

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
CAMBRIDGE, MASSACHUSETTS 02139
2.29 NUMERICAL FLUID MECHANICS — SPRING 2015

Problem Set 3

Issued: Wednesday, March 4, 2015

Due: Wednesday, March 18, 2015

Grading Note: Please provide your solutions either as hand-written/hard-copy solutions or by submitting via course website. MATLAB[®] codes should be submitted via course website. The bulk of the grades will be given to detailed explanations and to algorithms and numerical schemes that capture the essence of the numerical problems. We know that successful coding of numerical schemes can be time consuming and prone to small errors. Such small errors or omissions in a code will not be heavily penalized.

The goals of this Problem Set are to: (i) derive finite-difference schemes and utilize them in simple problems including differentiations from observations; and (ii) apply these schemes to solve differential equations in idealized fluid mechanics and heat transfer problems.

Problem 1

Do problem 23.19 from Chapra and Canale with the following three functions:

$$f_1(x) = e^{-2x} - x, \quad f_2(x) = e^{2x} - x, \quad f_3(x) = \sin(4x) - x$$

Problem 2 (Modified from 24.13 of Chapra and Canale). Bonus (not needed for full credit)

Videangiography is used to measure the blood flow and determine the status of circulatory function. In order to quantify the videoangiograms, blood vessel diameter and blood velocity are needed such that the total blood flow can be determined. One of your colleagues provides you with a densitometric profile taken from a videoangiogram of a blood vessel (see table), going through the center of the vessel. She tells you that you can estimate the edge of the blood vessel from these data if you can determine where the first derivative of the profile has extreme values.

Distance	Density	Distance	Density	Distance	Density	Distance	Density
0	26.013	28	38.273	56	39.124	84	37.331
4	26.955	32	39.103	60	38.813	88	35.980
8	26.351	36	39.025	64	38.925	92	31.936
12	28.343	40	39.432	68	38.804	96	28.843
16	31.100	44	39.163	72	38.806	100	26.309
20	34.667	48	38.920	76	38.666	104	26.146
24	37.251	52	38.631	80	38.658		

- a) With the data provided in the table, estimate the position of two edges and so the diameter of the blood vessel. Use both $O(h^2)$ and $O(h^4)$ centered difference formulas and compare the results.
- b) If the relative error on the data is 5%, what is the “standard error estimate” for your two centered difference formulas? Based on the results you obtain, which centered difference scheme is adequate?

Problem 3 Bonus (not needed for full credit)

- a) Do problem 24.13 from Chapra and Canale.
- b) If the relative error on the data is 5%, what is the “standard error estimate” for your two centered difference formulas? Based on the results you obtain, which centered difference scheme is adequate?

Problem 4 (Modified from Lomax et al, Prob. 3.1-3.2)

a) Derive a third-order finite-difference approximation to a first derivative in the form

$$\left(\frac{\partial u}{\partial x}\right)_j = \frac{1}{\Delta x} (a u_{j-2} + b u_{j-1} + c u_j + d u_{j+1})$$

b) Derive a finite-difference approximation to a first derivative in the form

$$\left(\frac{\partial u}{\partial x}\right)_j = \frac{1}{\Delta x} (a u_{j-1} + b u_j + c u_{j+1}) + d \left(\frac{\partial u}{\partial x}\right)_{j-1}$$

c) Compare the leading truncation error terms for a) and b).

d) Very briefly compare the compactness of the two numerical schemes and briefly explain how one would implement the finite differences obtained in a) and b) to compute the first derivative field over a finite domain ($x_0 \leq x \leq x_1$).

Problem 5: Solving a wave equation

The 1D Sommerfeld wave radiation equation is governed by: $\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$

This linear equation is one of the most studied for the derivation of finite difference schemes. It corresponds to a constant speed advection. Travelling waves in the form of $u(x,t) = e^{i(kx-\omega t)}$ are solutions of this equation if $\omega = c k$ which implies that the waves travel at constant amplitudes at fixed phase speed $\omega/k = c$ (non-dispersive waves).

a) Discretize this equation using only explicit finite-difference schemes, solving for $\hat{u}_i^{k+1} = \hat{u}(i \Delta x, (k+1) \Delta t)$: i) use a “first-order backward difference” in space and a “first-order forward difference” in time; ii) as i), but use a “second-order central difference” in space.

b) Using the two discretization schemes obtained in a), we wish to solve the Sommerfeld equation for the following conditions. The wave-speed, c , is set to 1 m/s, and the domain is 1 m long. The total time of interest is 1 s. The initial and boundary conditions are as follows. The initial condition $u(x,0)$ of the wave has the following shape:

$$u(x,0) = \begin{cases} 4(x-0.25) & 0.25 \leq x \leq 0.5 \\ 0 & \text{elsewhere} \end{cases}$$

At the left boundary, a pulse is generated starting every 1/4 of a second. That is, at $t = [0, 0.25, 0.5, 0.75, 1]$ s, a pulse is generated with the following shape:

$$u(0,t) = \begin{cases} \sin(\omega(t-t_s)) & 0 \leq t-t_s \leq 0.25 \\ 0 & \text{otherwise} \end{cases}$$

where $\omega = 4\pi$ and $t_s = [0, 0.25, 0.5, 0.75, 1]$ s. The duration of each pulse is 0.25 s.

Using MATLAB (or your preferred software), numerically solve for $u(x,t)$ using 100 nodes in time and 100 nodes in space:

- i) What time step (Δt) and space step () sizes do these choices correspond to?
 - ii) Implement and solve using the backwards difference discretization in space from a)
 - iii) Implement and solve using the centered difference discretization in space from a)
- Plot the solution at time-step 50 for each case. One of the discretization has issues, why?
- c) Using the acceptable discretization from b),
- i) Plot the solutions at $t = [0, 1/3, 2/3, 1]$ s (on the same graph) for velocities of $c = [0.9, 1, 1.05]$ (on separate graphs). This will result in a total of 3 graphs, one graph for each different velocity.
 - ii) Comment on the results. What happens when the value of c changes? What is the name of the condition that your numerical scheme needs to satisfy? Two of the three solutions seem fine, but one of them does not maintain a key property of exact solutions, why? Hint: Substitute $c = 1$ into your discretized equation. To better understand what's happening, download Waveanim.m from course website. If you save your solutions in a matrix such that $\mathbf{U} = [\mathbf{u}(x,t=0), \dots, \mathbf{u}(x,t=1)]$, where each $\mathbf{u}(x,t)$ is a column vector, then you can run the file Waveanim(\mathbf{U}) to create a movie!
- d) If c is increased to 4.061 m/s, what would you do to ensure an acceptable scheme without changing the total number of nodes? (the total number of nodes is 10 000, 100 time x 100 space).
- e) What would happen if c became negative, i.e. $c < 0$? Would your discretization scheme still be valid? What about the initial/boundary conditions, could they still be the same or would they need to be changed?
- f) For many realistic fields the background velocity fields (here denoted by c) will not be constant in the domain of interest. Hence one cannot determine a priori the range of the proper grid spacing for a stable solution: stability needs to be checked as the code is run and the solution computed. In this part of the problem, we investigate what happens when the stability criterion is violated. Let's assume that the background velocity varies and is equal to $c = A\sin(\pi t) + 0.5$, where A is either 0.6 or 0.8.
- i) Implement this change in your code and plot the solution at $t = [0, 1/3, 2/3, 1]$ s, for both values of $A = [0.6, 0.8]$. Also plot the parameter $C = c \frac{\Delta t}{\Delta x}$ for both cases.
 - ii) Comment your results. What do they imply for realistic advective flows, i.e. $c = u$?

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2.29 Numerical Fluid Mechanics

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