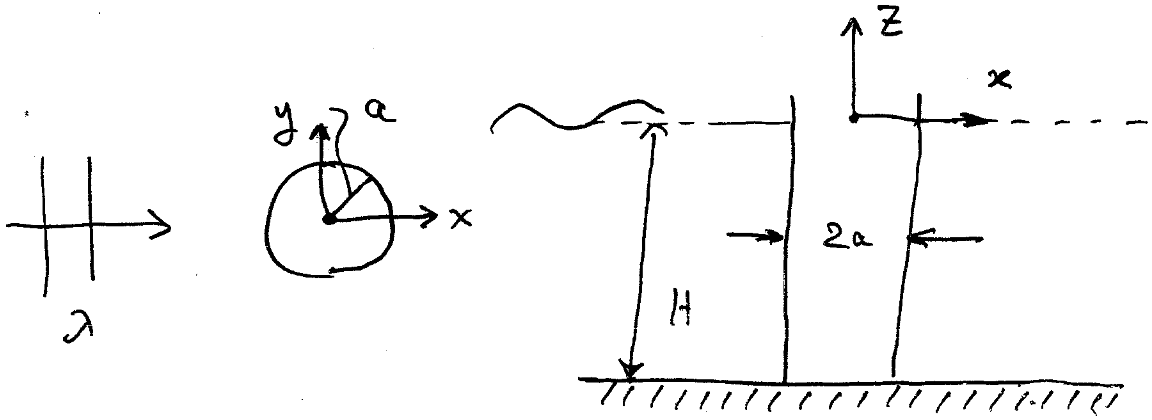
McCAMY – FUCHS ANALYTICAL SOLUTION OF THE SCATTERING OF REGULAR WAVES BY A VERTICAL CIRCULAR CYLINDER



THIS IMPORTANT FLOW ACCEPTS A CLOSED-FORM ANALYTICAL SOLUTION FOR ARBITRARY VALUES OF THE WAVELENGTH λ. THIS WAS SHOWN TO BE THE CASE BY McCAMY – FUCHS USING SEPARATION OF VARIABLES

$$\phi_I = \mathrm{Re}\{ \varphi_I e^{i\omega t} \}$$

$$\varphi_I = \frac{igA}{\omega} \frac{\cosh k(z+H)}{\cosh kH} e^{-ikx}$$

LET THE DIFFRACTION POTENTIAL BE:

$$\varphi_7 = \frac{igA}{\omega} \frac{\cosh k(z+H)}{\cos kH} \psi(x,y)$$

- FOR φ_7 TO SATISFY THE 3D LAPLACE EQUATION, IT IS EASY TO SHOW THAT ψ MUST SATISFY THE HELMHOLTZ EQUATION:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) \psi = 0$$

IN POLAR COORDINATES:

$$\left.\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \end{aligned}\right\} ; \quad \psi(R,\theta).$$

THE HELMHOLTZ EQUATION TAKES THE FORM:

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + k^2 \right) \psi = 0$$

ON THE CYLINDER:

$$\frac{\partial \varphi_7}{\partial n} = -\frac{\partial \varphi_I}{\partial n} \quad \text{OR}$$

$$\frac{\partial \psi}{\partial R} = -\frac{\partial}{\partial R}\left(e^{-ikx}\right)$$

$$= -\frac{\partial}{\partial R}\left(e^{-ikR\cos\theta}\right)$$

HERE WE MAKE USE OF THE FAMILIAR IDENTITY:

$$e^{-ikR\cos\theta} = \sum_{m=0}^{\infty} \epsilon_m J_m(kR) \cos m\theta$$

$$\epsilon_m = \begin{Bmatrix} 1, & m=0 \\ 2(-i)^m, & m>0 \end{Bmatrix}$$

TRY:

$$\psi(R,\theta) = \sum_{m=0}^{\infty} A_m F_m(kR) \cos m\vartheta$$

UPON SUBSTITUTION IN HELMHOLTZ'S EQUATION WE OBTAIN:

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{m^2}{R^2} + k^2 \right) F_m(kR) = 0$$

THIS IS THE BESSEL EQUATION OF ORDER m ACCEPTING AS SOLUTIONS LINEAR COMBINATIONS OF THE BESSEL FUNCTIONS

$$\begin{Bmatrix} J_m(kR) \\ Y_m(kR) \end{Bmatrix}$$

THE PROPER LINEAR COMBINATION IN THE PRESENT PROBLEM IS SUGGESTED BY THE RADIATION CONDITION THAT ψ MUST SATISFY:

AS $R \to \infty$: $\quad \psi(R,\theta) \propto e^{-ikR+i\omega t}$

ALSO AS $R \to \infty$:

$$J_m(kR) \sim \left(\frac{2}{\pi kR}\right)^{1/2} \cos\left(kR - \frac{1}{2}m\pi - \frac{\pi}{4}\right)$$

$$Y_m(kR) \sim \left(\frac{2}{\pi kR}\right)^{1/2} \sin\left(kR - \frac{1}{2}m\pi - \frac{\pi}{4}\right)$$

HENCE THE HANKEL FUNCTION:

$$H_m^{(2)}(kR) = J_m(kR) - iY_m(kR)$$

$$\sim \left(\frac{2}{\pi kR}\right)^{1/2} e^{-i\left(kR - \frac{1}{2}m\pi - \frac{\pi}{4}\right)}$$

SATISFIES THE FARFIELD CONDITION REQUIRED BY $\psi(R,\theta)$. SO WE SET:

$$\psi(r,\theta) = \sum_{m=0}^{\infty} \epsilon_m A_m H_m^{(2)}(kR) \cos m\theta$$

WITH THE CONSTANTS A_m TO BE DETERMINED.

THE CYLINDER CONDITION REQUIRES:

$$\left.\frac{\partial \psi}{\partial R}\right|_{R=a} = -\frac{\partial}{\partial R} \left.\sum_{m=0}^{\infty} \epsilon_m J_m(kR) \cos m\theta \right|_{r=a}$$

IT FOLLOWS THAT:

$$A_m H_m^{(2)'}(ka) = -J_m'(ka)$$

OR: $$A_m = -\frac{J_m'(ka)}{H_m^{(2)'}(ka)}$$

WHERE ($'$) DENOTES DERIVATIVES WITH RESPECT TO THE ARGUMENT. THE SOLUTION FOR THE TOTAL VELOCITY POTENTIAL FOLLOWS IN THE FORM

$$(\psi+\chi)(r,\theta) = \sum_{m=0}^{\infty} \epsilon_m \left[J_m(kR) - \frac{J_m'(ka)}{H_m^{(2)'}(ka)} H_m^{(2)}(kR) \right] \times \cos m\theta$$

AND THE TOTAL ORIGINAL POTENTIAL FOLLOWS:

$$\varphi = \varphi_I + \varphi_7 = \frac{igA}{\omega} \frac{\cosh k(z+H)}{\cosh kH} (\psi+\chi)(r,\theta)$$

IN THE LIMIT AS $H \to \infty$ $\frac{\cosh k(z+H)}{kH} \to e^{kz}$

AND THE SERIES EXPANSION SOLUTION SURVIVES.

SURGE EXCITING FORCE

THE TOTAL COMPLEX POTENTIAL, INCIDENT AND SCATTERED WAS DERIVED ABOVE.

THE HYDRODYNAMIC PRESSURE FOLLOWS FROM BERNOULLI:

$$p = \mathbb{R}e\left\{ \mathbb{P} e^{i\omega t} \right\}$$

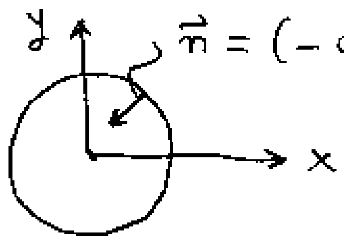
$$\mathbb{P} = -i\omega\rho\left(\varphi_I + \varphi_7 \right)$$

THE SURGE EXCITING FORCE IS GIVEN BY

$$X_1 = \iint_{S_B} p\, n_1\, ds = \mathbb{R}e\left\{ \mathbb{X}_1 e^{i\omega t} \right\}$$

$$\mathbb{X}_1 = \rho \int_{-\infty}^{0} dz \int_{0}^{2\pi} a\, d\theta \left(-i\omega \frac{igA}{\omega} \right) e^{kz}\, n_1 \left(\psi + \chi \right)\Big|_{R=a}$$

$$\vec{n} = (-\cos\theta, -\sin\theta) = (n_1, n_2)$$



SIMPLE ALGEBRA IN THIS CASE OF WATER OF INFINITE DEPTH LEADS TO THE EXPRESSION: