2.161 Signal Processing: Continuous and Discrete
Fall 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

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Lecture 22[1]

Reading:

- Proakis and Manolakis: Secs. 12,1 – 12.2
- Oppenheim, Schafer, and Buck:
- Stearns and Hush: Ch. 13

1 The Correlation Functions (continued)

In Lecture 21 we introduced the *auto-correlation* and *cross-correlation* functions as measures of self- and cross-similarity as a function of delay τ. We continue the discussion here.

1.1 The Autocorrelation Function

There are three basic definitions

(a) For an infinite duration waveform:

$$\phi_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)f(t+\tau)\,\mathrm{d}t$$

which may be considered as a "power" based definition.

(b) For an finite duration waveform: If the waveform exists only in the interval $t_1 \leq t \leq t_2$

$$\rho_{ff}(\tau) = \int_{t_1}^{t_2} f(t)f(t+\tau)\,\mathrm{d}t$$

which may be considered as a "energy" based definition.

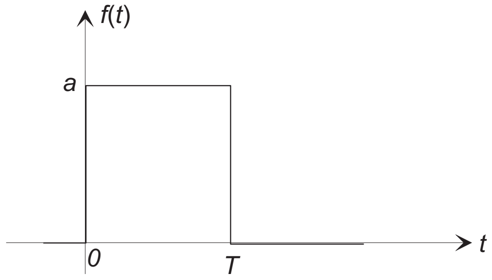
(c) For a periodic waveform: If $f(t)$ is periodic with period T

$$\phi_{ff}(\tau) = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)f(t+\tau)\,\mathrm{d}t$$

for an arbitrary t_0, which again may be considered as a "power" based definition.

■ Example 1

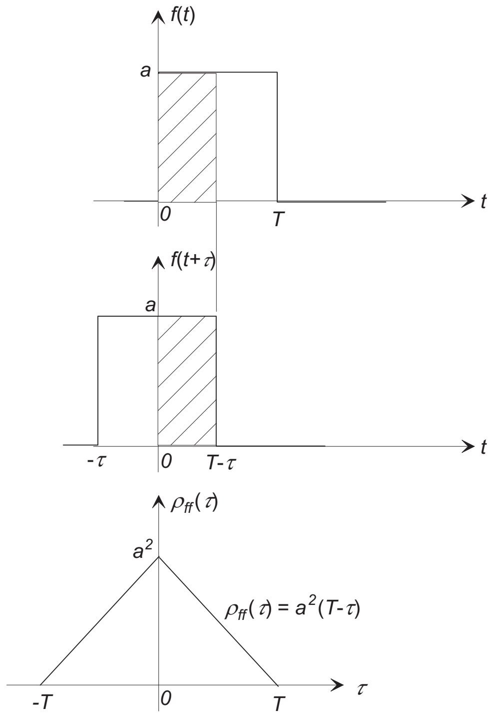
Find the autocorrelation function of the square pulse of amplitude a and duration T as shown below.



The wave form has a finite duration, and the autocorrelation function is

$$\rho_{ff}(\tau) = \int_0^T f(t)f(t+\tau)\,\mathrm{d}t$$

The autocorrelation function is developed graphically below



$$\begin{aligned}
\rho_{ff}(\tau) &= \int_0^{T-\tau} a^2\,\mathrm{d}t \\
&= a^2(T - |\tau|) \qquad -T \le \tau \le T \\
&= 0 \qquad\qquad\qquad \text{otherwise.}
\end{aligned}$$

■ Example 2

Find the autocorrelation function of the sinusoid $f(t) = \sin(\Omega t + \phi)$.

Since $f(t)$ is periodic, the autocorrelation function is defined by the average over one period

$$\phi_{ff}(\tau) = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)f(t+\tau)\,\mathrm{d}t.$$

and with $t_0 = 0$

$$\begin{aligned}\phi_{ff}(\tau) &= \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} \sin(\Omega t + \phi)\sin(\Omega(t+\tau)+\phi)\,\mathrm{d}t \\ &= \frac{1}{2}\cos(\Omega t)\end{aligned}$$

and we see that $\phi_{ff}(\tau)$ is periodic with period $2\pi/\Omega$ and is independent of the phase ϕ.

1.1.1 Properties of the Auto-correlation Function

(1) The autocorrelation functions $\phi_{ff}(\tau)$ and $\rho_{ff}(\tau)$ are even functions, that is

$$\phi_{ff}(-\tau) = \phi_{ff}(\tau), \quad \text{and} \quad \rho_{ff}(-\tau) = \rho_{ff}(\tau).$$

(2) A maximum value of $\rho_{ff}(\tau)$ (or $\phi_{ff}(\tau)$ occurs at delay $\tau = 0$,

$$|\rho_{ff}(\tau)| \le \rho_{ff}(0), \quad \text{and} \quad |\phi_{ff}(\tau)| \le \phi_{ff}(0)$$

and we note that

$$\rho_{ff}(0) = \int_{-\infty}^{\infty} f^2(d)\,\mathrm{d}t$$

is the "energy" of the waveform. Similarly

$$\phi_{ff}(0) = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} f^2(t)\,\mathrm{d}t$$

is the mean "power" of $f(t)$.

(3) $\rho_{ff}(\tau)$ contains no phase information, and is independent of the time origin.

(4) If $f(t)$ is periodic with period T, $\phi_{ff}(\tau)$ is also periodic with period T.

(5) If (1) $f(t)$ has zero mean ($\mu = 0$), and (2) $f(t)$ is non-periodic,

$$\lim_{\tau\to\infty}\rho_{ff}(\tau) = 0.$$

1.1.2 The Fourier Transform of the Auto-Correlation Function

Consider the transient case

$$\begin{aligned}
R_{ff}(\mathrm{j}\,\Omega) &= \int_{-\infty}^{\infty} \rho_{ff}(\tau)\,\mathrm{e}^{-\mathrm{j}\,\Omega\tau}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)f(t+\tau)\,\mathrm{d}t\right)\,\mathrm{e}^{-\mathrm{j}\,\Omega\tau}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{\mathrm{j}\,\Omega t}\,\mathrm{d}t.\int_{-\infty}^{\infty} f(\nu)\,\mathrm{e}^{-\mathrm{j}\,\Omega\nu}\,\mathrm{d}\nu \\
&= F(-\mathrm{j}\,\Omega)F(\mathrm{j}\,\Omega) \\
&= |F(\mathrm{j}\,\Omega)|^2
\end{aligned}$$

or

$$\boxed{\rho_{ff}(\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} R_{ff}(\mathrm{j}\,\Omega) = |F(\mathrm{j}\,\Omega)|^2}$$

where $R_{ff}(\Omega)$ is known as the *energy density spectrum* of the transient waveform $f(t)$. Similarly, the Fourier transform of the power-based autocorrelation function, $\phi_{ff}(\tau)$

$$\begin{aligned}
\Phi_{ff}(\mathrm{j}\,\Omega) &= \mathcal{F}\{\phi_{ff}(\tau)\} = \int_{-\infty}^{\infty} \phi_{ff}(\tau)\,\mathrm{e}^{-\mathrm{j}\,\Omega\tau}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty}\left(\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f(t)f(t+\tau)\,\mathrm{d}t\right)\,\mathrm{e}^{-\mathrm{j}\,\Omega\tau}\,\mathrm{d}\tau
\end{aligned}$$

is known as the *power density spectrum* of an infinite duration waveform.

From the properties of the Fourier transform, because the auto-correlation function is a real, even function of τ, the energy/power density spectrum is a real, even function of Ω, and contains no phase information.

1.1.3 Parseval's Theorem

From the inverse Fourier transform

$$\rho_{ff}(0) = \int_{\infty}^{\infty} f^2(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\mathrm{j}\,\Omega)\,\mathrm{d}\Omega$$

or

$$\int_{\infty}^{\infty} f^2(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\mathrm{j}\,\Omega)|^2\,\mathrm{d}\Omega,$$

which equates the total waveform energy in the time and frequency domains, and which is known as *Parseval's theorem.* Similarly, for infinite duration waveforms

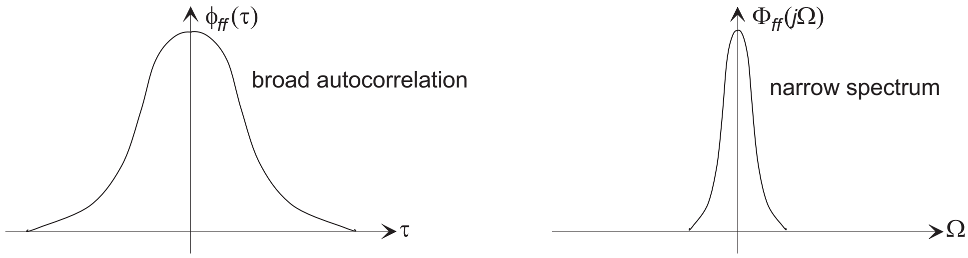
$$\lim_{T\to\infty}\int_{-T/2}^{T/2} f^2(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\mathrm{j}\,\Omega)\,\mathrm{d}\Omega$$

equates the signal power in the two domains.

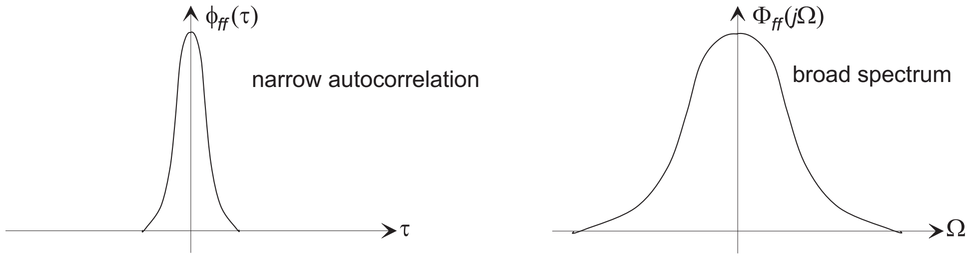
1.1.4 Note on the relative "widths" of the Autocorrelation and Power/Energy Spectra

As in the case of Fourier analysis of waveforms, there is a general reciprocal relationship between the width of a signals spectrum and the width of its autocorrelation function.

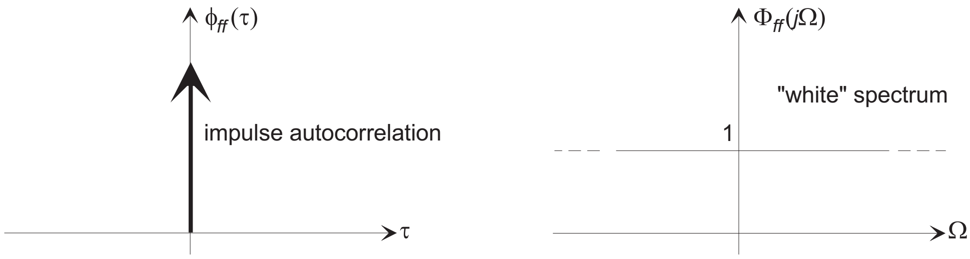
- A narrow autocorrelation function generally implies a "broad" spectrum



- and a "broad" autocorrelation function generally implies a narrow-band waveform.



In the limit, if $\phi_{ff}(\tau) = \delta(\tau)$, then $\Phi_{ff}(\mathrm{j}\,\Omega) = 1$, and the spectrum is defined to be "white".



1.2 The Cross-correlation Function

The cross-correlation function is a measure of self-similarity between two waveforms $f(t)$ and $g(t)$. As in the case of the auto-correlation functions we need two definitions:

$$\phi_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)g(t+\tau)\,\mathrm{d}\tau$$

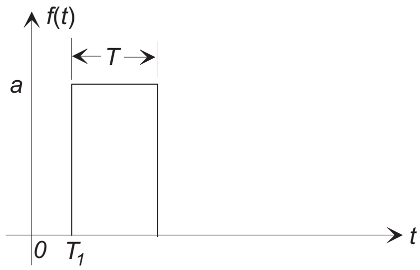
in the case of infinite duration waveforms, and

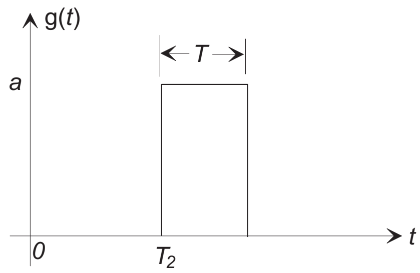
$$\rho_{fg}(\tau) = \int_{-\infty}^{\infty} f(t)g(t+\tau)\,\mathrm{d}\tau$$

for finite duration waveforms.

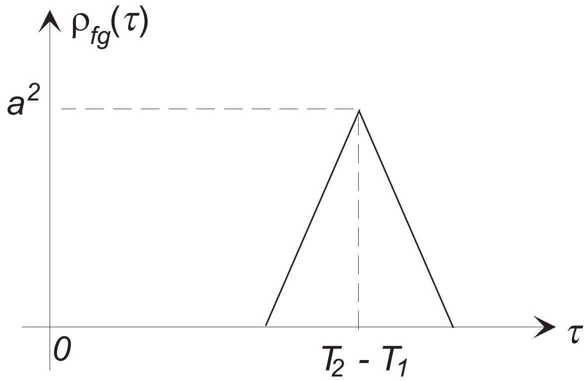
■ Example 3

Find the cross-correlation function between the following two functions





In this case $g(t)$ is a delayed version of $f(t)$. The cross-correlation is



where the peak occurs at $\tau = T_2 - T_1$ (the delay between the two signals).

1.2.1 Properties of the Cross-Correlation Function

(1) $\phi_{fg}(\tau) = \phi_{gf}(-\tau)$, and the cross-correlation function is not necessarily an even function.

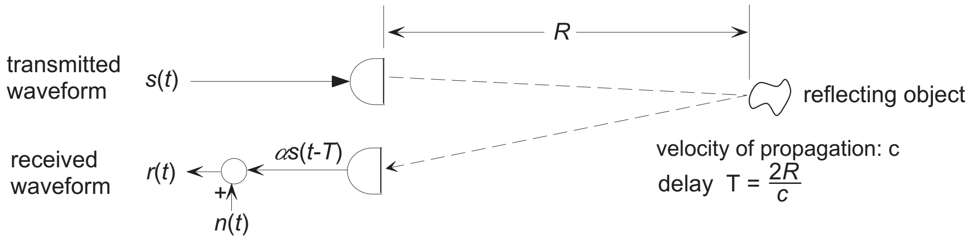
(2) If $\phi_{fg}(\tau) = 0$ for all τ, then $f(t)$ and $g(t)$ are said to be *uncorrelated.*

(3) If $g(t) = af(t - T)$, where a is a constant, that is $g(t)$ is a scaled and delayed version of $f(t)$, then $\phi_{ff}(\tau)$ will have its maximum value at $\tau = T$.

Cross-correlation is often used in optimal estimation of delay, such as in echolocation (radar, sonar), and in GPS receivers.

■ Example 4

In an echolocation system, a transmitted waveform $s(t)$ is reflected off an object at a distance R and is received a time $T = 2R/c$ sec. later. The received signal $r(t) = \alpha s(t-T)+n(t)$ is attenuated by a factor α and is contaminated by additive noise $n(t)$.



$$
\begin{aligned}
\phi_{sr}(\tau) &= \int_{\infty}^{\infty} s(t) r(t+\tau)\,\mathrm{d}t \\
&= \int_{\infty}^{\infty} s(t)(n(t+\tau) + \alpha s(t-T+\tau))\,\mathrm{d}t \\
&= \phi_{sn}(\tau) + \alpha\phi_{ss}(\tau - T)
\end{aligned}
$$

and if the transmitted waveform $s(t)$ and the noise $n(t)$ are uncorrelated, that is $\phi_{sn}(\tau) \equiv 0$, then

$$\phi_{sr}(\tau) = \alpha\phi_{ss}(\tau - T)$$

that is, a scaled and shifted version of the auto-correlation function of the transmitted waveform – which will have its peak value at $\tau = T$, which may be used to form an estimator of the range R.

1.2.2 The Cross-Power/Energy Spectrum

We define the cross-power/energy density spectra as the Fourier transforms of the cross-correlation functions:

$$
\begin{aligned}
R_{fg}(\mathrm{j}\,\Omega) &= \int_{-\infty}^{\infty} \rho_{fg}(\tau)\,\mathrm{e}^{-\mathrm{j}\,\Omega\tau}\,\mathrm{d}\tau \\
\Phi_{fg}(\mathrm{j}\,\Omega) &= \int_{-\infty}^{\infty} \phi_{fg}(\tau)\,\mathrm{e}^{-\mathrm{j}\,\Omega\tau}\,\mathrm{d}\tau.
\end{aligned}
$$

Then

$$
\begin{aligned}
R_{fg}(\mathrm{j}\,\Omega) &= \int_{-\infty}^{\infty} \rho_{fg}(\tau)\,\mathrm{e}^{-\mathrm{j}\,\Omega\tau}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(t+\tau)\,\mathrm{e}^{-\mathrm{j}\,\Omega\tau}\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{\mathrm{j}\,\Omega t}\,\mathrm{d}t \int_{-\infty}^{\infty} g(\nu)\,\mathrm{e}^{-\mathrm{j}\,\Omega\nu}\,\mathrm{d}\nu
\end{aligned}
$$

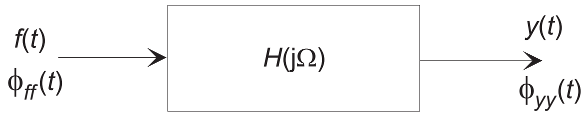
or

$$\boxed{R_{fg}(\mathrm{j}\,\Omega) = F(-\mathrm{j}\,\Omega)G(\mathrm{j}\,\Omega)}$$

Note that although $R_{ff}(\mathrm{j}\,\Omega)$ is real and even (because $\rho_{ff}(\tau)$ is real and even, this is not the case with the cross-power/energy spectra, $\Phi_{fg}(\mathrm{j}\,\Omega)$ and $R_{fg}(\mathrm{j}\,\Omega)$, and they are in general complex.

2 Linear System Input/Output Relationships with Random Inputs:

Consider a linear system $H(\mathrm{j}\,\Omega)$ with a random input $f(t)$. The output will also be random



Then

$$\begin{aligned} Y(\mathrm{j}\,\Omega) &= F(\mathrm{j}\,\Omega)H(\mathrm{j}\,\Omega), \\ Y(\mathrm{j}\,\Omega)Y(-\mathrm{j}\,\Omega) &= F(\mathrm{j}\,\Omega)H(\mathrm{j}\,\Omega)F(-\mathrm{j}\,\Omega)H(-\mathrm{j}\,\Omega) \end{aligned}$$

or

$$\boxed{\Phi_{yy}(\mathrm{j}\,\Omega) = \Phi_{ff}(\mathrm{j}\,\Omega)\left|H(\mathrm{j}\,\Omega)\right|^2.}$$

Also

$$F(-\mathrm{j}\,\Omega)Y(\mathrm{j}\,\Omega) = F(-\mathrm{j}\,\Omega)F(\mathrm{j}\,\Omega)H(\mathrm{j}\,\Omega),$$

or

$$\boxed{\Phi_{fy}(\mathrm{j}\,\Omega) = \Phi_{ff}(\mathrm{j}\,\Omega)H(\mathrm{j}\,\Omega).}$$

Taking the inverse Fourier transforms

$$\begin{aligned} \phi_{yy}(\tau) &= \phi_{ff}(\tau) \otimes \mathcal{F}^{-1}\left\{\left|H(\mathrm{j}\,\Omega)\right|^2\right\} \\ \phi_{fy}(\tau) &= \phi_{ff}(\tau) \otimes h(\tau). \end{aligned}$$

3 Discrete-Time Correlation

Define the correlation functions in terms of summations, for example for an infinite length sequence

$$\begin{aligned} \phi_{fg}(n) &= \mathcal{E}\left\{f_m g_{m+n}\right\} \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} f_m g_{m+n}, \end{aligned}$$

and for a finite length sequence

$$\rho_{fg}(n) = \sum_{m=-N}^{N} f_m g_{m+n}.$$

The following properties are analogous to the properties of the continuous correlation functions:

(1) The auto-correlation functions $\phi_(ff(n)$ and $\rho_{ff}(n)$ are real, even functions.

(2) The cross-correlation functions are not necessarily even functions, and

$$\phi_{fg}(n) = \phi_{gf}(-n)$$

(2) $\phi_{ff}(n)$ has its maximum value at $n = 0$,

$$|\phi_{ff}(n)| \leq \phi_{ff}(0) \qquad \text{for all } n.$$

(3) If $\{f_k\}$ has no periodic component

$$\lim_{n\to\infty} \phi_{ff}(n) = \mu_f^2.$$

(4) $\phi_{ff}(0)$ is the average *power* in an infinite sequence, and $\rho_{ff}(n)$ is the total *energy* in a finite sequence.

The discrete power/energy spectra are defined through the z-transform

$$\Phi_{ff}(z) = \mathcal{Z}\left\{\phi_{ff}(n)\right\} = \sum_{n=-\infty}^{\infty} \phi_{ff}(n) z^{-n}$$

and

$$\begin{aligned}
\phi_{ff}(n) &= Z^{-1}\left\{\Phi_{ff}(z)\right\} \\
&= \frac{1}{2\pi \mathrm{j}} \oint \Phi_{ff}(z) z^{n-1} dz \\
&= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \Phi_{ff}(\mathrm{e}^{\mathrm{j}\,\Omega T})\, \mathrm{e}^{\mathrm{j}\,n\Omega T}\, \mathrm{d}\Omega.
\end{aligned}$$

<u>Note on the MATLAB function `xcorr()`</u>: In MATLAB the function call `phi = xcorr(f,g)` computes the cross-correlation function, but reverses the definition of the subscript order from that presented here, that is it computes

$$\phi_{fg}(n) = \frac{1}{M} \sum_{-N}^{N} f_{n+m} g_m = \frac{1}{M} \sum_{-N}^{N} f_n g_{n-m}$$

where M is a normalization constant specified by an optional argument. Care must therefore be taken in interpreting results computed through `xcorr()`.

3.1 Summary of z-Domain Correlation Relationships

(The following table is based on Table 13.2 from Stearns and Hush)

Property	Formula
Power spectrum of $\{f_n\}$	$\Phi_{ff}(z) = \sum_{n=-\infty}^{\infty} \phi_{ff}(n) z^{-n}$
Cross-power Spectrum	$\Phi_{fg}(z) = \sum_{n=-\infty}^{\infty} \phi_{fg}(n) z^{-n} = \Phi_{gf}(z^{-1})$
Autocorrelation	$\phi_{ff}(n) = \frac{1}{2\pi \mathrm{j}} \oint \Phi_{ff}(z) z^{n-1} dz$
Cross-correlation	$\phi_{fg}(n) = \frac{1}{2\pi \mathrm{j}} \oint \Phi_{fg}(z) z^{n-1} dz$
Waveform power	$\mathcal{E}\left\{f_n^2\right\} = \phi_{ff}(0) = \frac{1}{2\pi \mathrm{j}} \oint \Phi_{fg}(z) z^{-1} dz$
Linear system properties	$Y(z) = H(z)F(z)$ $\Phi_{yy}(z) = H(z)H(z^{-1})\Phi_{ff}(z)$ $\Phi_{fy}(z) = H(z)\Phi_{ff}(z)$