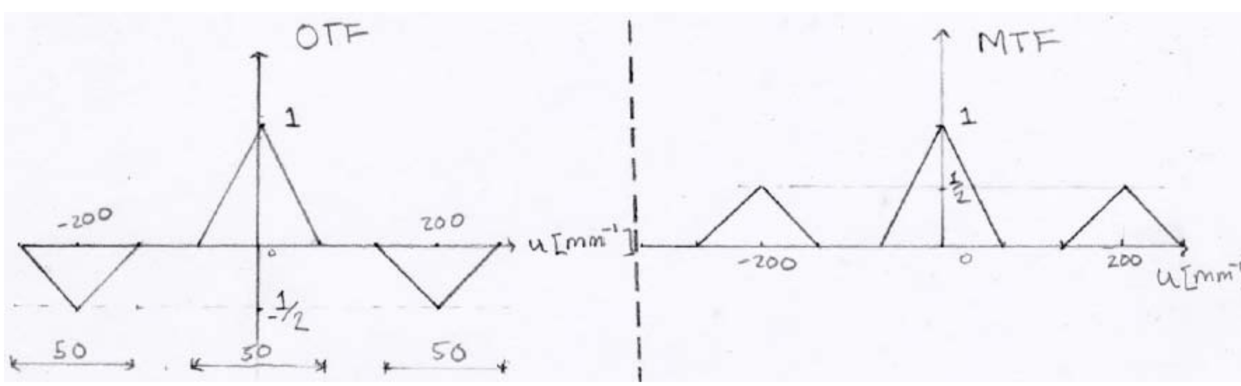


$$+ \int \text{rect}\left(\frac{u'-u_1}{\delta u}\right) \text{rect}\left(\frac{u'-u_1-u}{\delta u}\right) du' \rightarrow$$





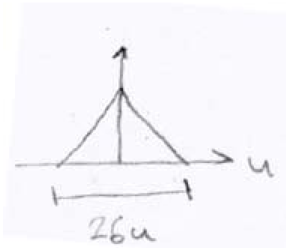
So the resultant OTF and MTF are (after normalization):



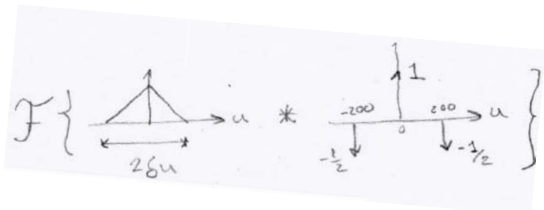
(b) Only the DC and the $\pm$ 1st harmonics at $u = \pm 200\text{mm}^{-1}$ (period $= 5\mu\text{m}$), i.e.

$$\begin{aligned} I(x') &= \frac{1}{2} - \frac{1}{2} \times \frac{1}{\pi} \times 2\cos\left(\frac{2\pi x'}{5\mu\text{m}}\right) \\ &= \text{DC term} - H(200mm^{-2}) \times \text{1st harmonic} \times 2\cos\left(\frac{2\pi x'}{5\mu\text{m}}\right) \end{aligned}$$

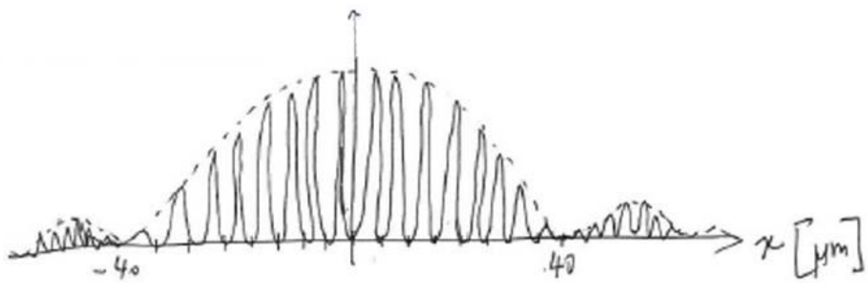
(c) Solution:



$\xrightarrow{\mathcal{F}}$ $\mathrm{sinc}^2(\delta u x)$, so the iPSF is:



$$= \mathrm{sinc}^2\left(\frac{x}{40\mu\mathrm{m}}\right) \times \left[1 - \cos\left(2\pi \frac{x}{5\mu\mathrm{m}}\right)\right]$$



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Spring 2009