Lecture 9 - u/p formulation

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We want to solve

Reading: Sec. 4.4.3

I. Equilibrium

$$\begin{cases} \tau_{ij,j} + f_i^B = 0 & \text{in Volume} \\ \tau_{ij} n_j = f_i^{S_f} & \text{on } S_f \end{cases} \tag{9.1}$$

II. Compatibility

III. Stress-strain law

Use the principle of virtual displacements

$$\int_V \overline{\boldsymbol{\epsilon}}^T \boldsymbol{C} \boldsymbol{\epsilon} \, dV = \mathcal{R} \tag{9.2}$$

We recognize that if $\nu \to 0.5$

$$\epsilon_V \to 0 \quad (\epsilon_V = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \tag{9.3}$$

$$\kappa = \frac{E}{3(1-2\nu)} \to \infty \tag{9.4}$$

$$p = -\kappa \epsilon_V \quad \text{must be accurately computed} \tag{9.5}$$

Solution

$$\tau_{ij} = \kappa \epsilon_V \delta_{ij} + 2G\epsilon'_{ij} \tag{9.6}$$

where

$$\delta_{ij} = \text{ Kronecker delta } = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{9.7}$$

Deviatoric strains:

$$\epsilon'_{ij} = \epsilon_{ij} - \frac{\epsilon_V}{3}\delta_{ij} \tag{9.8}$$

$$\tau_{ij} = -p\delta_{ij} + 2G\epsilon'_{ij} \quad \left(p = -\frac{\tau_{kk}}{3}\right) \tag{9.9}$$

(9.2) becomes

$$\int_V \overline{\boldsymbol{\epsilon}}'^T \boldsymbol{C}' \boldsymbol{\epsilon}' \, dV + \int_V \overline{\epsilon}_V \kappa \epsilon_V \, dV = \mathcal{R} \tag{9.10}$$

$$\int_V \overline{\boldsymbol{\epsilon}}'^T \boldsymbol{C}' \boldsymbol{\epsilon}' \, dV - \int_V \overline{\epsilon}_V^T p \, dV = \mathcal{R} \tag{9.11}$$

We need another equation because we now have another unknown p.

$$p + \kappa\epsilon_V = 0 \tag{9.12}$$

$$\int_V \overline{p}\left(p + \kappa\epsilon_V\right)\, dV = 0 \tag{9.13}$$

$$-\int_V \overline{p}\left(\epsilon_V + \frac{p}{\kappa}\right)\, dV = 0 \tag{9.14}$$

For an element,

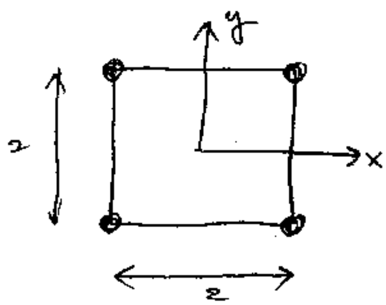
$$\boldsymbol{u} = \boldsymbol{H}\hat{\boldsymbol{u}} \tag{9.15}$$

$$\boldsymbol{\epsilon}' = \boldsymbol{B}_D\hat{\boldsymbol{u}} \tag{9.16}$$

$$\epsilon_V = \boldsymbol{B}_V\hat{\boldsymbol{u}} \tag{9.17}$$

$$p = \boldsymbol{H}_p\hat{\boldsymbol{p}} \tag{9.18}$$

Example



Plane strain ($\epsilon_{zz} = 0$)
4/1 element

Reading: Ex. 4.32 in the text

$$\epsilon_V = \epsilon_{xx} + \epsilon_{yy} \tag{9.19}$$

$$\boldsymbol{\epsilon}' = \begin{bmatrix} \epsilon_{xx} - \frac{1}{3}\left(\epsilon_{xx} + \epsilon_{yy}\right) \\ \epsilon_{yy} - \frac{1}{3}\left(\epsilon_{xx} + \epsilon_{yy}\right) \\ \gamma_{xy} \\ -\frac{1}{3}\left(\epsilon_{xx} + \epsilon_{yy}\right) \end{bmatrix} \tag{9.20}$$

Note: $\epsilon_{zz} = 0$ but $\epsilon'_{zz} \neq 0$!

$$p = \boldsymbol{H}_p\hat{\boldsymbol{p}} = [1]\{p_0\} \tag{9.21}$$

$$p(x, y) = p_0 \tag{9.22}$$

We obtain from (9.11) and (9.14)

$$\begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{up} \\ \boldsymbol{K}_{pu} & \boldsymbol{K}_{pp} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{bmatrix} \tag{9.23}$$

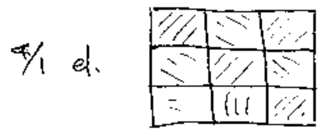
$$\boldsymbol{K}_{uu} = \int_V \boldsymbol{B}_D^T\boldsymbol{C}'\boldsymbol{B}_D\, dV \tag{9.24a}$$

$$\boldsymbol{K}_{up} = -\int_V \boldsymbol{B}_V^T\boldsymbol{H}_p\, dV \tag{9.24b}$$

$$\boldsymbol{K}_{pu} = -\int_V \boldsymbol{H}_p^T\boldsymbol{B}_V\, dV \tag{9.24c}$$

$$\boldsymbol{K}_{pp} = -\int_V \boldsymbol{H}_p^T\frac{1}{\kappa}\boldsymbol{H}_p\, dV \tag{9.24d}$$

In practice, we use elements that use pressure interpolations per element, not continuous between elements. For example:



Then, unless $\nu = 0.5$ (where $\boldsymbol{K}_{pp} = \boldsymbol{0}$), we can use static condensation on the pressure dof's.

Use $\hat{\boldsymbol{p}}$ equations to eliminate $\hat{\boldsymbol{p}}$ from the $\hat{\boldsymbol{u}}$ equations.

$$\left(\boldsymbol{K}_{uu} - \boldsymbol{K}_{up}\boldsymbol{K}_{pp}^{-1}\boldsymbol{K}_{pu}\right)\hat{\boldsymbol{u}} = \boldsymbol{R} \tag{9.25}$$

(In practice, ν can be 0.499999. . .)

The "best element" is the 9/3 element. (9 nodes for displacement and 3 pressure dof's).

$$p(x, y) = p_0 + p_1 x + p_2 y \tag{9.26}$$

The inf-sup condition

Reading: Sec. 4.5

$$\underbrace{\inf}_{q_h \in \mathrm{Q}_h} \; \underbrace{\sup}_{v_h \in \mathrm{V}_h} \left[\frac{\int_{\mathrm{Vol}} q_h \overbrace{\boldsymbol{\nabla} \cdot \boldsymbol{v}_h}^{=\epsilon_V} \, d\mathrm{Vol}}{\underbrace{\|q_h\| \, \|\boldsymbol{v}_h\|}_{\text{for normalization}}} \right] \geq \beta > 0 \tag{9.27}$$

Q_h: pressure space.

If "this" holds, the element is optimal for the displacement assumption used (ellipticity must also be satisfied).

Note:

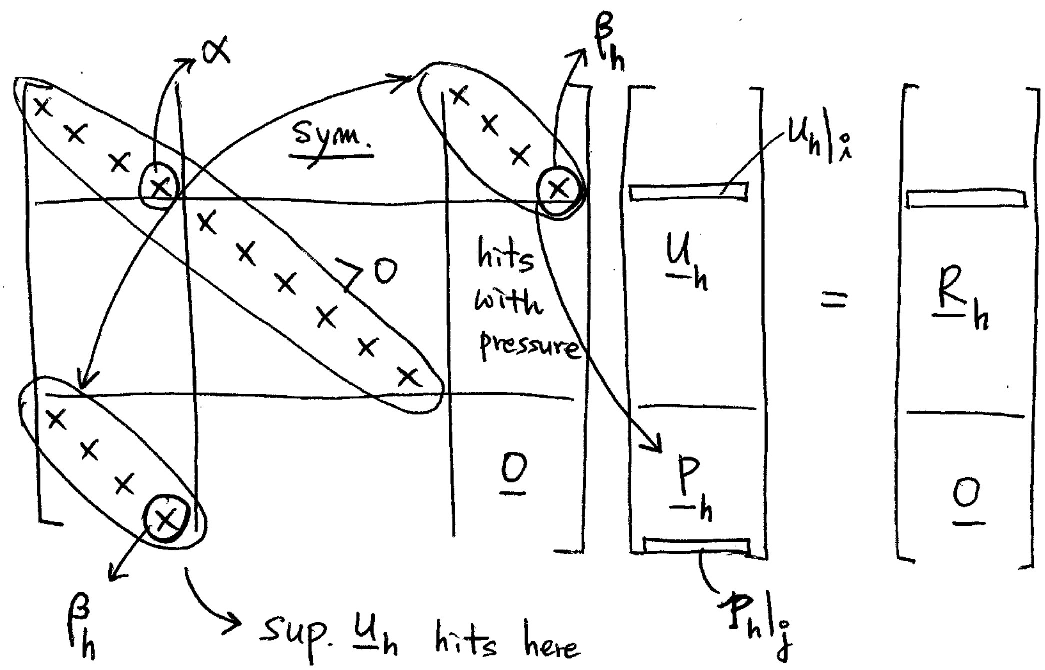
$$\begin{aligned} \text{infimum} &= \text{largest lower bound} \\ \text{supremum} &= \text{least upper bound} \end{aligned}$$

For example,

$$\begin{aligned} \inf\,\{1, 2, 4\} &= 1 \\ \sup\,\{1, 2, 4\} &= 4 \\ \inf\,\{x \in \mathrm{R}; \quad 0 < x < 2\} &= 0 \\ \sup\,\{x \in \mathrm{R}; \quad 0 < x < 2\} &= 2 \end{aligned}$$

(9.23) **rewritten** ($\kappa = \infty$, full incompressibility). Diagonalize using eigenvalues/eigenvectors.

For a mesh of element size h we want $\boxed{\beta_h > 0}$ as we refine the mesh, $h \to 0$



For  (entry [3,1] in matrix) assume the circled entry is the minimum (inf) of  .

Also, all entries in the matrix not shown are zero.

Case 1 $\beta_h = 0$

$$\Rightarrow \begin{cases} 0 \cdot u_h|_i = 0 & \text{(from the bottom equation)} \\ \underbrace{\alpha}_{\neq 0} \cdot u_h|_i + 0 \cdot p_h|_j = R_h|_i & \text{(from the top equation)} \end{cases}$$

$\Rightarrow$ no equation for $p_h|_j$

$\Rightarrow$ spurious pressure! (any pressure satisfies equation)

Case 2 $\beta_h = \text{small} = \epsilon$

$$\epsilon \cdot u_h|_i = 0 \quad \Rightarrow \boxed{u_h|_i = 0}$$

$$\therefore \epsilon \cdot \quad p_h|_j + u_h|_i \cdot \alpha = R_h|_i$$

$$\Rightarrow p_h|_j = \frac{R_h|_i}{\epsilon} \quad \Rightarrow \begin{pmatrix} \text{displ.} & = & 0 \\ \text{pressure} & \rightarrow & \text{large} \end{pmatrix} \text{ as } \epsilon \text{ is small}$$

The behavior of given mesh when bulk modulus increases: locking, large pressures. See Example 4.39 textbook.