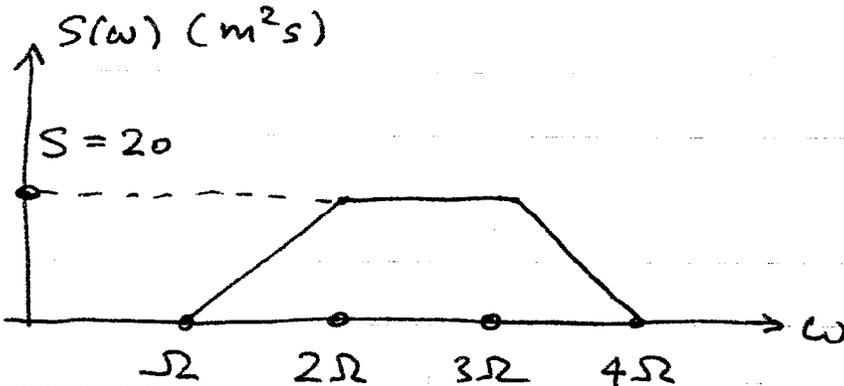13.022 — FALL 99
QUIZ #1 — SOLUTIONS
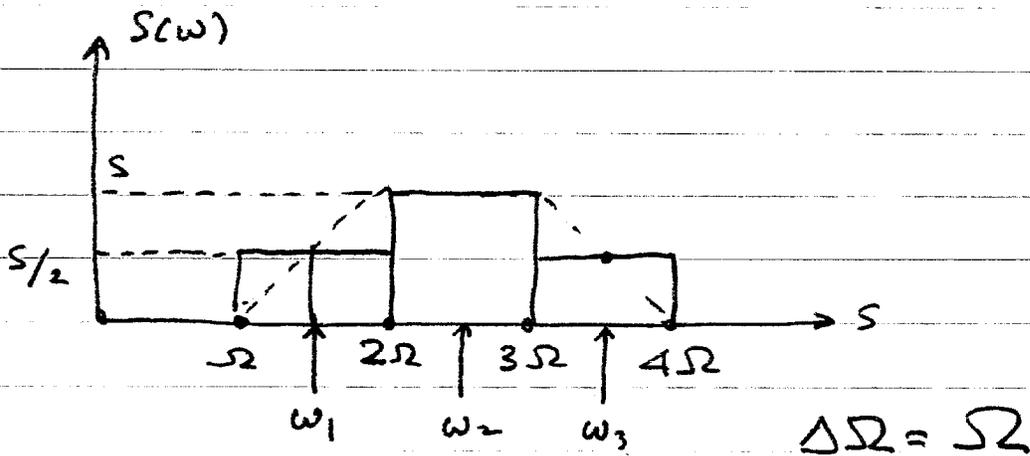
1.

a) $\quad \sigma^2 = \int_0^\infty d\omega\, S(\omega) = 2\Omega S = 2 \times 0.2 \times 20\ \text{m}^2$

$$= 8\ \text{m}^2$$

$$\sigma = 2.82\ \text{m}$$

b)

$$\Delta\Omega = \Omega$$

$$\omega_1 = \frac{3\Omega}{2}, \quad \omega_2 = \frac{5\Omega}{2}, \quad \omega_3 = \frac{7\Omega}{2}$$
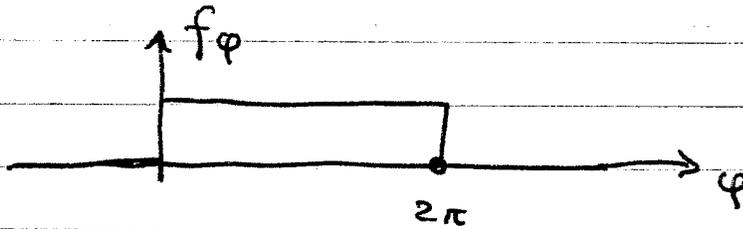
$$S(\omega_1) = \frac{S}{2}, \quad S(\omega_2) = S, \quad S(\omega_3) = \frac{S}{2}$$

$$\zeta(x,t) = \sum_{i=1}^{3} A_i \cos(k_i x - \omega_i t + \varphi_i)$$

- $\frac{1}{2} A_i^2 = S(\omega_i) \Delta\Omega \Rightarrow A_i = \{2\Omega S(\omega_i)\}^{1/2}$

- $k_i = \omega_i^2 / g$

- φ_i : RANDOM PHASE ANGLE UNIFORMLY DISTRIBUTED IN $(0, 2\pi)$

HENCE:

$$\zeta(x,t) = \sum_{i=1}^{3} \underbrace{\{2\Omega S(\omega_i)\}^{1/2}}_{A_i} \cos(k_i x - \omega_i t + \varphi_i)$$

c) $p = -\rho \dfrac{\partial \phi}{\partial t}$ (LINEAR TERM ONLY — DROP QUADRATIC + HYDROSTATIC)

$$\phi = \sum_{j=1}^{3} \mathbb{Re}\left\{ \frac{ig A_j}{\omega_j} e^{k_j z - i k_j x + i\omega_j t - \varphi_j} \right\}$$

CORRESPONDS TO WAVE ELEVATION DEFINED ABOVE. CHECK USING :

$$\zeta = -\frac{1}{g} \frac{\partial \phi}{\partial t}\Big|_{z=0}$$

d) $\quad \phi = \sum\limits_{j} \text{Re} \left\{ \dfrac{ig A_j}{\omega_j} e^{k_j z - i k_j x + i \omega_j t - i \varphi_j} \right\}$

ENERGY FLUX ACROSS PLANE AT $x=0$

$$\dfrac{\overline{dE}}{dt} = \rho \int\limits_{-\infty}^{0} \overline{\dfrac{\partial \phi}{\partial t} \dfrac{\partial \phi}{\partial x}}^{\,t} dz \quad , \quad \text{M.H., EQ. (75)}$$

$$\dfrac{\partial \phi}{\partial t} = \sum\limits_{j} \text{Re} \left\{ -g A_j e^{k_j z + i \omega_j t - i \varphi_j} \right\}_{x=0}$$

$$\dfrac{\partial \phi}{\partial x} = \sum\limits_{j} \text{Re} \left\{ \omega_j A_j e^{k_j z + i \omega_j t - i \varphi_j} \right\}_{x=0}$$

$$\int\limits_{-\infty}^{0} \overline{\dfrac{\partial \phi}{\partial t} \dfrac{\partial \phi}{\partial x}}^{\,t} dz = \int\limits_{-\infty}^{0} dz \, \overline{\sum\limits_{j} \text{Re} \left\{ -g A_j e^{k_j z + i \omega_j t - i \varphi_j} \right\}}$$

$$\times \sum\limits_{k} \text{Re} \left\{ \omega_k A_k e^{k_k z + i \omega_k t - i \varphi_k} \right\}^{\,t}$$

WHEN MEAN TIME VALUE IS TAKEN CROSS PRODUCTS
WITH FREQUENCIES $\omega_j \neq \omega_k$ AVERAGE OUT SINCE
$\overline{e^{i(\omega_j + \omega_k)t}}^{\,t} = 0$, $\omega_j \neq \omega_k$. HENCE ONLY THE
DIAGONAL TERMS SURVIVE LEADING TO AN
ENERGY FLUX

$$\dfrac{\overline{dE}}{dt}^{\,t} = \tfrac{1}{2} \rho g |A_1|^2 V_{g1} + \tfrac{1}{2} \rho g |A_2|^2 V_{g2}$$

$$+ \tfrac{1}{2} \rho g |A_3|^2 V_{g3}$$

$$V_{g1} = \dfrac{g}{2\omega_1} \quad , \quad V_{g2} = \dfrac{g}{2\omega_2} \quad , \quad V_{g3} = \dfrac{g}{2\omega_3} \quad \bullet \, -$$

e) WHEN PROPAGATING INTO WATER OF FINITE
DEPTH THE AMPLITUDES OF THE WAVE
COMPONENTS (A_i) OBEY THE ENERGY CONSERVATIO
RELATION:

$$\frac{1}{2} \rho g A_i^2 \frac{g}{2\omega_i} = \frac{1}{2} \rho g A_{iH}^2 V_{gH}$$

$$V_{gH} = \left(\frac{1}{2} + \frac{K_{iH} H}{\sinh 2 K_{iH} H} \right) \frac{\omega_i}{K_{iH}}$$

WHERE: $\omega_i^2 = g K_i \tanh K_i H \longrightarrow K_i$

IT FOLLOWS FROM ABOVE IDENTITY AND HOMEWORK/
RECITATION THAT $A_{iH} > A_{i\infty}$. MOREOVER
THIS INCREASE IN AMPLITUDE IS PROPORTIONALLY
GREATER THE LOWER THE FREQUENCY ω_i OR
THE LONGER THE WAVES. SO QUALITATIVELY
THE FINITE-WATER-DEPTH SPECTRUM IS OF
THE FORM, KEEPING Ω INVARIANT:
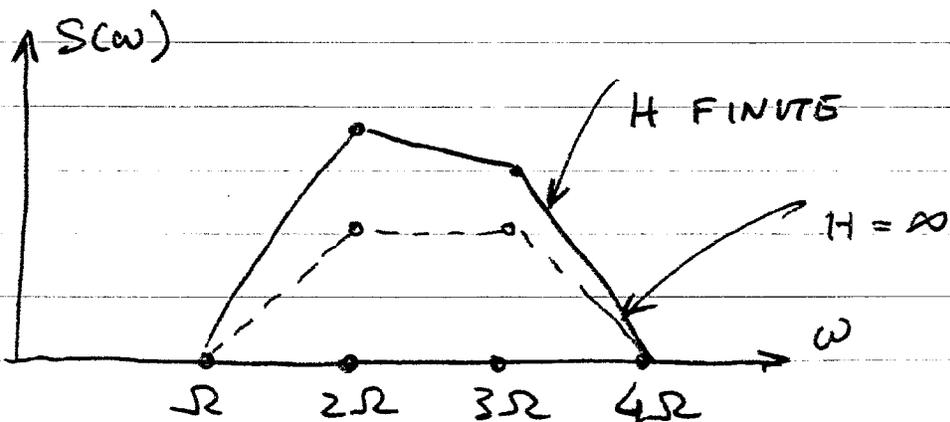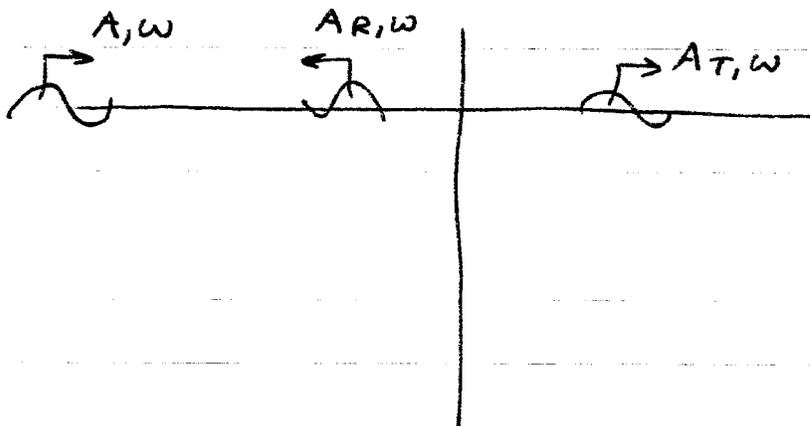
a) FREQUENCIES ARE CONSTANT BY VIRTUE OF
b) LINEARITY

INCIDENT ENERGY FLUX: $P_I = \frac{1}{2} \rho g A^2 \frac{g}{2\omega}$

TRANSMITTED ENERGY FLUX $P_T = \frac{1}{2} \rho g |A_T|^2 \frac{g}{2\omega}$

REFLECTED ENERGY FLUX $P_R = \frac{1}{2} \rho g |A_R|^2 \frac{g}{2\omega}$

FROM ENERGY BALANCE OF WAVES PROPAGATING
IN OPPOSITE DIRECTIONS

 ; NET ENERGY FLUX TOWARDS
THE SCREEN

$$\frac{1}{2} \rho g \left(A^2 - |A_R|^2 \right)$$

NET ENERGY FLUX PAST THE SCREEN

$$\frac{1}{2} \rho g |A_T|^2$$

IN THE ABSENCE OF ENERGY DISSIPATION:

$$|A|^2 = |A_R|^2 + |A_T|^2$$

WITH 10% ENERGY LOSS THE ENERGY BALANCE EQUATION BECOMES

$$E_{IN-NET} = \frac{1}{2}\rho g\left(|A|^2 - |A_R|^2\right)$$

$$E_{TRANSMITTER} = \frac{1}{2}\rho g|A_T|^2$$

$$E_{LOSS} = 0.1 \frac{1}{2}\rho g|A|^2$$

$$E_{IN-NET} = E_{TR} + E_{LOSS}$$

$$\frac{1}{2}\rho g\left(|A|^2 - |A_R|^2\right) = \frac{1}{2}\rho g\left(|A_T|^2 + 0.1A^2\right)$$

$$OR \quad 0.9|A|^2 = |A_R|^2 + |A_T|^2$$

NO FURTHER DETERMINATION OF A_R & A_T IS POSSIBLE IF ADDITIONAL INFORMATION IS NOT AVAILABLE. FOR QUESTIONS c) - e) ASSUME THAT MODULUS & PHASE OF A_T AND A_R ARE KNOWN.

c)

VELOCITY POTENTIAL $x < 0$:

$$\phi^- = \text{Re}\left\{ \frac{isA}{\omega} e^{kz - ikx + i\omega t} \right\}$$

$$+ \text{Re}\left\{ \frac{isA_R}{\omega} e^{kz + ikx + i\omega t} \right\}$$

$x > 0$: $\phi^+ = \text{Re}\left\{ \frac{isA_T}{\omega} e^{kz - ikx + i\omega t} \right\}$

HORIZONTAL LINEAR FORCE ON SCREEN

$$F_x = \int_{-\infty}^{0} (p^- - p^+)\Big|_{x=0} dz \times \underbrace{0.9}_{\%\,WET.}$$

$$P^- = -\rho \frac{\partial \phi^-}{\partial t} = \rho g \, \text{Re}\left\{ (A + A_R) e^{kz + i\omega t} \right\}$$

$$p^+ = -\rho \frac{\partial \phi^+}{\partial t} = \rho g \, \text{Re}\left\{ A_T e^{kz + i\omega t} \right\}$$

$$F_x = 0.9 \, \rho g \underbrace{\frac{1}{k}}_{\int_{-\infty}^{0} e^{kz} dz} \text{Re}\left\{ \underbrace{(A + A_R - A_T)}_{|A + A_R - A| e^{i\varphi}} e^{i\omega t} \right\}$$

$$F_x = \frac{0.9 \, \rho g}{k} |A + A_R - A_T| \cos(\omega t + \varphi) \, \circ \, ---$$

$$k = \omega^2 / g. \, ---$$

c)

$$\phi^- = Re\left\{\frac{isA}{\omega}e^{kz-ikx+i\omega t}\right\}$$

$$+ Re\left\{\frac{isA_R}{\omega}e^{kz+ikx+i\omega t}\right\}$$

$$\phi_x^- = Re\left\{\omega A\, e^{kz-ikx+i\omega t}\right\}$$

$$+ Re\left\{-\omega A_R\, e^{kz+ikx+i\omega t}\right\}$$
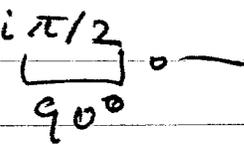
FOR SIMPLICITY SET X=0 SINCE PHASE
DIFFERECES ARE ACCOUNTED FOR UNIFORMLY
IN X. REPEAT FOR Z-DERIVATIVE

$$\phi_z^- = Re\left\{i\omega A\, e^{kz-ikx+i\omega t}\right\}$$

$$+ Re\left\{i\omega A_R\, e^{kz-ikx+i\omega t}\right\}$$

AT X=0: $\phi_x^- = e^{kz}\omega\, Re\left\{(A-A_R)e^{i\omega t}\right\}$

$$\phi_z^- = e^{kz}\omega\, Re\left\{i(A+A_R)e^{i\omega t}\right\}$$

NOTE THAT $A \equiv REAL$, $A_R = |A_R|e^{i\varphi}$

IF $A_R=0$ TRAJECTORIES CIRCULAR SINCE HORIZONTA
VELOCITY IS 90° OUT OF PHASE FROM VERTICAL
VELOCITY. NOTE THAT $i = e^{\underbrace{i\pi/2}_{90°}}$.

LET:
$$A - A_R = |A - A_R| e^{i\varphi_1} = M_1 e^{i\varphi_1}$$
$$i(A + A_R) = |A + A_R| e^{i\varphi_2} = M_2 e^{i\varphi_2}$$

$$\Rightarrow u \equiv \phi_x^- = e^{kz} \omega M_1 \cos(\omega t + \varphi_1) \equiv d\xi/dt$$
$$v \equiv \phi_z^- = e^{kz} \omega M_2 \cos(\omega t + \varphi_2) \equiv d\zeta/dt$$

WHERE $\xi(t)$, $\zeta(t)$ ARE HORIZONTAL, VERTICAL COORDINATES OF A FLUID PARTICLE. QUALITATIVELY THE SOLUTION OF THIS PAIR OF ORDINARY DIFFERENTIAL EQUATIONS WILL YIELD TWO SUPERIMPOSED CIRCULAR TRAJECTORIES WITH OPPOSITE DIRECTION OF ROTATION DUE TO THE TWO WAVES MOVING IN THE OPPOSITE DIRECTION WITH DIFFERENT AMPLITUDES. THUS:
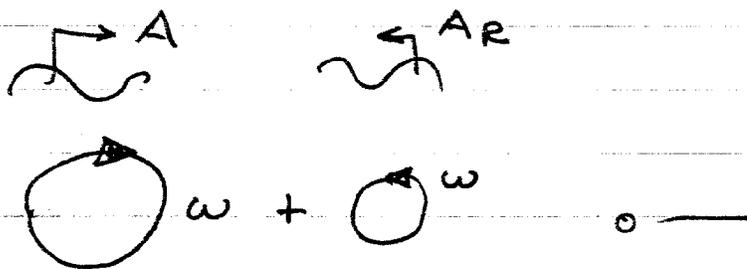
e) THE ANSWERS ABOUT A, A_T & A_R ARE THE SAME SINCE THE GROUP VELOCITY IN FINITE DEPTH FACTORS OUT.

THE HORIZONTAL FORCE WILL BECOME:

$$F_x = 0.9 \, \rho g \, \text{Re} \left\{ (A + A_R - A_T) e^{i\omega t} \right\}$$

$$x \underbrace{\int_{-H}^{0} dz \, \frac{\cosh k(z+H)}{\cosh kH}}_{\frac{1}{k} \tanh kH}$$

SO $\quad F_x = 0.9 \, \rho g \, \text{Re} \left\{ (A + A_R - A_T) e^{i\omega t} \right\} \frac{\tanh kH}{k}$

ASSUMING THAT THE MODULUS & PHASE OF
A_R, A_T IS THE SAME (NOT NECESSARILY SO BEFORE
FURTHER ANALYSIS) THE MODULUS OF THE
HORIZONTAL FORCE IN FINITE DEPTH WILL
BE SMALLER SINCE

$$\frac{\tanh kH}{k} \leq \frac{1}{k}$$

NOTE THAT AS $H \to 0$, $\tanh kH \to 1$
FROM VALUES SMALLER THAN 1