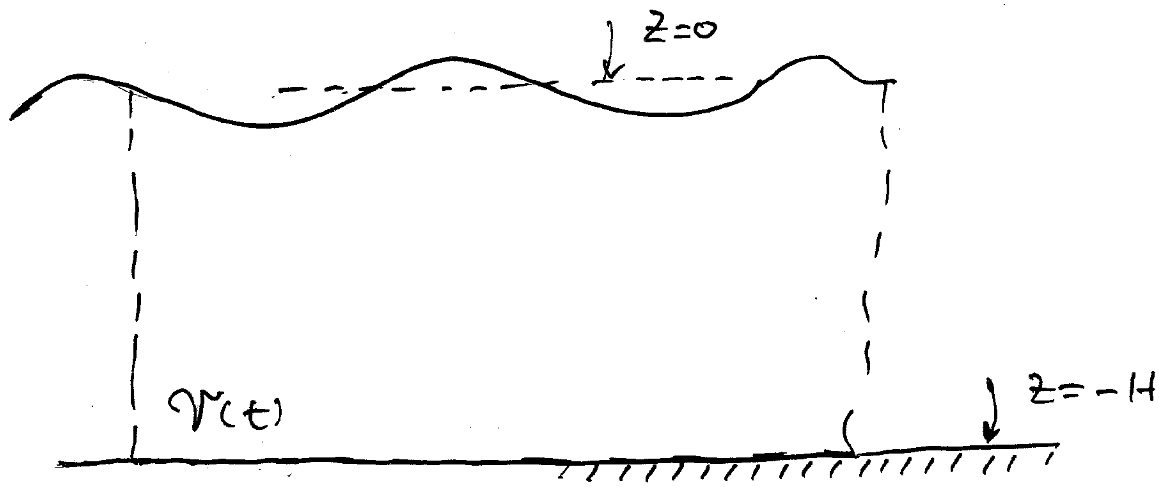
ENERGY DENSITY, ENERGY FLUX AND MOMENTUM FLUX OF SURFACE WAVES



$\mathcal{E}(t)$ = ENERGY IN CONTROL VOLUME $\mathcal{V}(t)$:

$$\mathcal{E}(t) = \rho \iiint_{\mathcal{V}} \Big(\underbrace{\tfrac{1}{2} v^2}_{\substack{\text{KINETIC}\\ \text{ENERGY}}} + \underbrace{g z}_{\substack{\text{POTENTIAL}\\ \text{ENERGY}}} \Big) \, dv$$

MEAN ENERGY OVER UNIT HORIZONTAL SURFACE AREA S :

$$\overline{\mathcal{E}} = \overline{\frac{\mathcal{E}(t)}{S}}^{\,t} = \rho \, \overline{\int_{-H}^{\zeta(t)} \left(\tfrac{1}{2} V^2 + g z \right) dz}^{\,t}$$

$$= \tfrac{1}{2} \rho \, \overline{\int_{-H}^{\zeta(t)} V^2 \, dz}^{\,t} + \tfrac{1}{2} \rho g \, \overline{\left(\zeta^2 - H^2 \right)}^{\,t}$$

WHERE $\zeta(t)$ IS FREE SURFACE ELEVATION.

IGNORE TERM $\sim \frac{1}{2}\rho g H^2$ WHICH REPRESENTS THE POTENTIAL ENERGY OF THE OCEAN AT REST.

THE REMAINING PERTURBATION COMPONENT IS THE SUM OF THE KINETIC + POTENTIAL ENERGY COMPONENTS

$$\bar{\mathcal{E}} = \bar{\mathcal{E}}_{KIN} + \bar{\mathcal{E}}_{POT}$$

$$\bar{\mathcal{E}}_{KIN} = \frac{1}{2}\rho \overline{\int_{-H}^{\zeta(t)} V^2 dz}^{\,t}, \quad V^2 = \nabla\phi\cdot\nabla\phi = \phi_x^2 + \phi_z^2$$

$$\bar{\mathcal{E}}_{POT} = \overline{\frac{1}{2}\rho g \zeta^2(t)}^{\,t}$$

CONSIDER NOW AS A SPECIAL CASE PLANE PROGRESSIVE WAVES DEFINED BY THE VELOCITY POTENTIAL IN DEEP WATER (FOR SIMPLICITY):

$$\phi = \mathrm{Re}\left\{ \frac{igA}{\omega} e^{kz - ikx + i\omega t} \right\}$$

$$\phi_x = \mathrm{Re}\left\{ \frac{igA}{\omega} \underset{\| \omega^2/g}{(-ik)} e^{kz - ikx + i\omega t} \right\}$$

$$= A\,\mathrm{Re}\left\{ \omega\, e^{kz - ikx + i\omega t} \right\}$$

$$\phi_z = \mathrm{Re}\left\{ \frac{igA}{\omega} k\, e^{kz - ikx + i\omega t} \right\}$$

$$= A\,\mathrm{Re}\left\{ i\omega\, e^{kz - ikx + i\omega t} \right\}. -$$

LEMMA

LET:
$$\mathrm{Re}\{\mathbb{A}e^{i\omega t}\} = \mathcal{A}(t)$$
$$\mathrm{Re}\{\mathbb{B}e^{i\omega t}\} = \mathcal{B}(t)$$

$$\overline{\mathcal{A}(t)\mathcal{B}(t)}^{t} = \frac{1}{2}\mathrm{Re}\{\mathbb{A}\mathbb{B}^*\}.-$$

$$\overline{\mathcal{E}}_{KIN} = \frac{1}{2}\rho\,\overline{\left(\int_{-\infty}^{0} + \int_{0}^{\zeta}\right)(\phi_x^2 + \phi_z^2)\,dz}^{t}$$
$$= \frac{1}{2}\rho\,\overline{\int_{-\infty}^{0}(\phi_x^2 + \phi_z^2)\,dz}^{t} + O(A^3)$$
$$= \rho\,\frac{\omega^2 A^2}{4k} = \frac{1}{4}\rho g A^2, \quad \text{FOR} \quad k = \omega^2/g$$

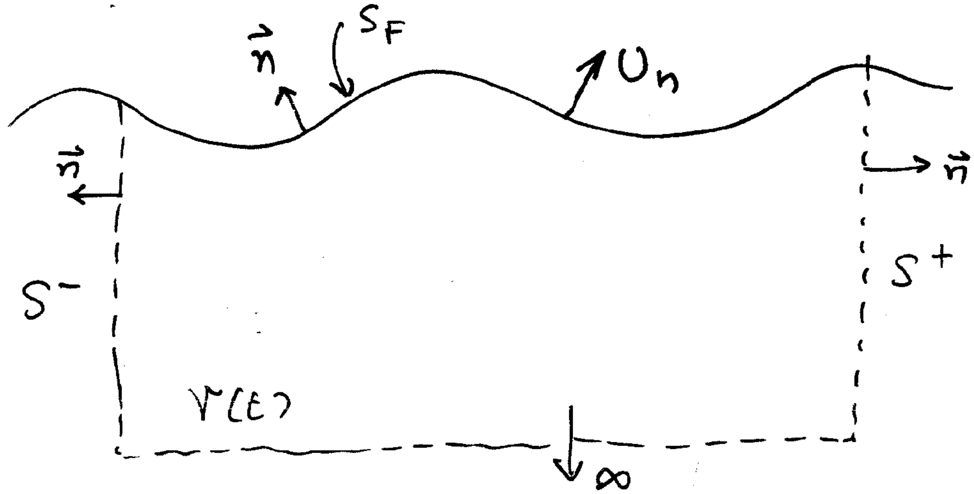
$$\overline{\mathcal{E}}_{POT} = \frac{1}{2}\rho g\,\overline{\zeta(t)^2} = \frac{1}{4}\rho g A^2.-$$

HENCE:

$$\overline{\mathcal{E}} = \underset{\frac{1}{4}\rho g A^2}{\overset{\shortparallel}{\overline{\mathcal{E}}_{KIN}}} + \underset{\frac{1}{4}\rho g A^2}{\overset{\shortparallel}{\overline{\mathcal{E}}_{POT}}} = \frac{1}{2}\rho g A^2.-$$

ENERGY FLUX = RATE OF CHANGE OF ENERGY DENSITY $\mathcal{E}(t)$

$$\mathcal{P}(t) \equiv \frac{d\,\mathcal{E}(t)}{dt}, \quad \mathcal{E}(t) = \iiint_{\mathcal{V}(t)} \underbrace{\left(\tfrac{1}{2}\rho V^2 + gz\right)}_{\epsilon(t)} d$$



$$\mathcal{P}(t) = \frac{d\,\mathcal{E}(t)}{dt} = \frac{d}{dt}\iiint_{\mathcal{V}(t)} \epsilon(t)\, dv = \oiint_{S(t)} \frac{\partial \epsilon(t)}{\partial t}\, dv + \oiint_{S(t)} \epsilon(t) U_n ds$$

TRANSPORT THEOREM WHERE U_n IS NORMAL VELOCITY OF SURFACE $S(t)$ OUTWARDS OF THE ENCLOSED VOLUME $\mathcal{V}$.

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho V^2 + \rho g z\right\} = \tfrac{1}{2}\rho \frac{\partial}{\partial t}(\nabla\phi\cdot\nabla\phi)$$

$$= \rho\,\nabla\cdot\left(\frac{\partial \phi}{\partial t}\nabla\phi\right) - \rho\frac{\partial\phi}{\partial t}\underbrace{\nabla^2\phi}_{0}$$

$$\mathcal{P}(t) = \frac{d\,\mathcal{E}(t)}{dt} = \rho \iiint_{\mathcal{V}(t)} \nabla\cdot\left(\frac{\partial\phi}{\partial t}\nabla\phi\right) dv$$

$$+\rho \oint\!\!\oint_{S(t)} \left(\tfrac{1}{2}V^2 + gz\right) U_n \, ds$$

INVOKING THE SCALAR FORM OF GAUSS'S THM IN THE FIRST TERM, WE OBTAIN:

- $$\mathcal{P}(t) = \rho \oint\!\!\oint_{S(t)} \frac{\partial\phi}{\partial t} \underbrace{\nabla\phi\cdot\vec{n}}_{\frac{\partial\phi}{\partial n}} \, ds + \rho \oint\!\!\oint_{S(t)} \left(\tfrac{1}{2}V^2 + gz\right) U_n \, ds$$

AN ALTERNATIVE FORM FOR THE ENERGY FLUX $\mathcal{P}(t)$ CROSSING THE CLOSED CONTROL SURFACE $S(t)$ IS OBTAINED BY INVOKING BERNOULLI'S EQUATION IN THE SECOND TERM. RECALL THAT:

$$\frac{p - p_a}{\rho} + \frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + gz = 0$$

AT ANY POINT IN THE FLUID DOMAIN AND ON BOUNDARIE

HERE WE DID ALLOW p_a = ATMOSPHERIC PRESSURE TO BE NON-ZERO FOR THE SAKE OF PHYSICAL CLARITY. UPON SUBSTITUTION IN $\mathcal{P}(t)$ WE OBTAIN THE ALTERNATNE FORM:

$$\mathcal{P}(t) = \rho \oiint_{S(t)} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial n} ds - \rho \oiint_{S(t)} \left(\frac{p - p_a}{\rho} + \frac{\partial \phi}{\partial t} \right) U_n \, ds$$

SO THE ENERGY FLUX ACROSS $S(t)$ IS GIVEN BY THE TERMS UNDER THE INTEGRAL SIGN. THEY CAN BE COLLECTED IN THE MORE COMPACT FORM:

- $$\mathcal{P}(t) = \oiint_{S(t)} \left\{ \rho \frac{\partial \phi}{\partial t} \left(\frac{\partial \phi}{\partial n} - U_n \right) - (p - p_a) U_n \right\} ds$$

NOTE THAT $\mathcal{P}(t)$ MEASURES THE ENERGY FLUX INTO THE VOLUME $\mathcal{V}(t)$ OR THE RATE OF GROWTH OF THE ENERGY DENSITY $\mathcal{E}(t)$.–

WE ARE READY NOW TO APPLY THE ABOVE FORMULAE TO THE SURFACE WAVE PROPAGATION PROBLEM.

BREAK $S(t)$ INTO ITS COMPONENTS AND DERIVE SPECIALIZED FORMS OF $\mathcal{P}(t)$ PERTINENT TO EACH.–

- S_F : NONLINEAR POSITION OF THE FREE SURFACE

$\frac{\partial \phi}{\partial n} = U_n$; NORMAL FLOW VELOCITY ≡ NORMAL VELOCITY OF FREESURFACE BOUNDARY ; KINEMATIC COND.

$p = p_a$; FLUID PRESSURE ≡ ATMOSPHERIC

THEREFORE OVER S_F ; $\mathcal{P}(t) \equiv 0$ AS EXPECTED. NO ENERGY CAN FLOW INTO THE ATMOSPHERE!

- S_B : <u>NON-MOVING</u> SOLID BOUNDARY

$$U_n = 0, \quad \frac{\partial \phi}{\partial n} = U_n \text{ ; NO-NORMAL FLUX CONDITION}$$

- $S^{\pm}$: FLUID BOUNDARIES FIXED IN SPACE RELATIVE TO AN EARTH FRAME

$$U_n = 0, \quad \frac{\partial \phi}{\partial n} \neq 0$$

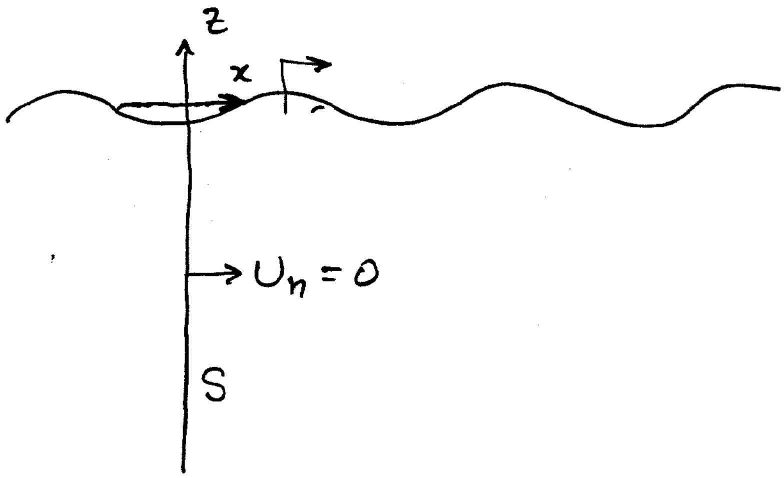
- S_U : FLUID BOUNDARIES MOVING W. VELOCITY $\vec{U}$ RELATIVE TO AN EARTH FRAME

$$U_n = \vec{U} \cdot \vec{n}, \quad \frac{\partial \phi}{\partial n} \neq 0$$

THIS CASE WILL BE OF INTEREST LATER IN THE COURSE WHEN WE CONSIDER SHIPS MOVING WITH CONSTANT VELOCITY U. —

THE FORMULAE DERIVED ABOVE ARE VERY GENERAL FOR POTENTIAL FLOWS WITH A FREE SURFACE AND SOLID BOUNDARIES. WE ARE NOW READY TO APPLY THEM TO PLANE PROGRESSIVE WAVES. —

• ENERGY FLUX ACROSS A VERTICAL FLUID BOUNDARY FIXED IN SPACE



$$\frac{P(t)}{\text{WIDTH}} = -\rho \int_{-\infty}^{\zeta(t)} \frac{\partial \phi}{\partial t} \phi_n \, dz = -\rho \left(\int_{-\infty}^{0} + \int_{0}^{\zeta} \right) \frac{\partial \phi}{\partial t} \phi_n \, dz$$

$$= -\rho \int_{-\infty}^{0} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial n} \, dz + O(A^3)$$

MEAN ENERGY FLUX FOR A PLANE PROGRESSIVE WAVE FOLLOWS UPON SUBSTITUTION OF THE REGULAR WAVE VELOCITY POTENTIAL AND TAKING MEAN VALUES:

$$\overline{P} = -\rho \, \overline{\int_{-\infty}^{0} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \, dz}^{\,t} = \underbrace{\frac{1}{2} \rho g A^2}_{\overline{\mathcal{E}}} \underbrace{\left(\frac{1}{2} \frac{g}{\omega} \right)}_{V_g}$$

OR

$$\overline{P} = \overline{\mathcal{E}} \, V_g , \quad V_g = \text{GROUP VELOCITY} = \frac{1}{2} V_p \equiv \frac{1}{2} C$$

IT FOLLOWS FROM THIS EXERCISE THAT THE MEAN ENERGY FLUX OF A PLANE PROGRESSIVE WAVE IS THE PRODUCT OF ITS MEAN ENERGY DENSITY TIMES A VELOCITY WHICH EQUALS $\frac{1}{2}$. THE PHASE VELOCITY IN DEEP WATER

WE CALL THIS THE <u>GROUP VELOCITY</u> OF DEEP WATER WAVES AND IT IS DEFINED AS:

$$V_g = \frac{1}{2} V_p = \frac{1}{2} \frac{g}{\omega}$$

A MORE FORMAL PROOF THAT THIS IS THE VELOCITY WITH WHICH THE ENERGY FLUX OF PLANE PROGRESSIVE WAVES PROPAGATES IS TO ASK THE FOLLOWING QUESTION:

→ WHAT NEEDS TO BE THE HORIZONTAL VELOCITY $U_n \equiv U$ OF A FLUID BOUNDARY SO THAT THE MEAN ENERGY FLUX ACROSS IT VANISHES?

THIS CAN BE FOUND FROM THE SOLUTION OF THE FOLLOWING EQUATION:

$$\overline{P(t)} = 0 = \rho \overline{\int_{-\infty}^{0} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \, dz}^{\,t} - U \overline{\int_{-\infty}^{0} \left(P/\rho + \frac{\partial \phi}{\partial t} \right) dz}^{\,t} = 0$$

WHERE TERMS OF $O(A^3)$ HAVE BEEN NEGLECTED.
NOTE THAT WITHIN LINEAR THEORY, ENERGY DENSITY AND ENERGY FLUX ARE QUANTITIES OF $O(A^2)$. IF HIGHER-ORDER TERMS ARE KEPT THEN WE NEED TO CONSIDER THE TREATMENT OF SECOND-ORDER SURFACE WAVE THEORY, AT LEAST. (SEE MEI).—

SOLVING THE ABOVE EQUATION FOR U WE OBTAIN:

$$U = \frac{\rho \overline{\int_{-\infty}^{0} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \, dz}^{\,t}}{\overline{\int_{-\infty}^{0} \left(P/\rho + \frac{\partial \phi}{\partial t} \right) dz}^{\,t}}$$

UPON SUBSTITUTION OF THE PLANE PROGRESSIVE WAVE VELOCITY POTENTIAL AND DEFINITION OF PRESSURE FROM BERNOULLI'S EQUATION WE OBTAIN:

$$U \equiv V_g = \frac{1}{2}\frac{g}{\omega} = \frac{1}{2} V_p$$

NOTE THAT $U \equiv V_g$ BY DEFINITION. IF THE ABOVE EXERCIZE IS REPEATED IN WATER OF FINITE DEPTH THE SOLUTION FOR U AFTER SOME ALGEBRA IS:

$$U = V_g = \left(\frac{1}{2} + \frac{KH}{\sinh 2KH}\right) \underset{\omega/K}{\underset{\parallel}{V_p}}$$

WITH

$$\omega^2 = gk \tanh KH \ldots$$

IT MAY BE SHOWN THAT THE GROUP VELOCITY V_g IS GIVEN IN TERMS OF ω & k BY THE RELATION

$$V_g = \frac{d\omega}{dk}$$

THIS RELATION FOLLOWS FROM THE VERY ELEGANT "DEVICE" DUE TO RAYLEIGH WHICH APPLIES TO ANY WAVE FORM:

CONSIDER TWO PLANE PROGRESSIVE WAVES OF NEARLY EQUAL FREQUENCIES AND HENCE WAVENUMBERS. THEIR JOINT WAVE ELEVATION IS GIVEN BY

$$\zeta(x,t) = A \cos(\omega_1 t - k_1 x) + A \cos(\omega_2 t - k_2 x)$$

WHERE THE AMPLITUDE IS ASSUMED TO BE COMMON AND:

$$\omega_2 = \omega_1 + \Delta\omega \quad , \quad |\Delta\omega| \ll \omega_1, \omega_2$$

$$k_2 = k_1 + \Delta k \qquad |\Delta k| \ll k_1, k_2$$

CONVERTING INTO COMPLEX NOTATION:

$$\zeta(x,t) = A \,\mathrm{Re}\left\{ e^{i\omega_1 t - ik_1 x} + e^{i\omega_2 t - ik_2 x} \right\}$$

$$= A \,\mathrm{Re}\left\{ e^{i\omega_1 t - ik_1 x} + e^{i\omega_1 t - ik_1 x + i\Delta\omega t - i\Delta k x} \right\}$$

$$= A \,\mathrm{Re}\left\{ e^{i\omega_1 t - ik_1 x} \underbrace{\left(1 + e^{i\Delta\omega t - i\Delta k x}\right)}_{F(x,t)} \right\}$$

THE COMBINED WAVE ELEVATION ζ VANISHES IDENTICALLY WHERE $F \equiv 0$

F=0 WHEN:

$$e^{i(\Delta\omega t - \Delta k x)} = -1$$

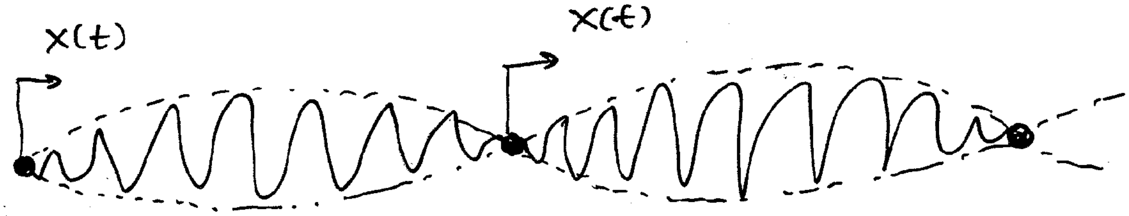
OR WHEN:

$$\Delta\omega t - \Delta k x = (2n+1)\pi \,, \quad n = 0,1,2,\ldots$$

SOLVING FOR x WE OBTAIN:

$$x = \frac{1}{\Delta k}\left\{(2n+1)\pi + t\,\Delta\omega\right\} \equiv x(t)$$

FOR VALUES OF $x(t)$ GIVEN ABOVE, $\zeta \equiv 0$
THESE ARE THE NODES OF THE BI-CHROMATIC WAVE TRAIN WHERE AT ALL TIMES THE ELEVATION VANISHES AND HENCE THE ENERGY DENSITY $\equiv 0$. THE WAVE GROUP HAS THE FORM



THE SPEED OF THE NODES IS $\dfrac{dx}{dt} = \dfrac{\Delta\omega}{\Delta k} \rightarrow \dfrac{d\omega}{dk}$

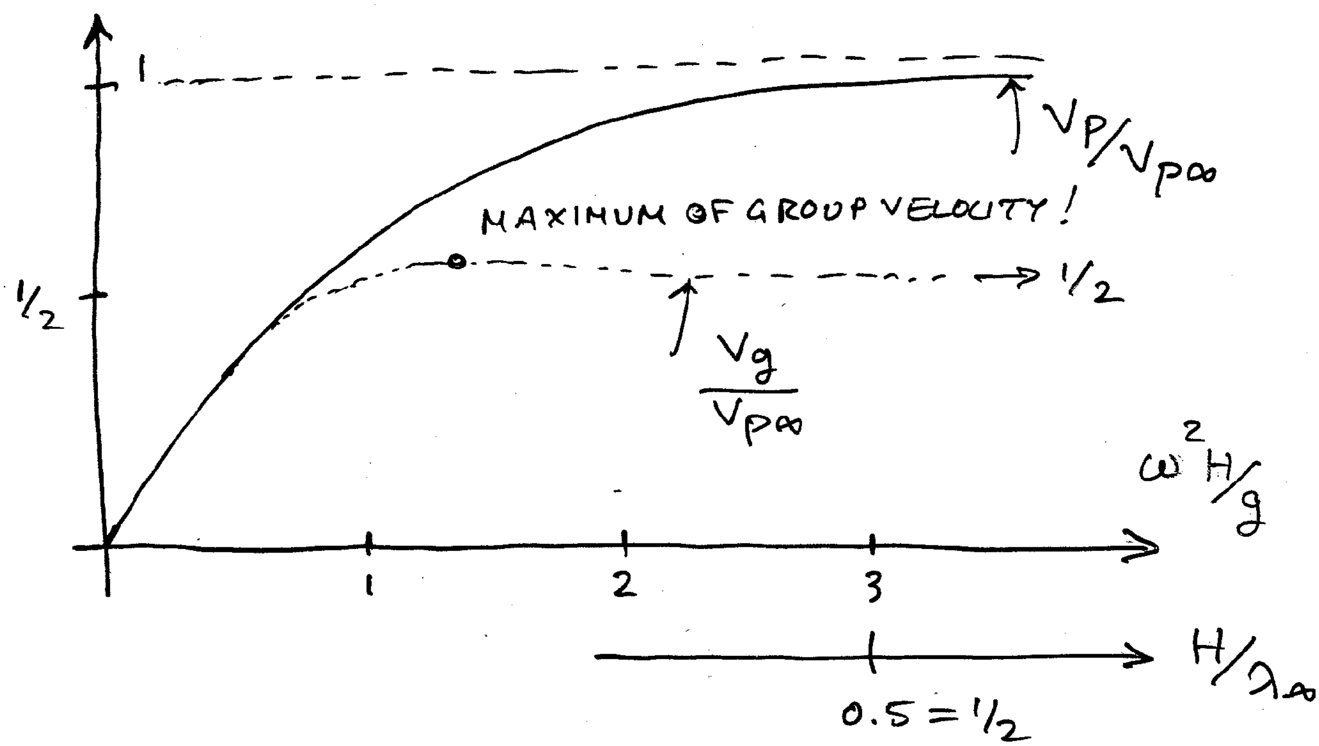
AND THE ENERGY TRAPPED WITHIN TWO CONSECUTIVE NODES CANNOT ESCAPE SO IT MUST TRAVEL AT THE GROUP VELOCITY: $V_g = \dfrac{dx}{dt} = \dfrac{d\omega}{dk}$!

NOTE THAT RAYLEIGH'S PROOF APPLIES EQUALLY TO WAVES IN FINITE DEPTH OR DEEP WATER AND IN PRINCIPLE TO ANY PROPAGATING WAVE FORM!

IN FINITE DEPTH IT CAN BE SHOWN AFTER SOME ALGEBRA THAT (SEE MH)

$$V_g = \frac{d\omega}{dk} = \left(\frac{1}{2} + \frac{kH}{\tanh kH}\right)\frac{\omega}{k} \cdot -$$

GRAPHICALLY, THE PHASE AND GROUP VELOCITIES MADE NON-DIMENSIONAL BY THE DEEP WATER PHASE VELOCITY $V_{p\infty} = g/\omega$ TAKE THE FORM:



THE FORMULAE FOR THE ENERGY FLUX DERIVED ABOVE ARE VERY GENERAL AND FOR POTENTIAL FLOW NONLINEAR SURFACE WAVES THAT ARE NOT BREAKING CONSTITUTE THE ENERGY CONSERVATION PRINCIPLE.

ENERGY FLUX (POWER) INPUT INTO THE FLUID DOMAIN BY ANY MECHANISM, WAVEMAKER WIND (IN A CONSERVATIVE MANNER), A SHIP OR ANY FLOATING BODY MUST BE "RETREIVED" AT SOME DISTANCE AWAY. DERIVING EXPRESSIONS OF THE ENERGY FLUX RETREIVED AT "INFINITY" IS A POWERFUL METHOD FOR ESTIMATING THE WAVE RESISTANCE OF SHIPS (MORE ON THIS LATER), THE WAVE DAMPING OF FLOATING BODIES ETC.

YET, THE ONLY GENERAL WAY OF EVALUATING WAVE FORCES ON FLOATING BODIES (MOVING OR NOT) OR ON SOLID BOUNDARIES IS BY APPLYING THE MOMENTUM CONSERVATION PRINCIPLE. →