Lecture 3 - Finite element formulation for solids and structures

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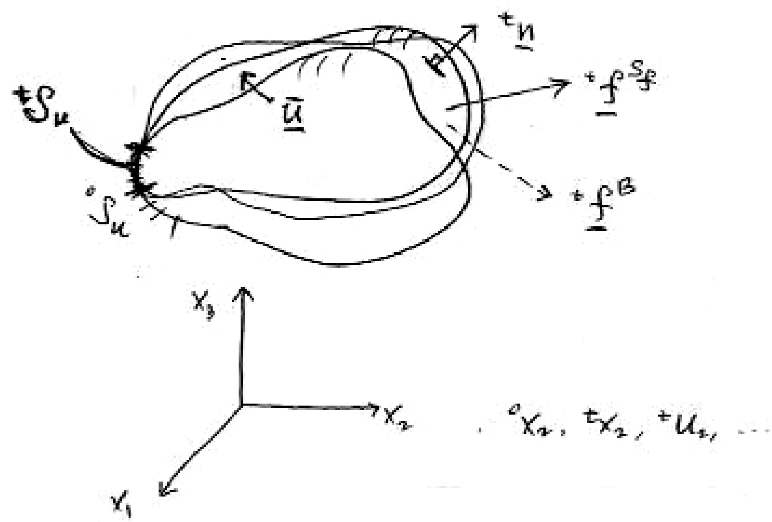


Reading:
Sec. 6.1-6.2

We need to satisfy at time t:

- *Equilibrium*

$$\frac{\partial^t \tau_{ij}}{\partial^t x_j} + {}^t f_i^B = 0 \quad (i = 1, 2, 3) \text{ in } {}^t V \tag{3.1}$$

$${}^t \tau_{ij} \, {}^t n_j = {}^t f_i^{S_f} \quad (i = 1, 2, 3) \text{ on } {}^t S_f \tag{3.2}$$

- *Compatibility*
- *Stress-strain law(s)*

Principle of virtual displacements

$$\int_{{}^t V} {}^t \tau_{ij} \, {}_t \bar{e}_{ij} \, d^t V = \int_{{}^t V} \bar{u}_i \, {}^t f_i^B \, d^t V + \int_{{}^t S_f} \bar{u}_i |_{{}^t S_f} \, {}^t f_i^{S_f} \, d^t S_f \tag{3.3}$$

$${}_t \bar{e}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial \, {}^t x_j} + \frac{\partial \bar{u}_j}{\partial \, {}^t x_i} \right) \tag{3.4}$$

- If (3.3) holds for any continuous virtual displacement (zero on ${}^t S_u$), then (3.1) and (3.2) hold and vice versa.
- Refer to Ex. 4.2 in the textbook.

Major steps

I. Take (3.1) and weigh with $\overline{u}_i$:

$$\left({}^t\tau_{ij,j} + {}^tf_i^B\right)\overline{u}_i = 0. \tag{3.5a}$$

II. Integrate (3.5a) over volume ${}^tV$:

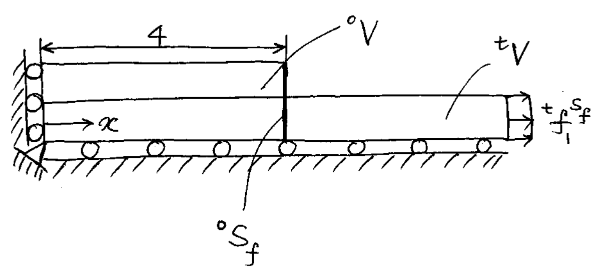
$$\int_{{}^tV}\left({}^t\tau_{ij,j} + {}^tf_i^B\right)\overline{u}_i \, d\,{}^tV = 0 \tag{3.5b}$$

III. Use divergence theorem. Obtain a boundary term of stresses times virtual displacements on ${}^tS = {}^tS_u \cup {}^tS_f$.

IV. But, on ${}^tS_u$ the $\overline{u}_i = 0$ and on ${}^tS_f$ we have (3.2) to satisfy.

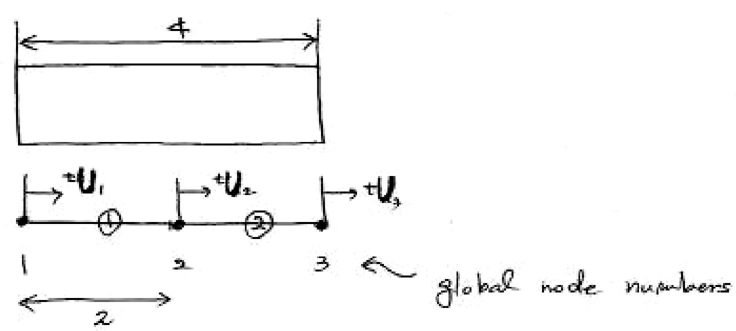
Result: (3.3).

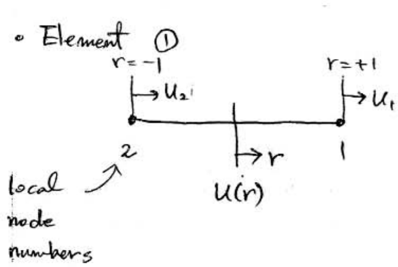
Example



$$\int_{{}^tV} {}^t\tau_{11}\,{}_t\overline{e}_{11}\, d^tV = \int_{{}^tS_f} \overline{u}_i\,{}^tf_1^{S_f}\, d^tS_f \tag{3.6}$$

One element solution:





$$u(r) = \frac{1}{2}(1+r)\,u_1 + \frac{1}{2}(1-r)\,u_2 \tag{3.7}$$

$$^{t}u(r) = \frac{1}{2}(1+r)\ {}^{t}u_1 + \frac{1}{2}(1-r)\ {}^{t}u_2 \tag{3.8}$$

$$\overline{u}(r) = \frac{1}{2}(1+r)\,\overline{u}_1 + \frac{1}{2}(1-r)\,\overline{u}_2 \tag{3.9}$$

Suppose we know ${}^{t}\tau_{11}$, ${}^{t}V$, ${}^{t}S_f$, ${}^{t}u$ ... use (3.6).

For element 1,

$${}_{t}\overline{e}_{11} = \frac{\partial \overline{u}}{\partial\, {}^{t}x} = \boldsymbol{B}^{(1)} \begin{pmatrix} \overline{u}_1 \\ \overline{u}_2 \end{pmatrix} \tag{3.10}$$

$$\int_{{}^{t}V} {}_{t}\overline{e}^{T}_{11\,t}\tau_{11} d\,{}^{t}V \xrightarrow{\text{for el. (1)}} [\overline{u}_1 \quad \overline{u}_2] \underbrace{\int_{{}^{t}V} \boldsymbol{B}^{(1)^T}\ {}^{t}\tau_{11}\ d\,{}^{t}V}_{=\,{}^{t}\boldsymbol{F}^{(1)}} \tag{3.11}$$

$$\xrightarrow{\text{for el. (1)}} [\overline{u}_1 \quad \overline{u}_2]\ {}^{t}\boldsymbol{F}^{(1)} \tag{3.12}$$

$$= \begin{bmatrix} \underbrace{\overline{U}_1}_{\overline{u}_2} & \underbrace{\overline{U}_2}_{\overline{u}_1} & \overline{U}_3 \end{bmatrix} \begin{bmatrix} {}^{t}\hat{\boldsymbol{F}}^{(1)} \\ 0 \end{bmatrix} \tag{3.13}$$

where

$${}^{t}\hat{F}^{(1)}_{1} = {}^{t}F^{(1)}_{2} \tag{3.14}$$

$${}^{t}\hat{F}^{(1)}_{2} = {}^{t}F^{(1)}_{1} \tag{3.15}$$

For element 2, similarly,

$$= \begin{bmatrix} \overline{U}_1 & \underbrace{\overline{U}_2}_{\overline{u}_2} & \underbrace{\overline{U}_3}_{\overline{u}_1} \end{bmatrix} \begin{bmatrix} 0 \\ {}^{t}\hat{\boldsymbol{F}}^{(2)} \end{bmatrix} \tag{3.16}$$

R.H.S.

$$\underbrace{\begin{bmatrix} \overline{U}_1 & \overline{U}_2 & \overline{U}_3 \end{bmatrix}}_{\overline{\boldsymbol{U}}^T} \begin{bmatrix} \text{(unknown reaction at left)} \\ 0 \\ {}^{t}S_f \cdot {}^{t}f_1^{S_f} \end{bmatrix} \tag{3.17}$$

Now apply,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{3.18}$$

then,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{3.19}$$

then,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{3.20}$$

This gives,

$$\begin{bmatrix} {}^t\hat{\boldsymbol{F}}^{(1)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ {}^t\hat{\boldsymbol{F}}^{(2)} \end{bmatrix} = \begin{bmatrix} \text{unknown reaction} \\ 0 \\ {}^t f_1^{\,{}^tS_f} \cdot {}^tS_f \end{bmatrix} \tag{3.21}$$

We write that as

$$ {}^t\boldsymbol{F} = {}^t\boldsymbol{R} \tag{3.22}$$

$$ {}^t\boldsymbol{F} = \mathrm{fn}\left({}^tU_1,\ {}^tU_2,\ {}^tU_3\right) \tag{3.23}$$

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