

Substitution Methods for Deriving System Equations

A system model represented by a bond graph with a complete and unambiguous causal assignment may be used to derive system equations by a straightforward method of successive substitution as follows.

Models with Linear Energy Storage Elements: Power Variables

In the important special case in which all of the model elements are linear, a convenient choice of state variables is the efforts and flows associated with independent energy storage elements — known as power state variables or circuit state variables. That choice is the basis of the following efficient, systematic procedure for deriving state equations by successive substitution.

5. Choose power state variables. These are the *output* variables from the independent energy storage elements, and the power conjugate variables are the *input* variables.
 - 5a. efforts associated with independent capacitors.
 - 5b. flows associated with independent inertias.
6. Using the constitutive equation for the corresponding energy storage element, write the rate of change of each state variable as an explicit function of its power conjugate variable.
 - 6a. independent capacitor:
$$\frac{de}{dt} = \frac{dq}{dt} \frac{1}{C} = \frac{f}{C}$$
 - 6b. independent inertia:
$$\frac{df}{dt} = \frac{dp}{dt} \frac{1}{I} = \frac{e}{I}$$
7. **Reading from the causal strokes on the graph**, write an expression for the *input* variable for each independent energy storing element as a function of the *output* variables from one or more of the other elements in the system.
 - 7a. This step proceeds by direct substitution using the constitutive equation(s) for each of the elements as needed, *expressed in the input/output form specified by the causal strokes on the graph*.
 - 7b. Continue substitution until the input to each independent energy storing element is expressed as a function *only* of one or more of the state variables or of the independent variables associated with sources.

This seven-step causality assignment and substitution procedure will result in state equations for the system model. With a little practice, in most cases state equations can be written by inspection from the causally augmented bond graph of the system model.

Example: Electric Circuit

An electric circuit schematic is shown in figure 6.1a. This schematic is already a model of the system and specifies the energetic elements to be used: an inertia, capacitor, resistor and effort source. It is evident from the schematic that all elements share the same flow and thus are connected by a single one-*junction*. A bond graph for the circuit, with sign convention included, is shown in figure 6.1b.

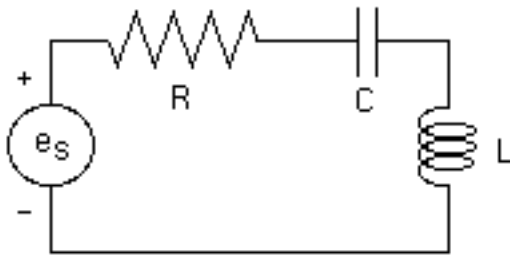


Figure 6.1a

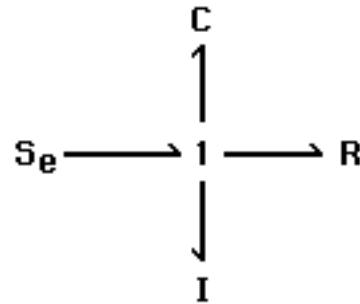


Figure 6.1b

Assume all elements have linear constitutive equations. The effort source must impress an effort on the rest of the system as shown in figure 6.1c. Assigning integral causality to the inertia and propagating that causal assignment through the one-*junction* completes the causal assignment as in figure 6.1d.

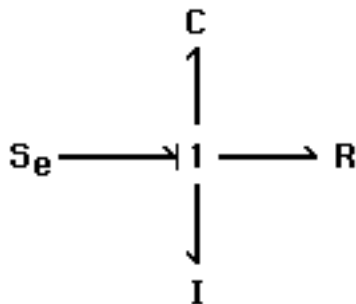


Figure 6.1c

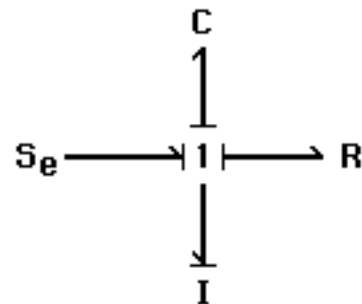


Figure 6.1d

From the causal graph, the capacitor and inductor are independent energy storage elements. Power (or circuit) state variables are the capacitor voltage, e_C , and the inductor current, i_L . From the constitutive equations for these elements:

$$\frac{d}{dt} e_C = \frac{i_C}{C} \tag{6.1}$$

$$\frac{d}{dt} i_L = \frac{e_L}{L} \tag{6.2}$$

Reading from the causal graph, the inductor determines the current associated with the one-junction, which in turn determines the current for the other connected elements. Thus,

$$i_C = i_L \quad (6.3)$$

Substituting, one state equation is:

$$\frac{d}{dt} e_C = \frac{i_L}{C} \quad (6.4)$$

Again reading from the causal graph, the inductor voltage is determined by the other three one-port elements.

$$e_L = e_S - e_C - e_R \quad (6.5)$$

Note that the signs in this junction equation are read directly from the half-arrows on the graph. In words, voltage into the inductor equals voltage into the junction from the source minus the voltages out of the junction into the capacitor and the resistor.

From the causal graph, the appropriate form of the constitutive equation for the dissipator is resistance causality.

$$e_R = R i_R \quad (6.6)$$

Again reading from the causal graph, the resistor current is determined by the inductor current.

$$i_R = i_L \quad (6.7)$$

Substituting, the second state equation is:

$$\frac{d}{dt} i_L = \frac{1}{L} (e_S - e_C - R i_R) \quad (6.8)$$

Note that the substitution process stops when the rate of change of a state variable has been expressed as a function of state and input variables. The state equations may be written in the conventional vector/matrix form as follows.

$$\frac{d}{dt} \begin{bmatrix} e_C \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} e_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e_S \quad (6.9)$$