**2.094 — Finite Element Analysis of Solids and Fluids** Fall '08

# Lecture 13 - Total Lagrangian formulation, cont'd

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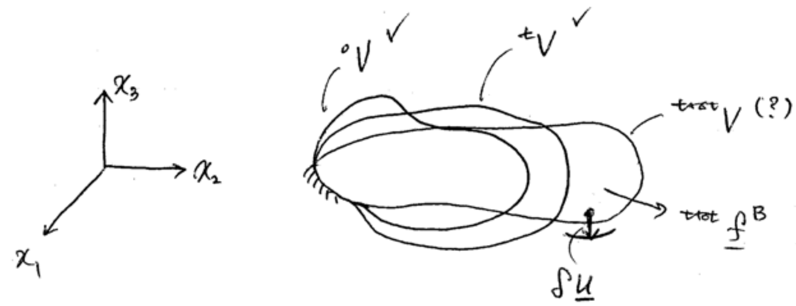
Example truss element. Recall:



Principle of virtual displacements applied at some time $t+\Delta t$:

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\delta_{t+\Delta t}e_{ij}d^{t+\Delta t}V = {}^{t+\Delta t}\mathcal{R} \tag{13.1}$$

$$\int_{^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}\delta^{t+\Delta t}_{\ \ 0}\epsilon_{ij}\delta^0V = {}^{t+\Delta t}\mathcal{R} \tag{13.2}$$

$${}^{t+\Delta t}_{\ \ 0}S_{ij} = {}^t_0S_{ij} + {}_0S_{ij} \tag{13.3}$$

$${}^{t+\Delta t}_{\ \ 0}\epsilon_{ij} = {}^t_0\epsilon_{ij} + {}_0\epsilon_{ij} \tag{13.4}$$

$${}_0\epsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij} \tag{13.5}$$

where ${}^t_0S_{ij}$ and ${}^t_0\epsilon_{ij}$ are known, but ${}_0S_{ij}$ and ${}_0\epsilon_{ij}$ are not.

$${}_0e_{ij} = \frac{1}{2}\left({}_0u_{i,j} + {}_0u_{j,i} + {}^t_0u_{k,i}\,{}_0u_{k,j} + {}^t_0u_{k,j}\,{}_0u_{k,i}\right) \tag{13.6}$$

$${}_0\eta_{ij} = \frac{1}{2}\left({}_0u_{k,i}\,{}_0u_{k,j}\right) \tag{13.7}$$

Substitute into (13.2) and linearize to obtain

$$\int_{^0V} \delta_0e_{ij\,0}C_{ijrs\,0}e_{rs}d^0V + \int_{^0V} {}^t_0S_{ij}\delta_0\eta_{ij}d^0V = {}^{t+\Delta t}\mathcal{R} - \int_{^0V} \delta_0e_{ij}{}^t_0S_{ij}d^0V \tag{13.8}$$

**F.E. discretization gives**

$$\left({}^t_0\boldsymbol{K}_L + {}^t_0\boldsymbol{K}_{NL}\right)\Delta\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^t_0\boldsymbol{F} \tag{13.9}$$

$$ {}_0^t\boldsymbol{K}_L = \int_{{}^0V} {}_0^t\boldsymbol{B}_L^T \, {}_0\boldsymbol{C} \, {}_0^t\boldsymbol{B}_L d^0V \tag{13.10} $$

$$ {}_0^t\boldsymbol{K}_{NL} = \int_{{}^0V} {}_0^t\boldsymbol{B}_{NL}^T \underbrace{{}_0^t\boldsymbol{S}}_{\text{matrix}} {}_0^t\boldsymbol{B}_{NL} d^0V \tag{13.11} $$

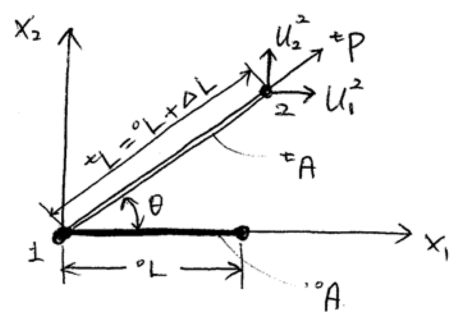
$$ {}_0^t\boldsymbol{F} = \int_{{}^0V} {}_0^t\boldsymbol{B}_L^T \underbrace{{}_0^t\hat{\boldsymbol{S}}}_{\text{vector}} d^0V \tag{13.12} $$

The iteration (full Newton-Raphson) is

$$ \left( {}^{t+\Delta t}_{\ \ \ 0}\boldsymbol{K}_L^{(i-1)} + {}^{t+\Delta t}_{\ \ \ 0}\boldsymbol{K}_{NL}^{(i-1)} \right) \Delta\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}_{\ \ \ 0}\boldsymbol{F}^{(i-1)} \tag{13.13} $$

$$ {}^{t+\Delta t}\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{U}^{(i-1)} + \Delta\boldsymbol{U}^{(i)} \tag{13.14} $$

**Truss element example** (p. 545)




Here we have to only deal with ${}_0^tS_{11}$, ${}_0e_{11}$, ${}_0\eta_{11}$

$$ {}_0e_{11} = \frac{\partial u_1}{\partial^0 x_1} + \frac{\partial^t u_k}{\partial^0 x_1} \cdot \frac{\partial u_k}{\partial^0 x_1} \tag{13.15} $$

$$ {}_0\eta_{11} = \frac{1}{2}\left( \frac{\partial u_k}{\partial^0 x_1} \cdot \frac{\partial u_k}{\partial^0 x_1} \right) \tag{13.16} $$

We are after

$$ {}_0e_{11} = {}_0^t\boldsymbol{B}_L \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{pmatrix} = {}_0^t\boldsymbol{B}_L \hat{\boldsymbol{u}} \tag{13.17} $$

$$ u_i = \sum_{k=1}^{2} h_k u_i^k \tag{13.18} $$

$$ {}^tu_i = \sum_{k=1}^{2} h_k \, {}^tu_i^k \tag{13.19} $$

$$
\begin{aligned}
{}_0e_{11} &= \frac{\partial u_1}{\partial^0 x_1} + \frac{\partial^t u_1}{\partial^0 x_1}\frac{\partial u_1}{\partial^0 x_1} + \frac{\partial^t u_2}{\partial^0 x_1}\frac{\partial u_2}{\partial^0 x_1} && (13.20a)\\
{}^t u_1^2 &= \left({}^0L + \Delta L\right)\cos\theta - {}^0L && (13.20b)\\
{}^t u_2^2 &= \left({}^0L + \Delta L\right)\sin\theta && (13.20c)
\end{aligned}
$$

$$
\begin{aligned}
{}_0e_{11} =& \frac{1}{{}^0L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\hat{\boldsymbol{u}}\\
&+ \left(\underbrace{\frac{{}^0L + \Delta L}{{}^0L}\cos\theta - 1}_{\frac{\partial^t u_1}{\partial^0 x_1}}\right)\cdot\frac{1}{{}^0L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\hat{\boldsymbol{u}}\\
&+ \left(\underbrace{\frac{{}^0L + \Delta L}{{}^0L}\sin\theta}_{\frac{\partial^t u_2}{\partial^0 x_1}}\right)\cdot\frac{1}{{}^0L}\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}\hat{\boldsymbol{u}} && (13.20d)\\
=& {}^t_0\boldsymbol{B}_L\hat{\boldsymbol{u}} && (13.20e)
\end{aligned}
$$

Hence,

$$
{}_0e_{11} = \boxed{\frac{{}^0L + \Delta L}{\left({}^0L\right)^2}\begin{bmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{bmatrix}}\,\hat{\boldsymbol{u}} \qquad (13.20f)
$$

where the boxed quantity above equals ${}^t_0\boldsymbol{B}_L$. In small strain but large rotation analysis we assume $\Delta L \ll {}^0L$,

$$
{}_0e_{11} = \frac{1}{{}^0L}\begin{bmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{bmatrix}\hat{\boldsymbol{u}} \qquad (13.20g)
$$

$$
{}_0\eta_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial^0 x_1}\frac{\partial u_1}{\partial^0 x_1} + \frac{\partial u_2}{\partial^0 x_1}\frac{\partial u_2}{\partial^0 x_1}\right) \qquad (13.21a)
$$

$$
\begin{aligned}
\delta_0\eta_{11} &= \frac{1}{2}\left(\frac{\partial\delta u_1}{\partial^0 x_1}\frac{\partial u_1}{\partial^0 x_1} + \frac{\partial u_1}{\partial^0 x_1}\frac{\partial\delta u_1}{\partial^0 x_1} + \frac{\partial\delta u_2}{\partial^0 x_1}\frac{\partial u_2}{\partial^0 x_1} + \frac{\partial u_2}{\partial^0 x_1}\frac{\partial\delta u_2}{\partial^0 x_1}\right) && (13.21b)\\
&= \left(\frac{\partial\delta u_1}{\partial^0 x_1}\frac{\partial u_1}{\partial^0 x_1} + \frac{\partial\delta u_2}{\partial^0 x_1}\frac{\partial u_2}{\partial^0 x_1}\right) && (13.21c)
\end{aligned}
$$

$$
{}^t_0S_{11}\delta_0\eta_{11} = \begin{bmatrix} \frac{\partial\delta u_1}{\partial^0 x_1} & \frac{\partial\delta u_2}{\partial^0 x_1} \end{bmatrix}\underbrace{\begin{pmatrix} {}^t_0S_{11} & 0\\ 0 & {}^t_0S_{11} \end{pmatrix}}_{{}^t_0\boldsymbol{S}}\begin{pmatrix} \frac{\partial u_1}{\partial^0 x_1}\\ \frac{\partial u_2}{\partial^0 x_1} \end{pmatrix} \qquad (13.21d)
$$

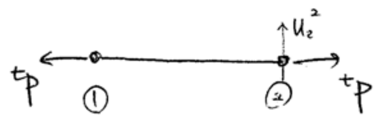
$$
\begin{pmatrix} \frac{\partial u_1}{\partial^0 x_1}\\ \frac{\partial u_2}{\partial^0 x_1} \end{pmatrix} = \underbrace{\frac{1}{{}^0L}\begin{bmatrix} -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{B}_{NL}}\hat{\boldsymbol{u}} \qquad (13.21e)
$$

$$ {}_0\boldsymbol{C} = E \tag{13.22}$$

$$ {}_0^t\hat{\boldsymbol{S}} = {}_0^tS_{11} \tag{13.23}$$

Assume small strains

$$\begin{aligned} {}_0^t\boldsymbol{K} &= \frac{EA}{{}^0L} \\ &= \underbrace{\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ & & \cos^2\theta & \sin\theta\cos\theta \\ \text{sym} & & & \sin^2\theta \end{bmatrix}}_{{}_0^t\boldsymbol{K}_L} \\ &\quad + \underbrace{\frac{{}^tP}{{}^0L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{{}_0^t\boldsymbol{K}_{NL}} \end{aligned} \tag{13.24}$$




When $\theta = 0$, ${}_0^t\boldsymbol{K}_L$ doesn't give stiffness corresponding to $u_2^2$, but ${}_0^t\boldsymbol{K}_{NL}$ does.